

The Ruelle Invariant And Convexity In Higher Dimensions

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Background

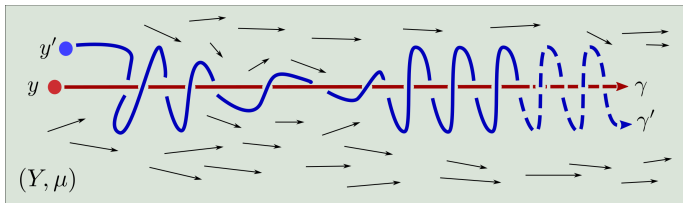
Consider a closed 3-manifold Y equipped with a smooth flow

$$\phi : \mathbb{R} \times Y \rightarrow Y \quad \text{preserving a measure } \mu$$

In 1985, Ruelle introduced his eponymous *Ruelle invariant*

$$\text{Ru}(Y, \phi) \in \mathbb{R}$$

This invariant is the integral of a function $\text{ru}(\phi)$ that “measures the linking of nearby trajectories” of ϕ in Y .



Invariant has since appeared in areas like

- Low-dimensional dynamics (Gambuodo-Ghys)
- Bifurcation theory (Parlitz)
- Sturm-Liouville theory (Schultz-Baldes)

Very recently, Ruelle invariant has been applied to 3-dimensional Reeb dynamics and 4-dimensional symplectic geometry.

- [1] M. Hutchings. *ECH capacities and the Ruelle invariant*
- [2] w/ O. Edtmair. *3d convex contact forms and the Ruelle invariant*
- [3] J. Dardennes et al. *Symplectic non-convexity of toric domains*

Today's talk: [arXiv:2205.00935](https://arxiv.org/abs/2205.00935) w/ O. Edtmair

- Generalizes R_u and some results of [2,3] to higher dimensions.
- Application: dynamical convexity \neq convexity in all dimensions.

(Generalized) Ruelle Invariant Of A Cocycle

Our version of the Ruelle invariant $Ru(\Phi, \tau)$ takes (as input) a symplectic cocycle Φ and a homotopy class of trivialization τ of ΛE .

Let Y be a compact manifold equipped with

a flow $\phi : \mathbb{R} \times Y \rightarrow Y$ preserving a measure μ

Definition

A *symplectic cocycle* Φ of (Y, ϕ) consists of a symplectic vector-bundle E and a family of symplectic linear maps

$$\Phi(t, x) : E_x \rightarrow E_{\phi(t, x)} \quad \text{for} \quad (t, x) \in \mathbb{R} \times Y$$

satisfying the following *cocycle property*.

$$\Phi(s + t, x) = \Phi(t, \phi(s, x)) \circ \Phi(s, x) \quad (1)$$

A cocycle on $E = \mathbb{C}^n$ is a map $\Phi : \mathbb{R} \times Y \rightarrow \text{Sp}(2n)$ satisfying (1).

Given a compatible complex structure, i.e. a symplectic bundle map

$$J : E \rightarrow E \quad \text{satisfying} \quad \omega(J-, -) > 0 \text{ and } J^2 = \text{Id}$$

We may associate a complex *determinant line bundle* to E .

$$\Lambda E = \Lambda_{\mathbb{C}}^n E \quad \text{where} \quad \text{rank}(E) = 2n$$

Space of J is contractible, so ΛE is independent of J up to isomorphism. If $c_1(E) = 0$, then can choose trivializations

$$\tau : \Lambda E \simeq \mathbb{C}$$

If $E = \mathbb{C}^n$, then there is a canonical $\tau_{\text{std}} : \Lambda \mathbb{C}^n \simeq \mathbb{C}$.

Definition

We let $\mathcal{T}(E)$ denote the set of homotopy classes

$$\mathcal{T}(E) = \{\tau : \Lambda E \simeq \mathbb{C}\} / \text{homotopy}$$

Note that $\mathcal{T}(E)$ is independent of J up to canonical bijection.

Theorem

Given a symplectic cocycle Φ on E and $\tau \in \mathcal{T}(E)$, there is a well-defined Ruelle density and accompanying Ruelle invariant

$$\text{ru}(\Phi, \tau) \in L^1(Y, \mu) \quad \text{and} \quad \text{Ru}(\Phi, \tau, \mu) := \int_Y \text{ru}(\Phi, \tau) \cdot \mu$$

with the following properties.

