Lagrangians and symplectomorphisms as zeroes of moment maps

Symplectic zoominar

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- No flow techniques in the PL context
- Symplectic PL geometry = Terra incognita
- Darboux, stability, etc... do not hold in the PL context.

A first result

Theorem 1 (Jauberteau-R.-Tapie, R.)

A smoothly immersed 2-torus of $\mathbb{C}^n$ can be approximated, in the C^0 sense, by immersed isotropic polyhedral tori. If the smooth torus is isotropic, the approximation can be done in the C^1 sense.

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- It involves moment map geometry
- Spin-off : flow techniques – effective constructions (experimental math)

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$$\mathscr{J} V = JV, \quad \Omega(V, W) = \int_\Sigma \omega(V, W)\sigma, \quad G(V, W) = \int_\Sigma g(V, W)\sigma$$

for every $V, W : \Sigma \to \mathbb{R}^{2n} \simeq T_f \mathscr{M}$.

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- $Z_u(f) = f_* X_u \in T_f\mathscr{M}$ is the induced vector field on $\mathscr{M}$

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In this context : consider the moment map flow and its discrete version.

$$\frac{\partial f}{\partial t} = -\mathscr{J} Z_{\mu(f)}.$$

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Dummy idea: we replace Ham with a torus (their Lie algebra look similar)

$$\mathrm{Ham}(\Sigma, \sigma) \Leftrightarrow \mathbb{T} = C^\infty(\Sigma, S^1)$$

$$\exp : \mathrm{Lie}\mathbb{T} \simeq C^\infty(\Sigma, \mathbb{R}) \to \mathbb{T}, \quad \xi \mapsto e^{i\xi}$$

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We define

$$\mu : \mathscr{F} \to C^\infty(\Sigma, \mathbb{R}) \simeq \mathrm{Lie}\mathbb{T}$$

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$$\mu(F) = \frac{\omega(F\cdot, F\cdot)}{\sigma} = \frac{F^*\omega}{\sigma}.$$

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Then we have a commutative diagram (abstract nonsense)

$$\begin{array}{ccccccc} \mathrm{Ham}(\Sigma, \sigma) & \longrightarrow & \mathscr{M} & \xrightarrow{\mathscr{D}} & \mathscr{F} & \longleftarrow & \mathbb{T} \\ & & & \searrow^{\mu^D} & \downarrow \mu & & \\ & & & & C^\infty(\Sigma, \mathbb{R}) & & \end{array}$$

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Proposition 2

The action of $\mathbb{T}$ on $\mathscr{F}$ is Hamiltonian, with moment map μ.

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We introduce the functional

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All the moment map geometry constructions have obvious analogues in the polyhedral setting.

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A computer program that runs this flow is in developpement (joint work with François Jauberteau).

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Question by Vincent Humilière (Symplectix seminar, IHP) $\rightsquigarrow$ prove it using a modified moment map flow technique !

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The space $\mathscr{F}$ admits a formal hyperKähler structure $(G, \mathscr{I}, \mathscr{J}, \mathscr{K}, \Omega_I, \Omega_J, \Omega_K)$, where

$$\mathscr{I}F = -F \circ I, \mathscr{J}F = -F \circ J, \mathscr{K}F = -F \circ K$$

Theorem 6

The action of $\mathbb{T}$ on $\mathscr{F}$ preserves the hyperKähler structure and is Hamiltonian w.r.t $\Omega_\bullet$ (for $\bullet = \mathscr{I}, \mathscr{J}, \mathscr{K}$), with moment map

$$\mu_\bullet(F) = \frac{(F^*\omega_{\mathbb{H}}) \wedge \omega_\bullet}{\omega_M^2} \in C^\infty(M, \mathbb{R})$$

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Proposition 7

If $\mu_\bullet(F) = 0$, then $F^*\omega_{\mathbb{H}}$ is selfdual.
If $f : M \to M$ satisfies $f^*[\omega_M] = [\omega_M]$ and $\mu_\bullet(\mathscr{D}f) = 0$, then f is a symplectomorphism.

Flow and symplectomorphisms

We can introduce a hyperKähler modified moment map flow along $\mathrm{Im}\mathscr{D}$ by considering the downward gradient flow of the map

$$\phi(F) = \|\mu_I(F)\|^2_{L^2} + \|\mu_J(F)\|^2_{L^2} + \|\mu_K(F)\|^2_{L^2}$$

The construction goes through in some polyhedral setting, and the moment map flow converges towards polyhedral symplectic maps.

Theorem 8

The polyhedral version of the modified moment map flow has long time existence and converge towards polyhedral symplectic maps of the 4-torus M. In particular the space of polyhedral symplectic maps is a deformation retract of the space of polyhedral maps.

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- See polyhedral deformations of Lagrangian and symplectic fibrations of the 4-torus.

Application

Theorem 9 (In progress)

The inclusion of the linear symplectic group

$$GL(M, \omega_M) \subset \mathrm{PLSymp}(M, \omega_M),$$

in the group of piecewise linear symplectic maps of the 4-torus (M, ω_M), is a homotopy equivalence.

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Thanks for your attention !