Integer-Valued Gromov–Witten Type Invariants

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Outline

1. Gromov–Witten invariants and virtual fundamental cycles
2. Normally polynomial sections of Fukaya–Ono
3. The integral Euler cycle and the integral invariants
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I. Gromov–Witten invariants and virtual fundamental cycles

Gromov–Witten invariants

- Fix a compact symplectic manifold (X, ω) and a compatible almost complex structure J. Fix $A \in H_2(X;\mathbb{Z})$ and $g, n \geq 0$.
- Consider the moduli space of J-holomorphic maps

$$\begin{array}{ccc} & \overline{\mathcal{M}}_{g,n}(X, J; A) & \\ {}^{\mathrm{ev}}\swarrow & & \searrow^{\mathrm{st}} \\ X^n & & \overline{\mathcal{M}}_{g,n} \end{array}.$$

- The Gromov–Witten invariants are defined via the virtual fundamental class

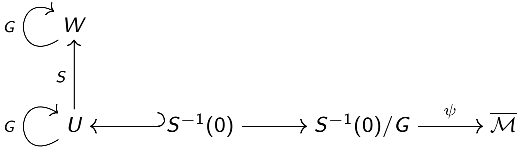
$$[\overline{\mathcal{M}}_{g,n}(X, J, A)]^{\mathrm{vir}} \in H_*(X^n \times \overline{\mathcal{M}}_{g,n}; \mathbb{Q}).$$

Finite dimensional reduction

- We need to additional structures to construct the virtual fundamental class.

Definition (Fukaya–Ono)

A Kuranishi chart on $\overline{\mathcal{M}}$ is (G, U, W, S, ψ) where G is a finite group, U is an (open) G-manifold, W is a G-representation, $S : U \to W$ is a G-equivariant map, and $\psi : S^{-1}(0)/G \to \overline{\mathcal{M}}$ is a homeomorphism onto an open set.



Virtual fundamental cycle

- One can cover $\overline{\mathcal{M}}$ by such Kuranishi charts $(G_\alpha, U_\alpha, W_\alpha, S_\alpha)$ with certain coordinate changes. The union of a nice collection of charts gives a virtual neighborhood of $\overline{\mathcal{M}}$

$$\overline{\mathcal{M}} = \bigcup_\alpha S_\alpha^{-1}(0)/G_\alpha \subset \bigcup_\alpha U_\alpha/G_\alpha.$$

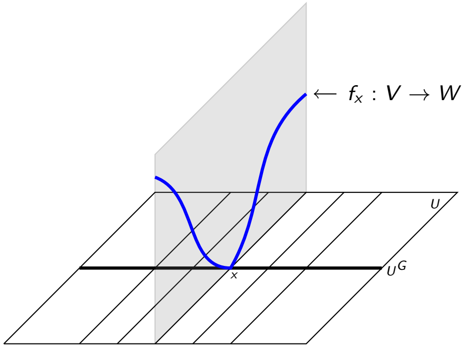
- If all $S_\alpha : U_\alpha \to W_\alpha$ are transverse to $0 \in W_\alpha$, then $\overline{\mathcal{M}}$ is an orbifold and hence carries an a priori rational fundamental class of degree $\dim U_\alpha - \mathrm{rank}\, W_\alpha$.
- If S_α is not transverse then we hope to slightly perturb it to achieve transversality.
- In general there are obstructions to achieve equivariant transversality by perturbation. One then uses multivalued perturbations to achieve transversality and define a rational fundamental class.
- Contributions by: Fukaya–Ono, Li–Tian, Ruan, Siebert, Pardon, Hofer–Wysocki–Zehnder, McDuff–Wehrheim, Fukaya–Oh–Ohta–Ono, etc.

II. Normally polynomial perturbations of Fukaya–Ono

Local Picture

- Consider a chart (G, U, W, S). Let $U^G \subset U$ be the G-fixed point set.
- Assume the normal bundle to U^G is trivial: $NU^G \cong U^G \times V$ where V is a G-representation containing no trivial G-summand.
- Near U^G the map S corresponds to a family of maps from V to W

$$f : U^G \to C^\infty(V, W)^G, \qquad f_x = S|_{\{x\} \times V}.$$



Equivariant transversality fails

- Decompose the obstruction space

$$W = W^{\text{trivial}} \oplus W^{\text{nontrivial}} = \mathring{W} \oplus \check{W}, \qquad S = (\mathring{S}, \check{S})$$

  where G acts trivially on $\mathring{W}$ and $\check{W}$ has no trivial summand.
- We can do single-valued perturbation to make $\mathring{S}$ transverse. Hence we can replace the chart by

