Higher algebra of A_∞-algebras in Morse theory

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Symplectic Zoominar, 28/01/2022

The results presented in this talk are taken from my two recent papers : *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory I* (arXiv:2102.06654) and *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory II* (arxiv:2102.08996).

Definition

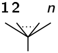
Let A be a cochain complex with differential m_1. An A_∞-algebra structure on A is the data of a collection of maps of degree $2-n$

$$m_n : A^{\otimes n} \longrightarrow A \ , \ \ n \geqslant 1,$$

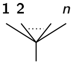
extending m_1 and which satisfy

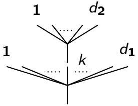
$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}).$$

These equations are called the A_∞-equations.

Representing m_n as 

, these equations can be written as

$[m_1,$ 

$] = \sum_{\substack{h+k=n+1 \\ 2 \leqslant h \leqslant n-1 \\ 1 \leqslant i \leqslant k}} \pm$ 

.

In particular,

$$[m_1, m_2] = 0 \ ,$$
$$[m_1, m_3] = m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) \ ,$$

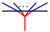
implying that m_2 descends to an associative product on $H^*(A)$. An A_∞-algebra is thus simply a correct notion of a dg-algebra whose product is associative up to homotopy.

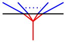
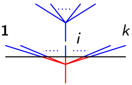
The operations m_n are the higher coherent homotopies which keep track of the fact that the product is associative up to homotopy.

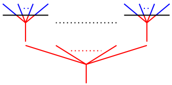
Definition

An A_∞-*morphism* between two A_∞-algebras A and B is a family of maps $f_n : A^{\otimes n} \to B$ of degree $1-n$ satisfying

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geqslant 2}} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} \pm m_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) \; .$$

Representing the operations f_n as , the operations m_n^B in red and the operations m_n^A in blue, these equations read as

$\Big[m_1,$  $\Big] = \sum_{\substack{h+k=n+1 \\ 1 \leqslant i \leqslant k \\ h \geqslant 2}} \pm$ 

$+ \sum_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} \pm$  .

We check that

$$[m_1, f_1] = 0$$
$$[m_1, f_2] = f_1 m_2^A - m_2^B(f_1 \otimes f_1) \ .$$

An A_∞-morphism between A_∞-algebras induces a morphism of associative algebras on the level of cohomology, and is a correct notion of morphism which preserves the product up to homotopy.

Our goal now : study the *higher algebra of* A_∞*-algebras*.

Considering two A_∞-morphisms F, G, we would like first to determine a notion giving a satisfactory meaning to the sentence "F and G are homotopic". Then, A_∞-homotopies being defined, what is now a good notion of a homotopy between homotopies ? And of a homotopy between two homotopies between homotopies ? And so on.

Definition ([LH02])

An A_∞-*homotopy* between two A_∞-morphisms $(f_n)_{n\geqslant 1}$ and $(g_n)_{n\geqslant 1}$ is a collection of maps

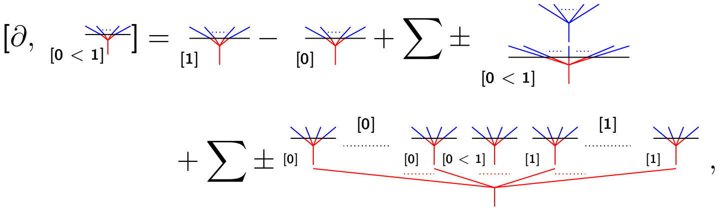
$$h_n : A^{\otimes n} \longrightarrow B \ ,$$

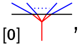
of degree $-n$, satisfying

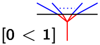
$$[\partial, h_n] = g_n - f_n + \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm h_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3})$$

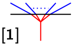
$$+ \sum_{\substack{i_1+\cdots+i_s+l \\ +j_1+\cdots+j_t=n \\ s+1+t\geqslant 2}} \pm m_{s+1+t}(f_{i_1} \otimes \cdots \otimes f_{i_s} \otimes h_l \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) \ .$$

In symbolic formalism,



where we denote 

, 

and 

respectively for the f_n, the h_n and the g_n.

The relation *being A_∞-homotopic* on the class of A_∞-morphisms is an equivalence relation. It is moreover stable under composition.

We denote the standard n-simplex Δ^n as $[0 < \cdots < n]$ and a subface of Δ^n as $[i_1 < \cdots < i_k]$.

Definition ([MS03])

Let I be a face of Δ^n. An *overlapping partition* of I is a sequence of faces $(I_l)_{1 \leqslant \ell \leqslant s}$ of I such that

(i) the union of this sequence of faces is I, i.e. $\cup_{1 \leqslant \ell \leqslant s} I_l = I$;

(ii) for all $1 \leqslant \ell < s$, $\max(I_\ell) = \min(I_{\ell+1})$.

