Lorentzian distance functions on the group of contactomorphisms

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Positivity on $\mathrm{Cont}_0(M, \xi)$

Consider a contact manifold $(M, \xi = \ker\alpha)$, the identity component of its group of contactomorphisms $\mathrm{Cont}_0(M, \xi)$.
ϕ_t is called **positive (non-negative)** if $\alpha\left(X_t^\phi\right) > 0$ (≥ 0), where X_t^ϕ denotes the contact Hamiltonian vector field of ϕ_t.
This induces two relations on $\mathrm{Cont}_0(M, \xi)$:

$$\phi \ll \psi :\Leftrightarrow \ \exists \text{ a positive isotopy from } \phi \text{ to } \psi$$

and

$$\phi \preccurlyeq \psi :\Leftrightarrow \ \exists \text{ a non-negative isotopy from } \phi \text{ to } \psi.$$

The **interval topology** is the topology induced by the open intervals of $\ll$, i.e. the sets

$$(\phi, \psi) := \{\tilde{\phi} \in \mathrm{Cont}_0(M, \xi) | \phi \ll \tilde{\phi} \ll \psi\}.$$

Causal Relations

Let (N, g) be a Lorentzian manifold, i.e. a pseudo-Riemannian manifold of index $(n-1)$.
Fix a vector field X with $g(X, X) > 0$ (time-orientation).
A smooth curve γ is called **future pointing timelike (causal)** if $g(\gamma', \gamma') > 0$ (≥ 0) and $g(X, \gamma') > 0$.
This induces two relations on (N, g):

$$p \ll q :\Leftrightarrow \; \exists \text{ a future pointing timelike curve from } p \text{ to } q$$

and

$$p \leq q :\Leftrightarrow \; \exists \text{ a future pointing causal curve from } p \text{ to } q.$$

(N, g) **strongly causal** if the interval topology of $\ll$ coincides with the manifold topology.

Lorentzian Distance Functions

Define

$$\tau_g \colon N \times N \to [0, \infty]$$
$$(p, q) \mapsto \begin{cases} \sup\limits_{\gamma} \int\limits_0^1 \sqrt{g(\gamma', \gamma')}\, dt, & \text{if } p \leq q \\ 0, & \text{otherwise} \end{cases}$$

Here the supremum is taken over all future pointing causal curves from p to q.

Basic Properties:

(i) $\tau_g(p, q) > 0 \Leftrightarrow p \ll q$

(ii) $\tau_g(p, q) \geq \tau_g(p, r) + \tau_g(r, q)$ for $p \leq r \leq q$

(iii) τ_g lower semi-continuous.

If (N, g) is strongly causal, g is uniquely determined by τ_g. In particular any bijection $f \colon N \to N$ with $\tau_g(f(p), f(q)) = \tau_g(p, q)$ is a smooth isometry.

Lorentzian Pre-Length Spaces

Definition (Kunzinger, Sämann)

Let $(X, \ll, \leq, d)$ be a metric space with a transitive relation $\ll$ and a reflexive and transitive relation $\leq$ such that $x \ll y \Rightarrow x \leq y$ (causal space).
Consider $\tau \colon X \times X \to [0, \infty]$.
$(X, \ll, \leq, d, \tau)$ is called a Lorentzian pre-length space if

(i) $\tau(x, y) > 0 \Leftrightarrow x \ll y$.

(ii) $\tau(x, z) \geq \tau(x, y) + \tau(y, z)$ for all $x \leq y \leq z$.

(iii) τ is lower semi-continuous.

The Metric on $\mathrm{Cont}_0(M, \xi)$

Consider a closed contact manifold (M, ξ). Shelukhin defined a Hofer-type norm on $\mathrm{Cont}_0(M, \xi)$:

$$|\phi|_\alpha := \inf_{\phi_t} \int_0^1 \max_M |\alpha(X_t^\phi)| \, dt,$$

where ϕ_t isotopy with $\phi_0 = id_M, \phi_1 = \phi$.

Theorem (Shelukhin 14)

The norm $|\cdot|_\alpha$ satisfies

(i) $|\phi|_\alpha = 0 \Leftrightarrow \phi = id_M$.

(ii) $|\phi\psi|_\alpha \leq |\phi|_\alpha + |\psi|_\alpha$.

(iii) $|\phi^{-1}|_\alpha = |\phi|_\alpha$.

(iv) $|\psi\phi\psi^{-1}|_\alpha = |\phi|_{\psi^*\alpha}$.

$d_\alpha(\phi, \psi) := |\psi^{-1}\phi|_\alpha$ defines a metric on $\mathrm{Cont}_0(M, \xi)$.

The Lorentzian Distance Function

Define

$$\tau_\alpha(\phi,\psi) := \begin{cases} \sup\limits_{\phi_t} \int\limits_0^1 \min\limits_M \alpha(X_t^\phi)\, dt, & \text{if } \phi \preccurlyeq \psi \\ 0, & \text{otherwise} \end{cases}$$

Here the supremum is taken over all non-negative ϕ_t with $\phi_0 = \phi$ and $\phi_1 = \psi$.

Theorem (H. 21)

$(\mathrm{Cont}_0(M,\xi), \ll, \preccurlyeq, d_\alpha, \tau_\alpha)$ is a Lorentzian pre-length space. Moreover τ_α is continuous with respect to d_α.
If $\preccurlyeq$ is a partial order, then $\tau_\alpha < \infty$.

Further Results

A similar proof shows

Theorem (H. 21)

The interval topology on $\mathrm{Cont}_0(M,\xi)$ *is contained in the topology induced by* d_α.

Theorem (H. 21)

Assume $\preccurlyeq$ *is a partial order. Let* ϕ_t^α *be the Reeb-flow of* α *and* $t \geq 0$. *Then*

$$|\phi_t^\alpha|_\alpha = t = \tau_\alpha(id_M, \phi_t^\alpha).$$

Questions

(i) Are there contact manifolds such that $\mathrm{Cont}_0(M,\xi)$ is strongly causal in the sense that the interval topology coincides with the topology of d_α?

(ii) τ_α is not bi-invariant. Are there bi-invariant Lorentzian distance functions on $\mathrm{Cont}_0(M,\xi)$ other than

$$\tau(\phi,\psi) := \begin{cases} \infty, & \text{if } \phi \ll \psi \\ 0, & \text{otherwise} \end{cases} ?$$

Thank you!