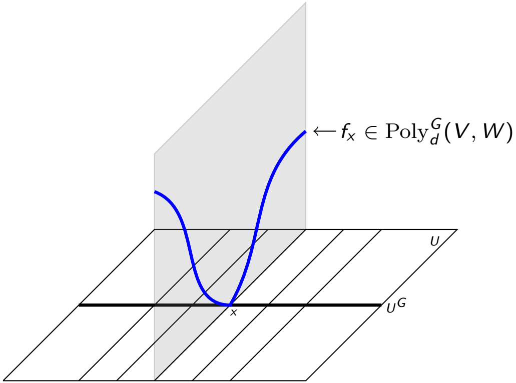
$$(G, \mathring{S}^{-1}(0), \check{W}, \check{S}|_{\mathring{S}^{-1}(0)}).$$

  Hence we can assume W has no trivial summand.
- Then any equivariant map $f \in C^\infty(V, W)^G$ must vanish at $0 \in V$. So for all single-valued perturbation $S_f : U \to W$, $U^G \subset S_f^{-1}(0)$.
- However the dimension of U^G might be larger than the virtual dimension.

Normally polynomial perturbations

Fukaya–Ono: If both V and W are complex representations, then consider single-valued perturbations S_f corresponding to

$$f : U^G \to \mathrm{Poly}^G_d(V, W) := \left\{ \text{Equivariant polynomial maps } P : V \to W \text{ of degrees } \leq d \right\}.$$



Normally polynomial perturbations

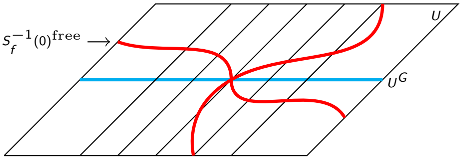
Lemma (Fukaya–Ono, 1997)

For d sufficiently large, for a generic normally polynomial perturbation S_f

1. *The free part $(S_f^{-1}(0))_{\mathrm{free}} := S_f^{-1}(0) \cap (NU^G)_{\mathrm{free}}$ is transverse.*
2. *The boundary of the free part*

$$\overline{(S_f^{-1}(0))_{\mathrm{free}}} \setminus (S_f^{-1}(0))_{\mathrm{free}}$$

is contained in the union of submanifolds of codimension two or higher.



Normally polynomial perturbations

Conjecture (Fukaya–Ono, 1997)

The moduli space of stable maps has a kind of "normally complex" Kuranishi structure. For a kind of "normally polynomial perturbation" one can define integer-valued Gromov–Witten invariants which morally counts curves with trivial automorphism groups.

Using similar type of perturbations one can define Hamiltonian Floer homology in integer coefficients and prove Arnold conjecture over $\mathbb{Z}$.

III. The integral Euler cycle and integral invariants

Integral Euler classes

- Let $\mathcal{U}$ be an effective orbifold and $\mathcal{W} \to \mathcal{U}$ is an orbifold vector bundle. Let $\mathcal{S} : \mathcal{U} \to \mathcal{W}$ be a section with $\mathcal{S}^{-1}(0)$ compact. We call $(\mathcal{U}, \mathcal{W}, \mathcal{S})$ a derived orbifold chart.
- Notice that locally around each $p \in \mathcal{S}^{-1}(0)$ one can find a Kuranishi chart (G_p, U_p, W_p, S_p).
- When $\mathcal{W}$ is (relatively) oriented, there is a well-defined Euler class

$$\chi^{\mathrm{orb}}(\mathcal{U}, \mathcal{W}, \mathcal{S}) \in H_*(\mathcal{U}; \mathbb{Q})$$

  represented the zero locus of a small transverse multivalued perturbation of $\mathcal{S}$.
- What Fukaya–Ono suggested is that when certain "normal complex structure" exists, there is a different class

$$\chi^{\mathbb{Z}}(\mathcal{U}, \mathcal{W}, \mathcal{S}) \in H_*(\mathcal{U}; \mathbb{Z})$$

  represented by a pseudocycle contained in the manifold part $\mathcal{U}_{\mathrm{mfd}} \subset \mathcal{U}$.