An overlapping 6-partition for $[0 < 1 < 2]$ is for instance

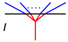
$$[0 < 1 < 2] = [0] \cup [0] \cup [0 < 1] \cup [1] \cup [1 < 2] \cup [2] .$$

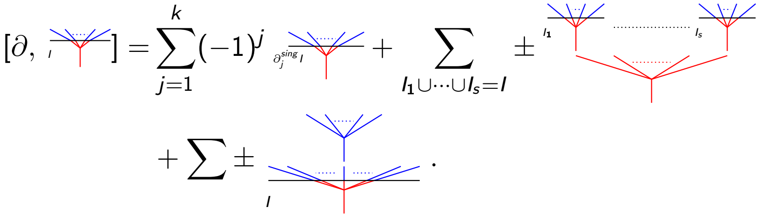
Definition ([Maz21b])

A *n-morphism* from A to B is defined to be a collection of maps $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ of degree $1-m-\dim(I)$ for $I \subset \Delta^n$ and $m \geqslant 1$, that satisfy

$$\left[\partial, f_I^{(m)}\right] = \sum_{j=0}^{\dim(I)} (-1)^j f_{\partial_j I}^{(m)} + \sum_{\substack{i_1+\cdots+i_s=m \\ I_1\cup\cdots\cup I_s=I \\ s\geqslant 2}} \pm m_s(f_{I_1}^{(i_1)} \otimes \cdots \otimes f_{I_s}^{(i_s)})$$

$$+ (-1)^{|I|} \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm f_I^{(i_1+1+i_3)}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) .$$

Equivalently and more visually, a collection of maps  satisfying



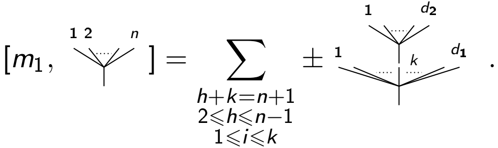
The sets $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A, B)_n$ of n-morphisms from A to B then fit into a HOM-simplicial set $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A, B)_\bullet$.

Theorem ([Maz21b])

For A and B two A_∞-algebras, the simplicial set $\mathrm{HOM}_{A_\infty}(A, B)_\bullet$ is a Kan complex.

The simplicial homotopy groups of the Kan complex $\mathrm{HOM}_{A_\infty}(A, B)_\bullet$ can moreover be explicitly computed.

There exists a collection of polytopes, called the *associahedra* and denoted $\{K_n\}$, which encode the A_∞-equations between A_∞-algebras. This means that K_n has dimension $n-2$ and that its boundary is modeled on the A_∞-equations read as



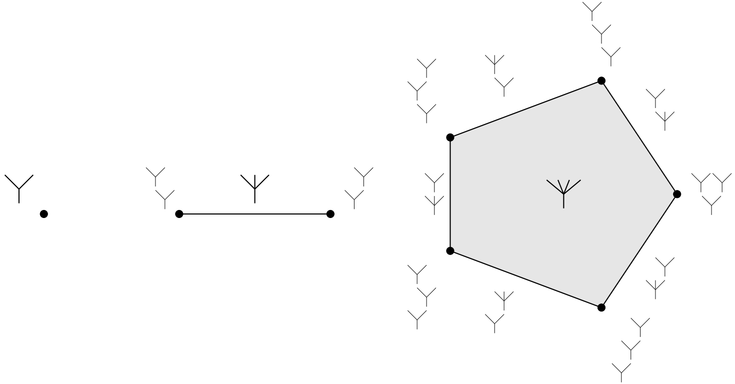
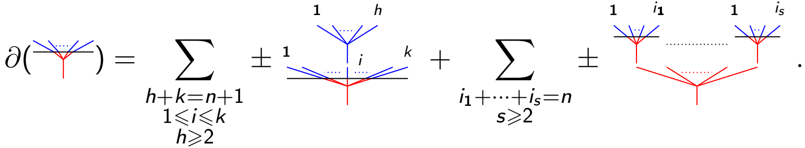


Figure: The associahedra K_2, K_3 and K_4, with cells labeled by the operations they define

1. A_∞-algebras and A_∞-morphisms
2. Higher algebra of A_∞-algebras
3. The n-multiplihedra
   - The associahedra
   - The multiplihedra
   - The n-multiplihedra
4. Higher algebra of A_∞-algebras in Morse theory

There exists a collection of polytopes, called the *multiplihedra* and denoted $\{J_n\}$, which encode the A_∞-equations for A_∞-morphisms. Again, J_n has dimension $n-1$ and the boundary of J_n is modeled on the A_∞-equations for A_∞-morphisms are



