Symplectic capacities of p-products

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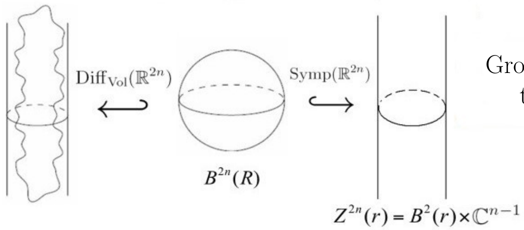
January, 2022

Symplectic capacities

A normalized symplectic capacity on $\mathbb{R}^{2n}$ is a map c from subsets $U \subset \mathbb{R}^{2n}$ to $[0,\infty]$ with the following properties.

- If $U \subseteq V$, $c(U) \leq c(V)$,
- $c(\phi(U)) = c(U)$ for any symplectomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$,
- $c(\alpha U) = \alpha^2 c(U)$ for $\alpha > 0$,
- $c(B^{2n}(r)) = c(B^2(r) \times \mathbb{C}^{n-1}) = \pi r^2$.



Gromov's non sqeezing theorem (1985):

iff $R \leq r$

The EHZ capacity

For a convex $K \subset \mathbb{R}^{2n}$ many normalized symplectic capacities coincide (Abbondandolo, Ekeland, Ginzburg, Gutt, Hofer, Hutchings, Irie, Kang, Shon, Viterbo, Zehnder):

$$c_{\mathrm{HZ}}(K) = c^1_{\mathrm{EH}}(K) = c^1_{\mathrm{GH}}(K) = c_{\mathrm{SH}}(K)$$

equals the minimal action of a closed characteristic on ∂K.

Denote this value by $c_{\mathrm{EHZ}}(K)$.

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H-K, 2019: Combinatorial formula for $c_{_{\mathrm{EHZ}}}$ of convex polytopes in $\mathbb{R}^{2n}$.

Chaidez–Hutchings, 2021: Algorithm to find closed characteristics up to a given action and C-Z index for polytopes in $\mathbb{R}^4$.

Viterbo's conjecture

The systolic ratio of $K \subset \mathbb{R}^{2n}$ is

$$sys_n(K) := \frac{c_{\mathrm{EHZ}}(K)}{(n!\mathrm{Vol}(K))^{\frac{1}{n}}}.$$

Conjecture (Viterbo, 2000)

For any convex body $K \subset \mathbb{R}^{2n}$,

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Viterbo, Artstein-Avidan–Milman–Ostrover:

Up to a constant independent of the dimension.

Abbondandolo–Bramham–Hryniewicz–Salomão,

Abbondandolo–Benedetti: Holds locally near the ball.

Artstein-Avidan–Karasev–Ostrover:

Viterbo's conjecture implies Mahler's conjecture.

Viterbo's conjecture in the asymptotic regime

Theorem (H-K, Ostrover)

If Viterbo's conjecture holds in dimension $2n$ for some $n > 1$, then it also holds in dimension $2m$ for every $m \leq n$.

Moreover, if there exists a sequence $\alpha(n) \xrightarrow{n\to\infty} 1$ such that for every convex body $K \subset \mathbb{R}^{2n}$ one has

$$sys_n(K) = \frac{c_{\mathrm{EHZ}}(K)}{(n!Vol(K))^{\frac{1}{n}}} \leq \alpha(n),$$

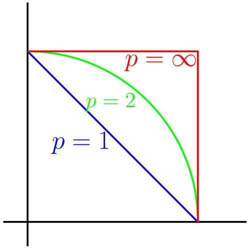
then Viterbo's conjecture holds in every dimension n.

p-products

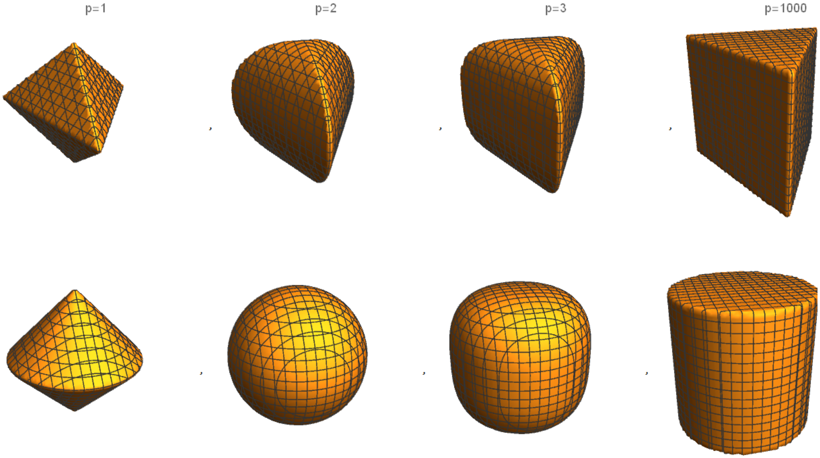
$K \subset \mathbb{R}^{2n}$, $T \subset \mathbb{R}^{2m}$ two convex bodies. A generalization of the Cartesian product is the p-product operation defined by

$$K \times_p T := \bigcup_{0 \le t \le 1} \left((1-t)^{1/p} K \times t^{1/p} T \right) \subset \mathbb{R}^{2n} \times \mathbb{R}^{2m}.$$