Normally polynomial sections

- For simplicity we assume both $T\mathcal{U}$ and $\mathcal{W}$ are complex.
- Any section $\mathcal{S} : \mathcal{U} \to \mathcal{W}$ locally is an equivariant map

$$S_p : U_p \to W_p \cong \mathring{W}_p \oplus \check{W}_p.$$

  We consider those normally polynomial ones.
- For each (small) chart (G_p, U_p, W_p, S_p) centered at p, we want to perturb $S_p = (\mathring{S}_p, \check{S}_p)$ to one for which $\check{S}_p$ is induced by a generic family

$$f_p : U_p^{G_p} \to \mathrm{Poly}_d^{G_p}(V_p, \check{W}_p).$$

- We need a notion of transversality for such global perturbation so that locally it behaves as Fukaya–Ono expected.

Strongly transverse sections

Following Fukaya–Ono and inspired by a subsequent work by Brett Parker, we defined a notion of transversality for such sections.

Theorem (Bai–X., 2022)

There is a notion of strong transversality for normally polynomial perturbations satisfying the following conditions.

1. *A generic normally polynomial perturbation is strongly transverse (we call them FOP perturbations).*
2. *For any FOP perturbation $\mathcal{S}' : \mathcal{U} \to \mathcal{W}$,*

$$(\mathcal{S}')^{-1}(0) \cap \mathcal{U}_{\rm mfd} \subset \mathcal{U}$$

is a pseudocycle whose homology class is a (cobordism) invariant of $(\mathcal{U}, \mathcal{W}, \mathcal{S})$, denoted by

$$\chi^{\rm FOP}(\mathcal{U}, \mathcal{W}, \mathcal{S}) \in H_*(\mathcal{U}; \mathbb{Z}).$$

Global chart in Gromov–Witten theory

Theorem (Abouzaid–McLean–Smith, 2021)

For a compact symplectic manifold X, $n \geq 0$, $A \in H_2(X;\mathbb{Z})$, there is a complex derived orbifold chart $(\mathcal{U}, \mathcal{W}, \mathcal{S})$ (a global Kuranishi chart), which is canonical up to certain equivalence relations, such that

$$\overline{\mathcal{M}}_{0,n}(X, J; A) = \mathcal{S}^{-1}(0).$$

Theorem (Abouzaid–McLean–Smith, 2021)

If $P \to S^2$ is a Hamiltonian fiber bundle with fibre being a compact symplectic manifold (X, ω), then there holds

$$H_*(P;\mathbb{Z}) \cong H_*(S^2;\mathbb{Z}) \otimes H_*(X;\mathbb{Z}).$$

Integral invariants

Corollary (Bai–X., 2022)

Recall the map

$$\mathrm{ev} \times \mathrm{st} : \overline{\mathcal{M}}_{0,n}(X, J; A) \to X^n \times \overline{\mathcal{M}}_{0,n}.$$

The homology class

$$[\overline{\mathcal{M}}_{0,n}(X, J, A)]^{\mathrm{FOP}} := (\mathrm{ev} \times \mathrm{st})_*(\chi^{\mathrm{FOP}}(\mathcal{U}, \mathcal{W}, \mathcal{S})) \in H_*(X^n \times \overline{\mathcal{M}}_{0,n}; \mathbb{Z})$$

is a symplectic deformation invariant, i.e., independent of almost complex structure, the global chart, and the perturbation. Therefore, its pairing with integral cohomology classes defines an integer-version of Gromov–Witten invariants

$$\Psi_{g,n,A} : H^*(\overline{\mathcal{M}}_{0,n}; \mathbb{Z}) \otimes \underbrace{H^*(X; \mathbb{Z}) \otimes \cdots \otimes H^*(X; \mathbb{Z})}_{n} \to \mathbb{Z}.$$

It should match the usual genus zero Gromov–Witten invariants for semi-positive X.

Counting curves with prescribed isotropy

Theorem (Bai–X., 2022)

For any finite group Γ, let $\mathcal{U}_\Gamma \subset \mathcal{U}$ be the set of points whose isotropy group is Γ. Then for any FOP perturbation $\mathcal{S}' : \mathcal{U} \to \mathcal{W}$,

$$(\mathcal{S}')^{-1}(0) \cap \mathcal{U}_\Gamma$$

is a pseudocycle. Hence there is a symplectic deformation invariant

$$[\mathcal{M}_{0,n}(X, J, A)]_\Gamma^{\mathrm{FOP}} \in H_*(X^n \times \overline{\mathcal{M}}_{0,n}; \mathbb{Z}).$$

Conjecture (Bai–X.)