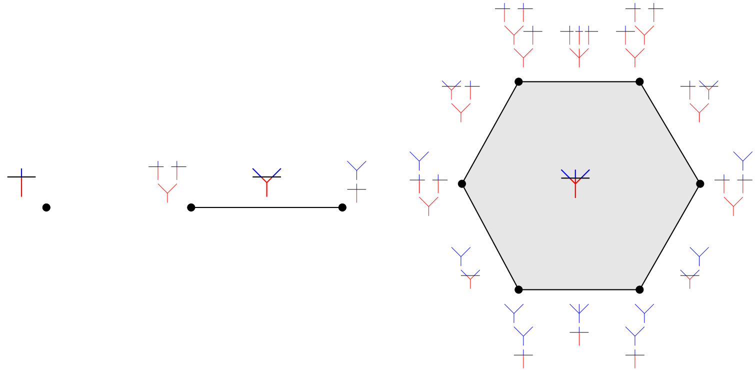


Figure: The multiplihedra J_1, J_2 and J_3 with cells labeled by the operations they define in $A_\infty - \mathrm{Morph}$

We would like to define a family of polytopes encoding n-morphisms between A_∞-algebras. The natural candidate is $\Delta^n \times J_m$.

We prove in [Maz21b] that there exists a refined polytopal subdivision of $\Delta^n \times J_m$ encoding the A_∞-equations for n-morphisms between A_∞-algebras. We define the *n-multiplihedra* to be the polytopes $\Delta^n \times J_m$ endowed with this polytopal subdivision and denote them $n - J_m$.

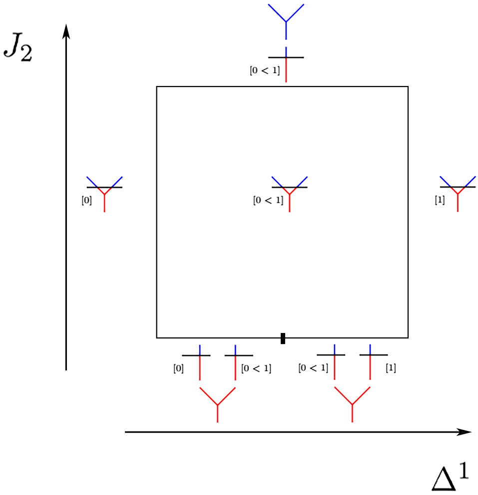


Figure: The 1-multiplihedron $\Delta^1 \times J_2$

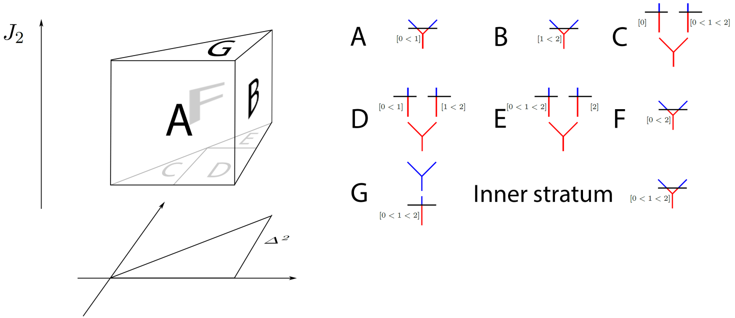


Figure: The 2-multiplihedron $\Delta^2 \times J_2$

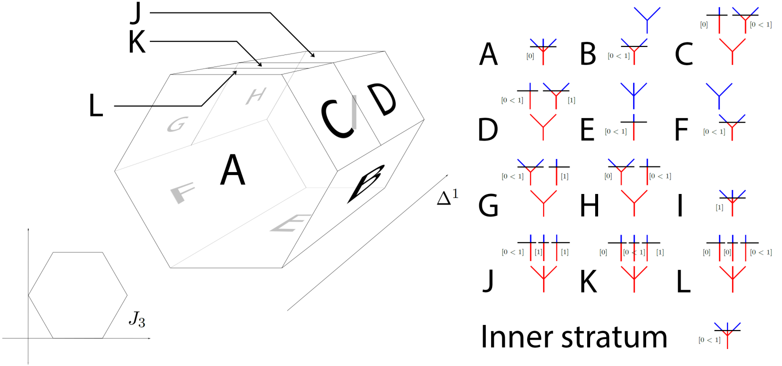


Figure: The 1-multiplihedron $\Delta^1 \times J_3$

Let M be an oriented closed Riemannian manifold endowed with a Morse function f together with a Morse-Smale metric. The Morse cochains $C^*(f)$ form a deformation retract of the singular cochains $C^*_{sing}(M)$ as shown in [Hut08].

$$h \circlearrowright (C^*_{sing}, \partial_{sing}) \underset{i}{\overset{p}{\rightleftarrows}} (C^*(f), \partial_{Morse}) \ .$$

The cup product naturally endows the singular cochains $C^*_{sing}(M)$ with a dg-algebra structure. The homotopy transfer theorem ensures that it can be transferred to an A_∞-algebra structure on the Morse cochains $C^*(f)$.