- (Naturality) If $\Psi : E \simeq E'$ is an isomorphism then

$$\text{ru}(\Psi^* \Phi, \Psi^* \tau) = \text{ru}(\Phi, \tau)$$

- (Direct Sum) They are additive under bundle sum

$$\text{ru}(\Phi \oplus \Phi', \tau \otimes \tau') = \text{ru}(\Phi, \tau) + \text{ru}(\Phi', \tau')$$

- (Trivial Bundle) If Φ is a symplectic cocycle on \mathbb{C}^n , then

$$\text{ru}(\Phi, \tau) = \lim_{T \rightarrow \infty} \frac{\rho \circ \tilde{\Phi}(T, -)}{T}$$

Here $\tilde{\Phi} : \mathbb{R} \times Y \rightarrow \widetilde{\text{Sp}}(2n)$ is the lift of Φ to the universal cover of $\text{Sp}(2n)$ and $\rho : \widetilde{\text{Sp}}(2n) \rightarrow \mathbb{R}$ is a rotation quasimorphism.

Rotation Quasimorphism

Definition

A *quasimorphism* $\rho : G \rightarrow \mathbb{R}$ on a group G is a map such that

$$|\rho(gh) - \rho(g) - \rho(h)| < C \text{ for some } C \text{ independent of } g, h$$

Two quasimorphisms ρ, ρ' are *equivalent* if $|\rho - \rho'|$ is bounded.

Example

Consider the map $\mathrm{Sp}(2n) \rightarrow \mathbb{R}$ given by

$$\underline{r} : \mathrm{Sp}(2n) \xrightarrow{\text{unitary part}} \mathrm{U}(n) \xrightarrow{\det_{\mathbb{C}}} \mathrm{U}(1) \quad (2)$$

The *determinant quasimorphism* $r : \widetilde{\mathrm{Sp}}(2n) \rightarrow \mathbb{R}$ is defined by

$$\underline{r}(A) = \exp(2\pi i \cdot r(\tilde{A})) \quad \text{if } \tilde{A} \in \widetilde{\mathrm{Sp}}(2n) \text{ lifts } A \in \mathrm{Sp}(2n)$$

A *rotation quasimorphism* is any quasimorphism equivalent to r .

Sketch Of Construction

- (1) Choose an almost complex structure J on E and a unitary trivialization $\tau : \Lambda E \simeq \mathbb{C}$ representing the homotopy class.
- (2) Given J , we have a polar decomposition

$$E_x \xrightarrow{P(t,x)} E_x \xrightarrow{U(t,x)} E_{\phi(t,x)} \quad \text{of the cocycle map } \Phi(t,x)$$

- (3) The unitary map $U(t,x)$ of Hermitian vector-spaces induces a map of determinant lines.

$$\Lambda U(t,x) : \Lambda E_x \rightarrow \Lambda E_{\phi(t,x)}$$

- (4) The trivialization $\tau : \Lambda E \simeq \mathbb{C}$ identifies $\Lambda U(t,x)$ with a map

$$u : \mathbb{R} \times Y \rightarrow U(1) \quad \text{lifting to a map} \quad \tilde{u} : \mathbb{R} \times Y \rightarrow \mathbb{R}$$

- (5) The Ruelle density $\text{ru}(\Phi, \tau) \in L^1(Y, \mu)$ is the limit

$$\text{ru}(\Phi, \tau) := \lim_{T \rightarrow \infty} \frac{\tilde{u}(T, -)}{T}$$

Special Cases

Example (Symplectic/Hamiltonian Flow)

A symplectic flow Φ on a compact symplectic manifold (X, ω) has

a *tangent cocycle* $T\Phi$ on the bundle TX

Since Φ preserves the measure ω^n , we get a Ruelle invariant

$$\text{Ru}(X, \Phi, \tau) \quad \text{for any} \quad \tau \in \mathcal{T}(TX)$$

We use notation $\text{Ru}(X, H, \tau)$ if Φ is generated by a Hamiltonian H .

