C^0 contact geometry of isotropic submanifolds

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Contact manifolds:

(M, ξ)

$\dim M = 2n+1$

ξ **contact structure** = maximally non-integrable hyperplane field $\xi \subset TM$

Locally $\xi = \ker \alpha$, for a 1-form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$

If **contact form** α is globally defined ξ is **coorientable**

Examples.

(1) $(\mathbb{R}^{2n+1}, \ker(dz - \sum_{j=1}^{n} y_j dx_j))$

(2) $(J^1M \cong T^*M \times \mathbb{R}(z), \ker \lambda \oplus dz)$ λ tautological 1-form on T^*M

(3) $(ST^*M, \ker \lambda|_{ST^*M})$ unit cotangent bundle

$\varphi : (M, \xi = \ker\alpha) \to (N, \eta = \ker\beta)$ is called **conactomorphism** if

$\varphi_* \xi = \eta \iff \varphi^* \beta = e^g \alpha$, $g \in C^\infty(M)$ **conformal factor**

Reeb vector field is given by: $d\alpha(R_\alpha, \cdot) = 0$, $\alpha(R_\alpha) = 1$

R_α generates **Reeb flow** ϕ_α^t

$L \subset (M,\xi)$ **isotropic** $\Leftrightarrow (\forall p \in L)\ T_pL \subset \xi_p$

(1) **Legendrian**: $\dim L = n$

(2) **Subcritical isotropic**: $\dim L < n$

A curve in a contact mfd is called **transverse** if it is tansverse to the contact structure.

Contact homeomorphisms:

$\varphi \in \mathrm{Homeo}(M)$ is called **contact homeomorphism** if there exists a sezuence of contactomorphisms $\{\varphi_i\}_{i\geq 1}$ which C^0-converges to φ.

Remark.

If M is coorientable ($\xi = \ker\alpha$) sezuence φ_i gives sezunce of conformal factors $\{g_i\}_{i\geq 1}$ ($\varphi_i^*\alpha = e^{g_i}\alpha$)

C^0-convergence of φ_i **does not** imply uniform conv. of g_i

If $g_i \xrightarrow{C^0} g \in C^\infty(M)$ φ is called **topological automorphism of ξ**

Theorem (C^0 rigidity of contactomorphisms).

$\mathrm{Cont}(M,\xi)$ is C^0-closed in $\mathrm{Diff}(M)$

$$\underset{\mathrm{Cont}}{\underset{\cap}{\varphi_k}} \xrightarrow{C^0} \underset{\mathrm{Diff}}{\underset{\cap}{\varphi}} \Rightarrow \varphi_*\xi = \xi$$

Smooth contact homeomorphisms $=$ contactomorphisms

Questions and results:

Let $\varphi \in \overline{Cont}(M,\xi)$ be a contact homeomorphism, $L \subset M$ isotropic submanifold, such that $\varphi(L)$ is smooth submanifold of M.
Must $\varphi(L)$ be isotropic?

Conjecture.
If L is Legendrian, the answer is **YES**, otherwise it is **NO**.

Subcritical isotropic case:

Theorem (S. '22).

(M,ξ) $\dim M \geqslant 5$ $\Rightarrow$ $\exists$ isotropic embedding $\gamma : S^1 \to M$ and $\varphi \in \overline{Cont}(M,\xi)$ such that $\varphi \circ \gamma$ is smooth and transverse

The main tool: **quantitative h-principle** for subcritical isotropic embeddings

Theorem (S. '22).

$\gamma_0, \gamma_1 : S^1 \to (\mathbb{R}^{2n+1}, \xi_{STD})$ subcritical isotropic embeddings $(n \geqslant 2)$

$\exists$ homotopy $\gamma_t : S^1 \to \mathbb{R}^{2n+1}$ of size $< \varepsilon$ $\left(\operatorname{diam}\{\gamma_t(\theta) \mid t \in [0,1]\} < \varepsilon \text{ for all } \theta \in S^1 \right)$

$\Rightarrow$ $\exists$ contact isotopy $(\Psi^t)_{t\in[0,1]}$ such that

(1) $\Psi^1 \circ \gamma_0 = \gamma_1$

(2) $\max\limits_{t\in[0,1]} d_{C^0}(\Psi^t, \mathrm{Id}) < \varepsilon$

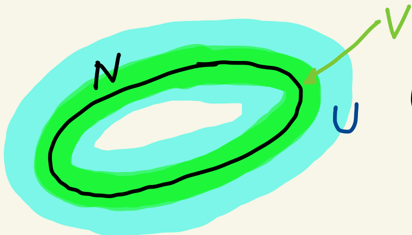
Remark. theorem remains true if we consider closed subcritical isotropic discs of any subcritical dimension.