The FOP cycles can be constructed for all genera; for each finite group Γ, $[\overline{\mathcal{M}}_{g,n}(X, J, A)]_\Gamma^{\mathrm{FOP}}$ is morally supported in $\overline{\mathcal{M}}_{g,n}(X, J, A)_\Gamma$, the locus of stable maps of automorphism group Γ. Moreover, there exists a certain virtual sheaf $(V_\Gamma \to W_\Gamma)$ and an integral characteristic class $e(V_\Gamma \to W_\Gamma)$ such that

$$[\overline{\mathcal{M}}_{g,n}(X, J, A)]^{\mathrm{vir}} = \sum_\Gamma \frac{1}{|\Gamma|} \Big\{ [\overline{\mathcal{M}}_{g,n}(X, J, A)]_\Gamma^{\mathrm{FOP}} \cap e(V_\Gamma \to W_\Gamma) \Big\}.$$

Remarks

- We reproved Abouzaid–McLean–Smith's result without using generalized cohomology theories.
- The Gromov–Witten axioms for genus zero (such as WDVV equation/quantum cohomology) can be verified for this new invariant.
- It is straightforward to define the quantum Steenrod operation on general compact symplectic manifolds, extending the monotone case defined by Seidel and Wilkins.
- Our construction needs smooth Kuranishi structures. So far we do not have a sufficiently simple way to establish smoothness for higher genus case.
- If we can construct smooth Kuranishi structures on all moduli spaces of Floer trajectories, then one can define an integer-version of Hamiltonian Floer homology and prove the Arnold conjecture over $\mathbb{Z}$.

IV. The transversality condition

Revisiting the local picture

- For a Kuranishi chart $G \circlearrowright U \xrightarrow{S} W \circlearrowleft G$. Assume W has no trivial summand.
- Recall the universal zero set is an affine variety

$$Z := Z_d^G(V, W) := \left\{(v, P) \in V \times \mathrm{Poly}_d^G(V, W) \mid P(v) = 0\right\}.$$

  Denote

$$Z_{\mathrm{free}} := Z \cap \left(V_{\mathrm{free}} \times \mathrm{Poly}_d^G(V, W)\right).$$

- Because $Z_{\mathrm{free}} \subset Z$ is a Zariski open set, the boundary

$$\overline{Z_{\mathrm{free}}} \setminus Z_{\mathrm{free}}$$

  is the union of smooth algebraic sets of codimension two or higher.
- We need maps to be transverse to both Z_{free} and boundary strata.

Whitney stratification

Theorem (Thom, Whitney, Łojasiewicz)

Given a smooth complex algebraic variety M and an algebraic subset $Z \subset M$, there exists a "canonical" Whitney stratification, i.e., a decomposition

$$Z = \bigsqcup_{\alpha} Z_\alpha$$

into strata, which satisfy "Whitney condition (b)". Moreover, each Z_α is a smooth algebraic set.

Lemma (Bai–X.)

There exists a canonical Whitney stratification on $Z \subset V \times \mathrm{Poly}_d^G(V, W)$, such that each stratum is contained $V_H \times \mathrm{Poly}_d^G(V, W)$ where

$$V_H = \{v \in V \mid G_v = H\}, \ H < G.$$

Strong transversality

- A family of polynomial maps $f : U^G \to \mathrm{Poly}_d^G(V, W)$ is strongly transverse if the associated map $F : V \times U^G \to V \times \mathrm{Poly}_d^G(V, W)$ is transverse to each stratum of Z.
- Then f is strongly transverse implies that

$$\overline{(S_f^{-1}(0))_{\mathrm{free}}} \setminus (S_f^{-1}(0))_{\mathrm{free}} \subseteq \bigcup \text{manifolds of codimension } \geq 2.$$

- We chose the Whitney stratification in such a careful way in order to settle the following two subtle issues.

1. Does strong transversality for d imply strong transversality for $d' > d$?
2. Does strong transversality for G imply strong transversality for $H < G$?

Strong transversality is intrinsic

- When considering Fukaya–Ono's proposal, Brett Parker (2013) introduced a class of "nice" Whitney stratifications on Z and found a method of relating nice Whitney stratifications on Z for different d and G.
- However Parker did not provide a coherent way to choose nice Whitney stratifications nor provide enough details for constructing global transverse perturbations with his version of Kuranishi structure.
- By modifying Parker's method, we proved that our strong transversality condition is intrinsic, i.e., being independent of large d and being an open condition.

THANKS!