Example (Reeb Flow)

The Reeb flow ϕ on a compact contact manifold (Y, ξ) with contact form α preserves $\alpha \wedge d\alpha^{n-1}$ and has a cocycle

$$T\phi|_{\xi} \text{ on the bundle } \xi$$

Thus have a Ruelle invariant $\text{Ru}(Y, \alpha, \tau)$ for every $\tau \in \mathcal{T}(\xi)$.

Example (Star-Shaped Domains)

Let $X \subset \mathbb{C}^n$ be a star-shaped domain with boundary ∂X transverse to the standard Liouville vector-field

$$Z = \frac{1}{2} \cdot \sum_i x_i \cdot \partial_{x_i} + y_i \cdot \partial_{y_i}$$

The *canonical Hamiltonian* $H_X : \mathbb{C}^n \rightarrow \mathbb{R}$ of X is defined by

$$H_X^{-1}(1) = \partial X \quad \text{and} \quad ZH_X = H_X$$

Since $c_1(X) = 0$ and $H^1(X; \mathbb{Z}) = 0$, we have a unique trivialization τ up to homotopy, and thus a Ruelle invariant $\text{Ru}(X) := \text{Ru}(X, H_X)$.

Lemma

If X is a star-shaped domain, then

$$\text{Ru}(X) = \text{Ru}(\partial X, \lambda|_{\partial X})$$

Moreover, $\text{Ru}(X)$ is invariant under symplectomorphism.

Toric Domains

Definition

A *toric domain* $X_\Omega \subset \mathbb{C}^n$ with *moment region* $\Omega \subset [0, \infty)^n$ is a domain of the form $\mu^{-1}(\Omega)$ where μ is the moment map

$$\mu : \mathbb{C}^n \rightarrow [0, \infty)^n \quad \text{with} \quad \mu(z_1 \dots z_n) = \pi \cdot (|z_1|^2 \dots |z_n|^2)$$

The canonical Hamiltonian of a star-shaped toric X_Ω factors as

$$H_X := f_\Omega \circ \mu$$

Here $f_\Omega : [0, \infty)^n \rightarrow \mathbb{R}$ is the unique function satisfying

$$f_\Omega^{-1}[0, 1] = \Omega \quad \text{and} \quad \sum_i x_i \cdot \partial_i f_\Omega(x) = f_\Omega(x)$$

Theorem

The Ruelle density $\text{ru}(X_\Omega)$ and Ruelle invariant $\text{Ru}(X_\Omega)$ are given by

$$\text{ru}(X_\Omega) = \left(\sum_i \partial_i f_\Omega \right) \circ \mu \quad \text{and} \quad \int_\Omega n! \cdot \sum_i \partial_i f_\Omega \cdot dx^n$$

Example

The standard ellipsoid $E = E(a_1, \dots, a_n)$ has moment region $\Delta = \Delta(a_1, \dots, a_n)$, and f_Δ is given by

$$f_\Delta(x_1, \dots, x_n) = \sum_i \frac{x_i}{a_i}$$

Therefore the Ruelle invariant is given by

$$\text{Ru}(E) = n! \cdot \int_\Delta \sum_i \frac{1}{a_i} \cdot dx^n = \left(\sum_i \frac{1}{a_i} \right) \cdot \prod_i a_i$$

Proof

The moment map $\mu = (\mu_1, \dots, \mu_n)$ extends to toric coordinates

$$(\mu, \theta) : (\mathbb{C}^\times)^n \simeq (0, \infty)^n \times T^n$$

In these coordinates, the Hamiltonian vector-field is given by

$$V_H(z) = \sum_i \partial_i f_\Omega(\mu(z)) \cdot \partial_{\theta_i}$$

The flow of V_H is given by $\Phi(t, z) = U(t, \mu(z))z$ where $U(t, x)$ is the unitary matrix on \mathbb{C}^n with diagonal entries

$$u(t, x) = \exp(2\pi i t \cdot \partial_i f_\Omega(x))$$

The differential $T\Phi(t, x) \in \text{Sp}(2n)$ can then be calculated to be

$$T\Phi(t, z) = U(t, \mu(z))Q(t, z) \quad \text{where} \quad Q(t, z) = \text{Id} + t \cdot M(z)$$

Here $M(z)$ is a t -independent matrix-valued function of z .