Legendrian case:

Definition.

A compact subset $N \subset (M,\xi)$ is called **nearly Reeb invariant** if every open neighbourhood $U \supset N$ contains an open subset $V \subset U$ (and also $N \subset V$) such that $\phi_\alpha^t(V) = V$, $\xi = \ker \alpha$

α depends on U



$(\forall U \supset N)(\exists V \subset U, \alpha \in \Omega^1(U))$
$\xi|_U = \ker \alpha \;\; \& \;\; (\forall t)\, \phi_\alpha^t(V) = V$

Examples.

(1) Transverse knots in contact 3-manifolds

$(S^1 \times \mathbb{R}^2, \ker(\underbrace{d\theta - x\,dy}_{\alpha}))$, $(\forall \varepsilon > 0)\; \phi_\alpha^t(S^1 \times D(\varepsilon)) = S^1 \times D(\varepsilon)$ $\left(R_\alpha = \frac{\partial}{\partial \theta}\right)$

(2) $T \subset (M^3, \ker \lambda_M)$ transverse, $(N^{2n-2}, d\lambda_N)$ exact symplectic

$\Rightarrow$ $T \times L \subset (M \times N, \ker \lambda_M \oplus \lambda_N)$ is nearly Reeb inv. for $L^{n-1} \subset N$.

Theorem (S. '22).

$L \subset (M,\xi)$ closed Legendrian, $\varphi \in \overline{\mathrm{Cont}}(M,\xi)$ contact homeo.

$\Rightarrow$ $\varphi(L)$ is not nearly Reeb invariant.

Proposition (S. '22).

$K \subset (M^3, \xi)$ non-Legendrian knot $(\exists p \in K,\ T_pK \pitchfork \xi_p)$

$\Rightarrow K$ is nearly Reeb invariant

Corollary (Dimitroglou Rizell-Sullivan '22).

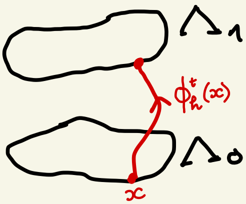
$L \subset (M^3, \xi)$ Legendrian knot, $\varphi \in \overline{\mathrm{Cont}}(M^3, \xi)$, $\varphi(L)$ smooth

$\Rightarrow \varphi(L)$ is Legendrian

The main tool:

Contact interlinkig (Entov, Polterovich):

$(M, \xi = \ker\lambda)$



(Λ_0, Λ_1) interlinked Legendrians

$\Updownarrow$

Every bounded contact Hamiltonian h on M with $h \geq c > 0$ possesses an orbit of time-length $\leq \mu/c$ from Λ_0 to Λ_1

$(\mu = \mu(\Lambda_0, \Lambda_1, \lambda))$

Theorem (Entov, Polterovich '21).

Q-closed manifold, $J^1Q = T^*Q(p,q) \times \mathbb{R}(z)$ $\lambda_{std} = dz - pdq$, $\psi : Q \to \mathbb{R}_+$,

$\underbrace{\Lambda_0 := \{p = 0, z = 0\}}_{\text{zero section}}$, $\underbrace{\Lambda_1 := \{z = \psi(q), p = \psi'(q)\}}_{\text{graph of 1-jet of } \psi}$ $\Rightarrow (\Lambda_0, \Lambda_1)$ interlinked.

Related results:

Theorem (Rosen, Zhang '20).

$L \subset (M, \xi = \ker\alpha)$ Legendrian, $\varphi : M \circlearrowleft$ topological automorphism of ξ

$\varphi(L)$ is smooth $\Rightarrow$ $\varphi(L)$ is Legendrian

$\phi \in \overline{Cont}(M, \xi = \ker\alpha)$ is **bounded below near L** if the approx. sequence ϕ_m has conformal factors with positive lower bounds on some neighbourhood of L.

Theorem (Usher '20).

$L \subset (M, \xi = \ker\alpha)$ Legendrian, $\phi \in \overline{Cont}(M, \xi)$ bounded below near L, $\phi(L)$ smooth $\Rightarrow$ $\phi(L)$ is Legendrian.