The Lagrangian capacity of toric domains

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Table of Contents

Goal

Basics

Results

Table of Contents

Goal

Basics

Results

Conjecture ([Per22, Conjecture 6.24])

If X_Ω is a convex or concave toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Goal

To **motivate** and **prove** the conjecture (in some special cases).

Table of Contents

Goal

Basics

Results

Definition 2.1

1. The **moment map** is the map $\mu\colon \mathbb{C}^n \longrightarrow \mathbb{R}^n_{\geq 0}$ given by $\mu(z_1, \ldots, z_n) := \pi(|z_1|^2, \ldots, |z_n|^2)$.
2. A **toric domain** is a star-shaped domain X of the form $X = X_\Omega := \mu^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^n_{\geq 0}$.
3. The **diagonal** of X_Ω is $\delta_\Omega := \sup\{a \mid (a, \ldots, a) \in \Omega\}$.

Example 2.2

$$P(a) := \{z \in \mathbb{C}^n \mid \forall j = 1, \ldots, n\colon \pi|z_j|^2 \leq a\} \quad \textbf{(cube)}$$

$$N(a) := \{z \in \mathbb{C}^n \mid \exists j = 1, \ldots, n\colon \pi|z_j|^2 \leq a\}$$

$$\textbf{(nondisjoint union of cylinders)}$$

Definition 2.3 ([CM18, Section 1.2])

Let (X, ω) be a symplectic manifold. If L is a Lagrangian submanifold of X, then we define the **minimal symplectic area of** L by

$$A_{\min}(L) \coloneqq \inf\{\omega(\sigma) \mid \sigma \in \pi_2(X, L),\, \omega(\sigma) > 0\}.$$

Definition 2.4 ([CM18, Section 1.2])

The **Lagrangian capacity** of (X, ω) is

$$c_L(X) \coloneqq \sup\{A_{\min}(L) \mid L \subset X \text{ is an embedded Lagrangian torus}\}.$$

Definition 2.5

The **cube capacity** is given by

$$c_P(X, \omega) \coloneqq \sup\{a \mid \exists \text{ symplectic embedding } P^{2n}(a) \longrightarrow X\}.$$

Lemma 2.6

If X is a star-shaped domain, then $c_L(X) \geq c_P(X)$.

Proof.

Let $\iota \colon P(a) \longrightarrow X$ be a symplectic embedding, for some $a > 0$. We want to show that $c_L(X) \geq a$. Define $T := \mu^{-1}(a, \ldots, a) \subset \partial P(a)$ and $L := \iota(T) \subset X$. Then, $c_L(X) \geq A_{\min}(L) = A_{\min}(T) = a$. □

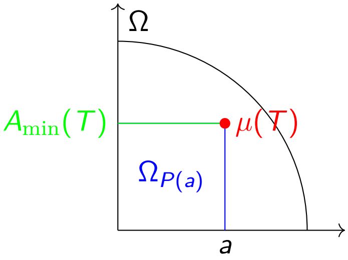

Figure: Proof of $c_L(X) \geq c_P(X)$ for $X = X_\Omega$

Lemma 2.7

If X_Ω is a convex or concave toric domain, then $c_P(X_\Omega) \geq \delta_\Omega$.

Proof.

Since X_Ω is convex or concave, we have $P(\delta_\Omega) \subset X_\Omega \subset N(\delta_\Omega)$. The result follows since $c_P(X_\Omega) := \sup\{a \mid \exists\, P(a) \hookrightarrow X_\Omega\}$. □

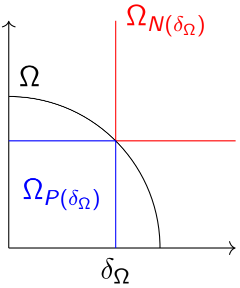

Figure: If X_Ω is convex or concave then $P(\delta_\Omega) \subset X_\Omega \subset N(\delta_\Omega)$

We now consider the results by Cieliebak–Mohnke for the Lagrangian capacity of the ball and the cylinder.

Proposition 2.8 ([CM18, Corollary 1.3])

The Lagrangian capacity of the ball is

$$c_L(B^{2n}(1)) = \frac{1}{n} = \delta_{\Omega(B^{2n}(1))}.$$

Proposition 2.9 ([CM18, p. 215-216])

The Lagrangian capacity of the cylinder is

$$c_L(Z^{2n}(1)) = 1 = \delta_{\Omega(Z^{2n}(1))}.$$

Conclusion

$$\begin{aligned} X_\Omega \text{ is a convex or concave toric domain} &\implies c_L(X_\Omega) \geq \delta_\Omega \\ X_\Omega \text{ is the ball or the cylinder} &\implies c_L(X_\Omega) = \delta_\Omega \end{aligned}$$

Conjecture 2.10 ([Per22, Conjecture 6.24])

If X_Ω is a convex or concave toric domain then

$$c_L(X_\Omega) = \delta_\Omega.$$

Table of Contents

Goal

Basics

Results

To prove our results about the Conjecture 2.10, we will need to use the following symplectic capacities.

McDuff–Siegel capacities	$\tilde{\mathfrak{g}}_k^{\leq \ell}$	[MS22]
Higher symplectic capacities	$\mathfrak{g}_k^{\leq \ell}$	[Sie20]
Gutt–Hutchings capacities	c_k^{GH}	[GH18]

for $k, \ell \in \mathbb{Z}_{\geq 1}$. We will only need to consider these capacities for $\ell = 1$, i.e. $\tilde{\mathfrak{g}}_k^{\leq 1}, \mathfrak{g}_k^{\leq 1}$.

