Constructions of High dimensional Legendrians and Isotopies

– AGNIVA ROY (Georgia Tech)

Legendrian submanifolds : Tangent to contact hyperplane arrangement.

$L^n \subset M^{2n+1}$

Front projections : In $\mathbb{R}^{2n+1}$, $\xi_{std} = \ker\left(dz - \sum_{i=1}^{n} y_i dx_i\right)$, a Legendrian is uniquely determined by its image under

more generally in $J^1(M)$, $\Pi_F : J^1(M) \to \mathbb{R}\times M$

$$\Pi_F : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{n+1}$$

$$(z, x_1, y_1, \ldots, x_n, y_n) \longmapsto (z, x_1, x_2, \ldots, x_n).$$

- Legendrians in high dimensions

$$L^n \subset M^{2n+1} \quad ; \quad n \geq 2$$

- Motivating question:

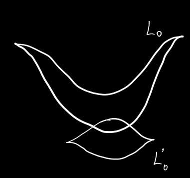
How do you construct interesting Legendrian submanifolds, and compute their invariants?

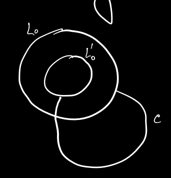
— focus on S^n

- Existing work:

1. Infinite family in $(\mathbb{R}^{2n+1}, \xi_{st})$ by "stabilising" the unknot away from cusps.

(Ekholm - Etnyre - Sullivan '06)





2. N-weave calculus for $n=2$ in $J^1(C)$ for a smooth surface C.

(Casals - Zaslow '20)

3. Legendrian lifts of doubled Lagrangian fillings.

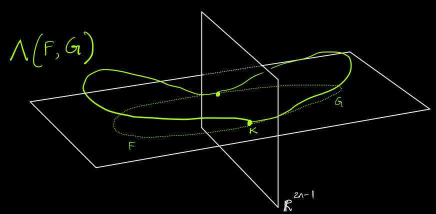
(Ekholm '16)

- 3. <u>Ekholm's construction</u> :

Consider a Legendrian knot $K \subset (\mathbb{R}^{2n-1}, \xi_{std})$

Consider two Lagrangian disk fillings $F, G \subset (\mathbb{R}^{2n}, \omega_{st})$ of K

Join them and consider the Legendrian lift.



- **Thm** (Ekholm-Murphy '16, Courte-Ekholm '18)

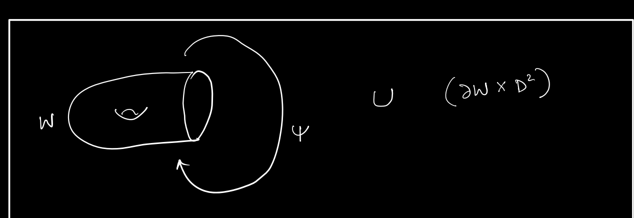
i) If F, G are different, $\Lambda(F,G)$ is loose.

ii) $\Lambda(F,F)$ is isotopic to the unknot.

- <u>Open books and Stabilisations</u> :

(Giroux) : Every closed (M^{2n+1}, ξ) has a supporting open book decomposition.

<u>i.e.</u> $\exists$ Liouville manifold (W^{2n}, λ) and a symplectomorphism $\Psi : W \to W$ such that $(M, \xi) \underset{\text{contacto.}}{\simeq} OB(W, \lambda, \Psi)$.
(Ψ identity near ∂W)



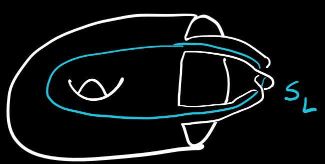
- **Stabilisation**: Given Lagrangian disk $L \subset W$, one can attach a Weinstein n-handle to ∂W along L to get (W_L, λ_L)

S_L : Lagrangian sphere $\subset W_L$. Then,

$$(M,\xi) \underset{\text{contacto}}{\simeq} OB(W,\lambda,\psi) \underset{\text{contacto}}{\overset{*}{\simeq}} OB(W_L, \lambda_L, \psi \circ \tau_{S_L})$$

τ_{S_L} : Dehn-twist along S_L.

(*) Folk theorem, but a helpful resource is "Lecture Notes" by Otto Van Koert.



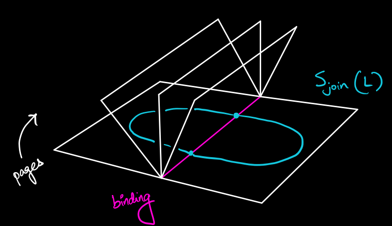
- **Constructions** (R.)

Input : i) $(M,\xi) \simeq OB(W,\lambda,\psi)$.

ii) $L \subset W$ Lagrangian n-disk.

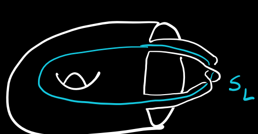
① $S_{join}(L)$

Take two copies of L, L' in different pages. Join them through the binding and take the Legendrian lift.



② $S_{stab}(L)$

Stabilise $OB(W,\lambda,\psi)$ along ∂L.
Consider S_L.



- **Theorem 1** (R.)