Proof (Continued)

Now by the (Trivial Bundle) property of the Ruelle density, we write

$$\text{ru}(X_\Omega)(z) = \lim_{T \rightarrow \infty} \frac{r(\widetilde{T\Phi}(T, z))}{T}$$

By the quasimorphism property for r , we have

$$\lim_{T \rightarrow \infty} \frac{r(\widetilde{T\Phi}(T, z))}{T} = \lim_{T \rightarrow \infty} \frac{r(\widetilde{U}(T, \mu(z)))}{T} + \lim_{T \rightarrow \infty} \frac{r(\widetilde{Q}(T, z))}{T}$$

Since r is the lift of $\det_{\mathbb{C}}$ on $\widetilde{U}(n)$, we find that

$$r(\widetilde{U}(T, \mu(z))) = T \cdot \sum_i \partial_i f_\Omega(\mu(z))$$

On the otherhand, easy exercise to show that

$$\text{Id} + t \cdot M(z) \in \text{Sp}(2n) \text{ for all } t \implies M(z) \text{ is nilpotent}$$

Thus $Q(t, z)$ is sheer for all t . This implies that $r(\widetilde{Q}(t, z))$ is bounded (easier to see this using another rotation quasimorphism). \square

Symplectically Convex Domains

Definition

A star-shaped domain X is *symplectically convex* if there is a convex domain $X' \subset \mathbb{C}^n$ such that X is symplectomorphic to X' .

The Ruelle invariants of symplectically convex domains satisfy a certain systolic inequality. To be specific, let

$$c(X) := \min \{ T : T \text{ is the period of a closed Reeb orbit on } \partial X \}$$

Moreover, let $\text{vol}(X)$ be the (symplectic) volume of X . Then

Theorem (C-Edtmair)

There is a constant $C(n)$ so that, for any convex domain $X \subset \mathbb{C}^n$

$$c(X) \cdot \text{Ru}(X) \leq C(n) \cdot \text{vol}(X)$$

This generalizes our upper bound from [2] in dimension four.

Trace Bound

The proof of the Ruelle bound has a few ingredients.

A path Φ in $\text{Sp}(2n)$ is *generated* by a path of symmetric matrices S if

$$\frac{d\Phi}{dt} = J_0 S(t) \Phi(t) \quad \text{for all } t \text{ in the domain } [0, T]$$

Any path based at Id represents an element of $\widetilde{\text{Sp}}(2n)$.

Lemma (Trace Bound)

If $\tilde{\Phi}$ is generated by path of positive definite matrices S , then

$$r(\tilde{\Phi}) \leq \int_0^T \text{tr}(S(t)) \cdot dt$$

Laplacian Integral

The linearized flow of H_X is generated by the Hessian $\nabla^2 H_X$. By applying the trace bound, we can prove that

Lemma (Laplacian Integral)

The Ruelle invariant of a convex, star-shaped $X \subset \mathbb{C}^n$ satisfies

$$\text{Ru}(X) \leq S(H_X) := \int_X \Delta H_X \cdot \omega^n$$

By a direct analysis of this Laplacian integral, one can show that

Lemma (Sandwiching)

There is a constant $C(n, L)$ such that if $X \subset W \subset L \cdot X \subset \mathbb{C}^n$ then

$$S(H_X) \leq C(n, L) \cdot S(H_W)$$

John Ellipsoid

On the otherhand, we know by John's ellipsoid theorem that

Lemma (John Ellipsoid)

If X is convex, then there is an affine symplectomorphism Ψ and a standard ellipsoid $E = E(a_1, \dots, a_n)$ such that

$$X \subset E \subset 2n \cdot X$$

By combining the Laplacian bound, Sandwiching lemma and the John ellipsoid lemma, we find that for any convex domain X ,

$$\begin{aligned} c(X) \cdot \text{Ru}(X) &\leq c(X) \cdot S(H_X) \leq C(n, 2n) \cdot c(E) \cdot S(H_E) \\ \text{vol}(E) &\leq (2n)^{2n} \cdot \text{vol}(X) \end{aligned}$$

where E is a standard ellipsoid. This reduces the proof of the Ruelle bound to the case of ellipsoids, where it can be checked explicitly.