The differential on the Morse cochains is defined by a count of moduli spaces of gradient trajectories. Is it then possible to define higher multiplications m_n on $C^*(f)$ by a count of moduli spaces such that they fit in a structure of A_∞-algebra ?

Question solved by Abouzaid in [Abo11], drawing from earlier works by Fukaya ([Fuk97] for instance), using moduli spaces of perturbed Morse gradient trees.

We prove in [Maz21a] and [Maz21b] that given two Morse functions f and g, one can in fact construct n-morphisms between their Morse cochain complexes $C^*(f)$ and $C^*(g)$ through a count of geometric moduli spaces of perturbed Morse gradient trees.

These constructions stem from the fact that ...

... the associahedra can be realized as the compactified moduli spaces of stable metric ribbon trees ...

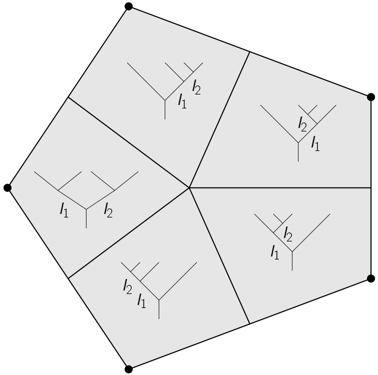
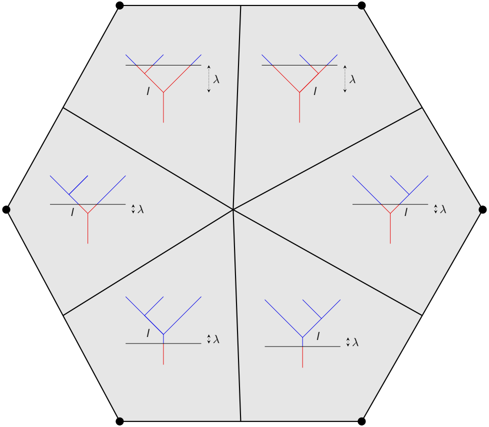


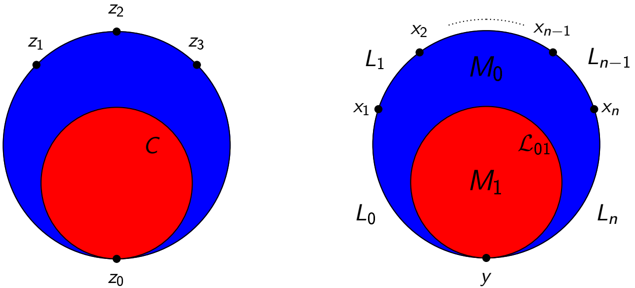
Figure: The compactified moduli space $\overline{\mathcal{T}}_4$

... and the multiplihedra can be realized as the compactified moduli spaces of stable two-colored metric ribbon trees.



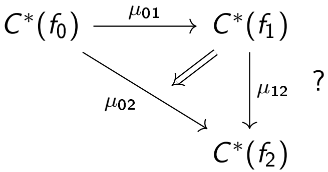
The compactified moduli space $\overline{\mathcal{CT}}_3$

1. It is quite clear that given two compact symplectic manifolds M and N, one should be able to construct n-morphisms between their Fukaya categories $\mathrm{Fuk}(M)$ and $\mathrm{Fuk}(N)$ through counts of moduli spaces of quilted disks (see [MWW18] for the $n = 0$ case).



2. Given three Morse functions f_0, f_1, f_2 and geometrical A_∞-morphisms $\mu_{ij} : C^*(f_i) \to C^*(f_j)$, can we construct an A_∞-homotopy such that $\mu_{12} \circ \mu_{01} \simeq \mu_{02}$ through this homotopy ?

That is, can the following diagram be filled in the A_∞ realm



Work in progress, see also [MWW18] for a similar question.

3. Links between the n-multiplihedra and the 2-associahedra of Bottman (see [Bot19a] and [Bot19b] for instance) ?

Thanks for your attention !

Acknowledgements : Alexandru Oancea, Bruno Vallette, Jean-Michel Fischer, Guillaume Laplante-Anfossi, Brice Le Grignou, Geoffroy Horel, Florian Bertuol, Thomas Massoni, Amiel Peiffer-Smadja and Victor Roca Lucio.

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