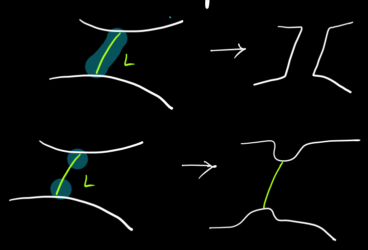
$$S_{join}(L) \underset{\text{isotopic}}{\overset{\text{smooth Legendrian}}{=\!=}} S_{stab}(L)$$

Technicalities:

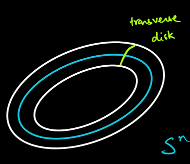
1) They are defined in distinct contactomorphic manifolds.
2) Need to first understand stabilisation as an embedded operation.
3) Re-interpret Dehn twist as Legendrian surgery and construct isotopy through appropriate local models.

- **Stabilisation as an embedded operation**

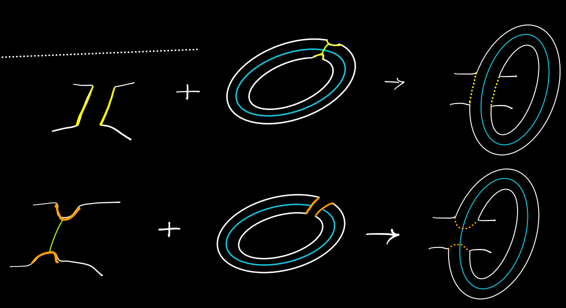
(generalising Vértesi - Licata's idea to high dimensions)



- surger out a neighbourhood of L from some pages and that of ∂L from the rest of the pages

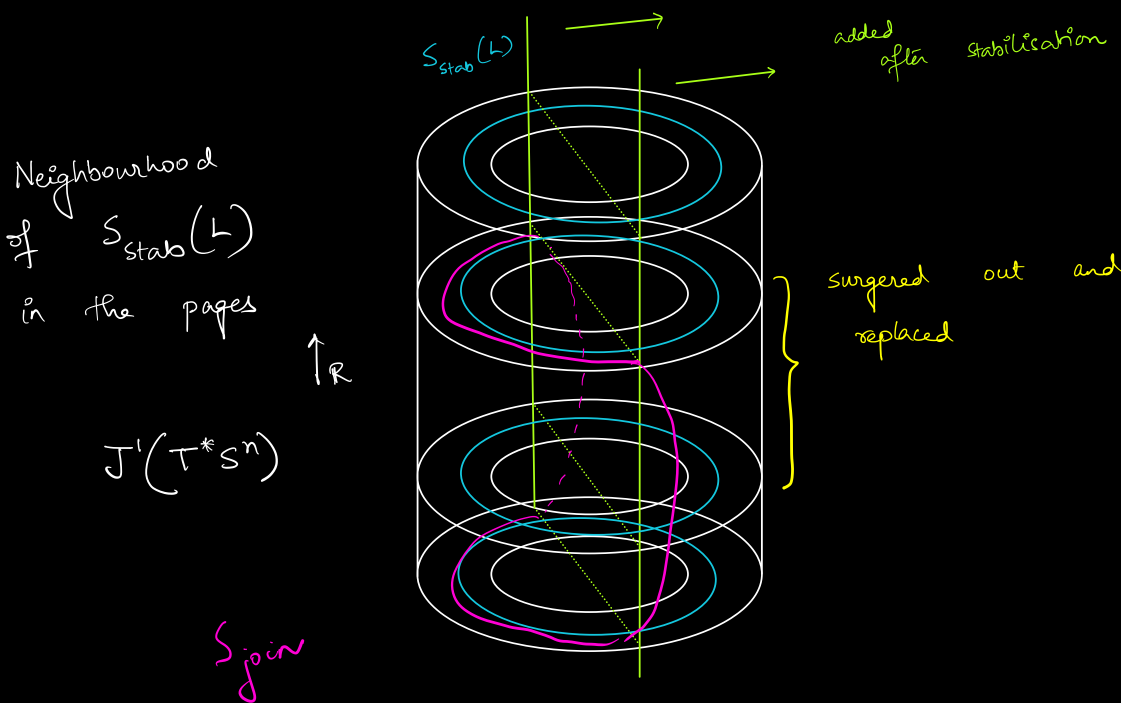


- identify with neighbourhood D of "transverse" disk in page of $OB(D(T^*S^n), \tau) \cong (S^{2n+1}, \xi_{st})$



- glue in $S^{2n+1} \setminus D$, "twisting" the pages.

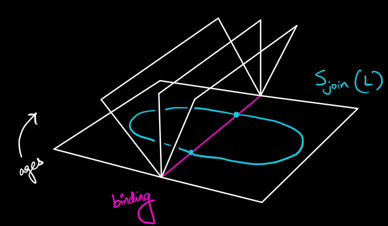
- Isotopy through Legendrian surgery



- Work in progress :

Result 2 : $S_{join}(L)$, and hence $S_{stab}(L)$, is isotopic to the unknot.

Idea : Can isotope L and L' to lie on pages close to each other, thus $S_{join}(L)$ is just L and its pushoff, joined along their boundaries.



- Work in progress:

Result 3: Removing a point from $(S^{2n+1}, \xi_{st}) \simeq OB(B^{2n}, \lambda_{st}, id)$, takes $S_{join}(L)$ to a sphere isotopic to $\Lambda(L, L)$.

(This is a way to reprove Courte-Ekholm's result that $\Lambda(L,L)$ is the unknot.)

In summary:

1. $S_{join}(L) \overset{\text{smooth}}{\underset{\text{isotopic}}{\overset{\text{Legendrian}}{=}}} S_{stab}(L)$

2.* They are all unknots

3.* In $S^{2n+1} \setminus \{pt\}$, this recovers the fact that $\Lambda(L, L)$ is always standard.

Thank you!