Theorem 3.1 ([Per22, Theorem 6.40])

If (X, λ) *is a Liouville domain and* $k \geq 1$ *then* $c_L(X) \leq \tilde{\mathfrak{g}}_k^{\leq 1}(X)/k$*.*

Proof sketch.

1. By definition of c_L, it suffices to assume that $L \subset X$ is an embedded Lagrangian torus and to prove that there exists a disk D with boundary on L with “small” symplectic area.
2. By definition of $\tilde{\mathfrak{g}}_k^{\leq 1}$, there exists a sequence u_t of J_t-holomorphic curves with bounded energy and satisfying a tangency constraint.
3. By the SFT compactness theorem, u_t converges to a broken holomorphic curve $F = (F_1, \ldots, F_N)$ (neck stretching along $S^* L$).
4. One of the components of the broken holomorphic curve F will be the desired disk. □

Theorem 3.2 ([Per22, Theorem 6.41])

If X_Ω is a 4-dimensional convex toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Proof.

For every $k \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned}
\delta_\Omega &\leq c_P(X_\Omega) && \text{[by Lemma 2.7]} \\
&\leq c_L(X_\Omega) && \text{[by Lemma 2.6]} \\
&\leq \tilde{\mathfrak{g}}_k^{\leq 1}(X_\Omega)/k && \text{[by Theorem 3.1]} \\
&= c_k^{\mathrm{GH}}(X_\Omega)/k && \text{[dim 4 and [MS22, Proposition 5.6.1]]} \\
&\leq c_k^{\mathrm{GH}}(N(\delta_\Omega))/k && \text{[}X_\Omega \text{ is convex, hence } X_\Omega \subset N(\delta_\Omega)\text{]} \\
&= \delta_\Omega(k+1)/k && \text{[by [GH18, Lemma 1.19]].}
\end{aligned}$$

□

Theorem 3.3 ([Per22, Theorem 7.64])

If X is a Liouville domain such that $\pi_1(X) = 0$ and $2c_1(TX) = 0$ then $\mathfrak{g}_k^{\leq 1}(X) = c_k^{\mathrm{GH}}(X)$.

Proof sketch.

1. Let $E = E(a_1, \ldots, a_n)$ be a "skinny" ellipsoid such that there exists a strict exact symplectic embedding $\phi \colon E \longrightarrow X$.
2. By definition of c_k^{GH} and $\mathfrak{g}_k^{\leq 1}$ (and the Bourgeois–Oancea isomorphism), it suffices to show that $\#^{\mathrm{vir}} \mathcal{M}_E^J(\gamma)\langle \mathcal{T}^{(k)} x \rangle \neq 0$.
3. Show that $\mathcal{M}_E^J(\gamma)\langle \mathcal{T}^{(k)} x \rangle$ is transversely cut out. This implies that $\#^{\mathrm{vir}} \mathcal{M}_E^J(\gamma)\langle \mathcal{T}^{(k)} x \rangle = \# \mathcal{M}_E^J(\gamma)\langle \mathcal{T}^{(k)} x \rangle$.
4. Compute explicitly that $\# \mathcal{M}_E^J(\gamma)\langle \mathcal{T}^{(k)} x \rangle \neq 0$ (curves in this moduli space are polynomials). □

Theorem 3.4 ([Per22, Theorem 7.65])

Assume that a suitable virtual perturbation scheme exists. If X_Ω is a convex or concave toric domain then $c_L(X_\Omega) = \delta_\Omega$.

Proof.

$$\begin{aligned}
\delta_\Omega &\leq c_P(X_\Omega) && \text{[by Lemma 2.7]} \\
&\leq c_L(X_\Omega) && \text{[by Lemma 2.6]} \\
&\leq \tilde{\mathfrak{g}}_k^{\leq 1}(X_\Omega)/k && \text{[by Theorem 3.1]} \\
&\leq \mathfrak{g}_k^{\leq 1}(X_\Omega)/k && \text{[by [MS22, Section 3.4]]} \\
&= c_k^{\mathrm{GH}}(X_\Omega)/k && \text{[by Theorem 3.3]} \\
&\leq c_k^{\mathrm{GH}}(N(\delta_\Omega))/k && [X_\Omega \text{ is convex, hence } X_\Omega \subset N(\delta_\Omega)] \\
&= \delta_\Omega(k+n-1)/k && \text{[by [GH18, Lemma 1.19]].}
\end{aligned}$$

□

$$c_L(X) \leq \inf_k \frac{\tilde{\mathfrak{g}}_k^{\leq 1}(X)}{k}$$

$$\mathfrak{g}_k^{\leq 1}(X) = c_k^{\mathrm{GH}}(X)$$

$$c_L(X_\Omega) = \delta_\Omega$$

Thank you for listening!

References

[CM18] Kai Cieliebak and Klaus Mohnke. Punctured holomorphic curves and Lagrangian embeddings. *Inventiones mathematicae*, 212(1):213–295, April 2018.

[GH18] Jean Gutt and Michael Hutchings. Symplectic capacities from positive S^1-equivariant symplectic homology. *Algebraic & Geometric Topology*, 18(6):3537–3600, October 2018.

[MS22] Dusa McDuff and Kyler Siegel. Symplectic capacities, unperturbed curves, and convex toric domains. *arXiv:2111.00515 [math]*, February 2022.

[Per22] Miguel Pereira. *Equivariant symplectic homology, linearized contact homology and the Lagrangian capacity*. PhD thesis, Universität Augsburg, March 2022.

[Sie20] Kyler Siegel. Higher symplectic capacities. *arXiv:1902.01490 [math-ph]*, February 2020.