Note that $K \times_\infty T = K \times T$ is the Cartesian product, and $K \times_1 T = K \oplus T$ is the free sum of K and T.



p-products



p-products

For two convex bodies $K \subset \mathbb{R}^{2n}$, $T \subset \mathbb{R}^{2m}$, and $1 \leq p \leq \infty$,

Lemma

$$Vol(K \times_p T) = \frac{\Gamma(\frac{2n}{p}+1)\Gamma(\frac{2m}{p}+1)}{\Gamma(\frac{2m+2n}{p}+1)} Vol(K) Vol(T).$$

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Theorem (H-K, Ostrover)

$$c_{\mathrm{EHZ}}(K \times_p T) = \begin{cases} \min\{c_{\mathrm{EHZ}}(K), c_{\mathrm{EHZ}}(T)\}, & 2 \leq p \leq \infty \\ \left(c_{\mathrm{EHZ}}(K)^{\frac{p}{p-2}} + c_{\mathrm{EHZ}}(T)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}, & 1 \leq p < 2 \end{cases}$$

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Corollary (H-K, Ostrover)

$$sys_{n+m}(K \times_p T)^{m+n} \leq sys_n(K)^n sys_m(T)^m,$$

where equality holds if and only if $c_{\mathrm{EHZ}}(K) = c_{\mathrm{EHZ}}(T)$ *and* $p = 2$.

Higher order capacities

Conjecture (H-K, Ostrover)

For star-shaped domains $K \subset \mathbb{R}^{2n}$, $T \subset \mathbb{R}^{2m}$, and $p \geq 1$,

$$c^k(K \times_p T) = \begin{cases} \min\limits_{i+j=k} \left[c^i(K)^{\frac{p}{p-2}} + c^j(T)^{\frac{p}{p-2}} \right]^{\frac{p-2}{p}}, p \geq 2 \\ \max\limits_{\substack{i+j=k+1 \\ i,j \neq 0}} \left[c^i(K)^{\frac{p}{p-2}} + c^j(T)^{\frac{p}{p-2}} \right]^{\frac{p-2}{p}}, 1 \leq p \leq 2 \end{cases}$$

It is conjectured that $c^k_{\mathrm{EH}}(K) = c^k_{\mathrm{GH}}(K)$ (Gutt–Hutchings).

Gutt–Hutchings: True for $c^k_{\mathrm{GH}}(K \times T)$ for convex toric domains.

Cieliebak–Hofer–Latschev–Schlenk, Chekanov:

True for $c^k_{\mathrm{EH}}(K \times T)$ in general.

Kerman–Liang: True for p-products of discs.

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Theorem (H-K, Ostrover)

The conjecture above holds for $\{c^k_{GH}(K \times_p T)\}_{k=1}^{\infty}$ when K and T are convex toric domains and $p \geq 2$, or when K and T are concave toric domains and $1 \leq p \leq 2$.

p-decomposition of the ball

Theorem (H-K, Ostrover)

Assume that the conjecture holds. Then for any two convex bodies $K \subset \mathbb{R}^{2n}$, $T \subset \mathbb{R}^{2m}$, $p \neq 2$, one has $B^{2n+2m}(r) \not\cong K \times_p T$. Moreover, if a symplectic image of the ball $\widetilde{B}^{2(n+m)}(r) \subset \mathbb{R}^{2(n+m)}$ can be written as $\widetilde{B}^{2(n+m)}(r) = K \times_2 T$ for some convex bodies $K \subset \mathbb{R}^{2n}$, $T \subset \mathbb{R}^{2m}$, then one has $c^k_{\mathrm{EH}}(K) = c^k_{\mathrm{EH}}(B^{2n}(r))$ and $c^k_{\mathrm{EH}}(T) = c^k_{\mathrm{EH}}(B^{2m}(r))$.

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Ginzburg, Gürel: If $c^1_{\mathrm{EH}}(K) = c^n_{\mathrm{EH}}(K)$ then K is symplectic Zoll.

Abbondandolo, Bramham, Hryniewicz, Salomão: In $\mathbb{R}^4$ every symplectic Zoll body is symplectomorphic to the ball.

Abbondandolo, Benedetti: In higher dimensions symplectic Zoll bodies are local maximizers of the systolic ratio.

c_∞ of p-products

Denote

$$c_\infty(K) = \lim_{k\to\infty} \frac{c^k(K)}{k}.$$

Cieliebak, Hofer, Latschev, Schlenk:

$$c_\infty(E(a_1,\dots,a_n)) = \frac{1}{1/a_1 + \cdots + 1/a_n},$$

and

$$c_\infty(P(a_1,\dots,a_n)) = \min\{a_1,\dots,a_n\}.$$

Theorem (H-K, Ostrover)

If the conjecture holds then for $1 \le p \le \infty$, and convex domains $K_1,\dots,K_m \subset \mathbb{R}^{2n}$ such that $c_\infty(K_1),\dots,c_\infty(K_m)$ exist, one has

$$c_\infty(K_1 \times_p \cdots \times_p K_m) = \left(c_\infty(K_1)^{\frac{-p}{2}} + \cdots + c_\infty(K_m)^{\frac{-p}{2}}\right)^{\frac{-2}{p}}.$$