Dynamical Convexity

Convexity has many dynamical implications (e.g. Viterbo's conjecture) but no definition in purely symplectic terms.

Question

What is symplectic convexity?

One candidate answer, introduced by Hofer-Wysocki-Zehnder, was

Definition

A contact form α on $(S^{2n-1}, \xi_{\text{std}})$ is *dynamically convex* if

$$\text{CZ}(\gamma) \geq n + 1 \quad \text{for all closed Reeb orbits } \gamma$$

A star-shaped domain X is dynamically convex if ∂X is.

It is simple to check that strictly convex domains are dynamically convex. The converse, however, remained open for about 25 years.

Toric Counter-Example

Our results provide an easy source of non-convex, dynamically convex toric domains (introduced by Dardannes-Gutt-Zhang [3] in 4d).

Lemma

Consider the very flat ellipsoid $X(a) = X_{\Delta(a)}$ with moment region

$$\Delta(a) := \{x \in [0, \infty)^n : a^n \cdot x_1 + a^{-1} \cdot x_2 + \cdots + a^{-1} \cdot x_n \leq 1\}$$

Then the Ruelle invariant and volume of $X_{\Delta(a)}$ is given by

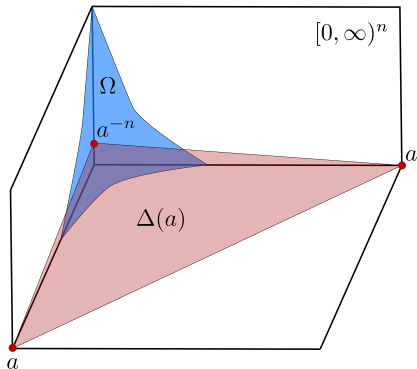
$$\text{Ru}(X_{\Delta(a)}) = \frac{a^n + (n-1) \cdot a^{-1}}{a} \quad \text{and} \quad \text{vol}(X) = \frac{1}{a}$$

In particular, $\text{Ru}(X(a)) \rightarrow \infty$ and $\text{vol}(X(a)) \rightarrow 0$ as $a \rightarrow \infty$.

Now let Ω be any concave, star-shaped moment region. That is a region where

$$[0, \infty)^n \setminus \Omega \quad \text{is convex}$$

Let $\Omega(\epsilon)$ be a smoothing of $\Omega \cup \Delta(\epsilon)$ that (1) with $\Delta(\epsilon)$ away from a small neighborhood of $\Omega \cap \Delta(a)$ and (2) contains $\Omega \cup \Delta(\epsilon)$.



Theorem (The Counter-Examples, C-Edtmair)

For any $\epsilon, C > 0$, there exists an a such that

$$c(X_\Omega) \leq c(X_{\Omega(a)}) \quad \text{Ru}(X_{\Omega(a)}) \geq C$$

$$\text{vol}(X_\Omega) \leq \text{vol}(X_{\Omega(a)}) \leq \text{vol}(X_\Omega) + \epsilon$$

Proof.

For concave toric domains, the minimum action and volume are monotonic under inclusion of moment region. Thus

$$c(X_\Omega) \leq c(X_{\Omega(a)}) \quad \text{and} \quad \text{vol}(X_\Omega) \leq \text{vol}(X_{\Omega(a)}) \quad \text{for any } a$$

Moreover, we approximately have

$$\text{Ru}(X_{\Omega(a)}) \sim \text{Ru}(X_\Omega) + \text{Ru}(X_{\Delta(a)}) \geq \text{Ru}(X_\Omega) + a^{n-1}$$

$$\text{vol}(X_{\Omega(a)}) \leq \text{vol}(X_\Omega) + \text{vol}(X_{\Delta(a)}) = \text{vol}(X_\Omega) + \frac{1}{a}$$

These become the desired inequalities as $a \rightarrow \infty$. □

To conclude, we simply recall the following fact due to Gutt-Hutchings (and generalized by us).

Lemma (Gutt-Hutchings, C-Edtmair)

Any concave (and more generally, strictly monotone) toric domain is dynamically convex.

Since the counter-examples theorem states that $X_{\Omega(a)}$ violates the Ruelle bound for large a , we acquire the following corollary.

Corollary

There exist dynamically convex domains in \mathbb{C}^n that are not symplectically convex.

Thank you!