Sections and unirulings of families over $\mathbb{P}^1$

Alex Pieloch

Columbia University

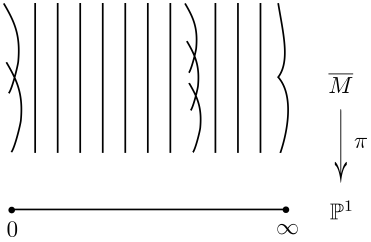
pieloch@math.columbia.com

November 5, 2021

Setup

Notation

Let $\pi : \overline{M} \to \mathbb{P}^1$ be a morphism of smooth projective varieties over $\mathbb{C}$.



Theorem (Griffiths)

If $\pi : \overline{M} \to \mathbb{P}^1$ has at most two singular fibres, then the variation of the Hodge structures of the fibres of π is trivial.

Main results

Question

Is this Hodge theoretic triviality the shadow of algebraic cycles or actual complex geometry features?

Theorem (P.)

(i) If $\pi : \overline{M} \to \mathbb{P}^1$ has at most one singular fibre, then $\overline{M}$ is uniruled and admits sections.

(ii) If $\pi : \overline{M} \to \mathbb{P}^1$ has at most two singular fibres (say at 0 and ∞), and

$$c_1(M = \pi^{-1}(\mathbb{P}^1 \setminus \infty)) = 0,$$

then $\overline{M}$ is uniruled and admits genus zero multisections.

Idea of proof

(i) Local symplectic cohomology: associate to each compact subset $K \subset M$ a chain complex that is constructed from Hamiltonian dynamics near K (generators) and holomorphic curves in M (differentials).

(ii) Show that local symplectic cohomology of

$$\pi^{-1}(\mathbb{D}_a) \subset M = \overline{M} \setminus \pi^{-1}(\infty)$$

vanishes for each $a > 0$.

(iii) Vanishing is witnessed by holomorphic (multi)sections of π over $\mathbb{D}_a$.

(iv) Use a degeneration to the normal cone argument to produce (multi)sections of π over $\mathbb{P}^1$ from the (multi)sections of π over $\mathbb{D}_a$.

Arranging to do Floer Theory

$M_a = \pi^{-1}(\mathbb{D}_a) \subset M = \overline{M} \setminus \pi^{-1}(\infty)$.

So $\pi|_M : M \to \mathbb{C}$ and $\pi|_{M_a} : M_a \to \mathbb{D}_a$.

Lemma

Given a Kähler form $\Omega_{\mathbb{C}}$ on M there exists a symplectic embedding

$$\psi : (M, \Omega_{\mathbb{C}}) \hookrightarrow (M, \Omega)$$

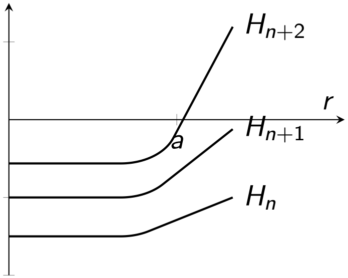
such that:

- *(i) The end of (M, Ω) is convex and satisfies an integrated maximum principle with respect to Floer trajectories.*
- *(ii) The orbits of the radial function $r = |\pi(\cdot)|$ are Reeb orbits that wrap positively around origin in $\mathbb{C}$ when projected by π.*

Strategy: push-forward the (integrable) complex structure on M along ψ, produce curves in (M, Ω), and pull-back the curves along ψ.

Defining Local Symplectic Cohomology (1)

Let $r = |\pi(\cdot)|$, and consider (radially admissible) Hamiltonians



(i) H_n is C^2-small Morse inside interior of M_a,

(ii) $H_n < H_{n+1}$,

(iii) $\partial_r H_n < \partial_r H_{n+1}$, and

(iv)

$$\lim_n H_n(x) = \begin{cases} 0 & x \in M_a \\ +\infty & x \in M \setminus M_a \end{cases}$$

Defining Local Symplectic Cohomology (2)

Associated to each H_n is a chain complex:

(i) $CF(H_n) = \Lambda_{\geq 0} \cdot \langle x \mid x \text{ is a 1-periodic orbit of } H_n \rangle$

(ii) $\partial : CF(H_n) \to CF(H_n)$:

$$\partial(x_+) = \sum_{\substack{\overline{\partial}_{H_n} u = 0 \\ u \text{ rigid cylinder} \\ u(\pm\infty, t) = x_\pm(t)}} \left(\#_{vir}(u) \cdot x_- \cdot T^{E_{top}(A)} \right)$$

where

$$E_{top}(u) = \int u^*\Omega + \int H(x_-) - H(x_+)\, dt.$$

Defining Local Symplectic Cohomology (3)

There are continuation maps

$$CF(H_n) \to CF(H_{n+1})$$

defined over $\Lambda_{\geq 0}$.

Definition

The *local symplectic cohomology of M_a in M* is

$$\widehat{SH}(M_a \subset M) = H\left(\varprojlim_R \left(\operatorname{colim}_n CF(H_n) \otimes_{\Lambda_{\geq 0}} \Lambda_{\geq 0}/\Lambda_{\geq R}\right)\right) \otimes_{\Lambda_{\geq 0}} \Lambda,$$

where Λ is the Novikov field.

Heuristically, the completion "kills" orbits that lie away from M_a.

Producing (multi)sections over $\mathbb{D}_a$

Proposition

If $\widehat{SH}(M_a \subset M) \equiv 0$, then there exists a holomorphic disk $u : \mathbb{D} \to M_a$ such that $\pi \circ u$ covers $\mathbb{D}_a$ (ie u is a multisection).

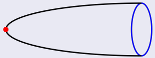
Proof.

(i) Via the integrated maximum principle there is a LES:

$$\longrightarrow H^*(M;\Lambda) \longrightarrow \widehat{SH}(M_a \subset M) \longrightarrow \widehat{SH}_+(M_a \subset M) \longrightarrow ,$$

where $\widehat{SH}_+(M_a \subset M)$ is generated by Reeb orbits.

(ii) $\widehat{SH}(M_a \subset M) \equiv 0 \implies$ there is a Floer trajectory connecting a Reeb orbit to a critical point that corresponds to the unit in $H^*(M;\Lambda)$.



(iii) Our Reeb orbits project under $\pi : M \to \mathbb{C}$ to curves that wrap positively around $\partial\mathbb{D}_a$.

(iv) So the corresponding Floer trajectory covers $\mathbb{D}_a$.

(v) Use a Gromov compactness argument to "turn off" the Hamiltonian perturbation in Floer's equation to obtain genuine holomorphic disk that covers $\mathbb{D}_a$.

□

Proposition

If $\pi : M \to \mathbb{C}$ has no singular fibres or has one singular fibre with $c_1(M) = 0$, then $\widehat{SH}(M_a \subset M) = 0$ for all $a > 0$.

Vanishing of local symplectic cohomology

Proof.

(i) A neighborhood of $\pi^{-1}(0)$ is stably displaceable inside of M (McLean).

(ii) So $\widehat{SH}(M_\varepsilon \subset M) = 0$ for some $\varepsilon > 0$ sufficiently small (Varolgunes, McLean).

(iii) However, it could be the case that $M_\varepsilon \subset M_a$.

(iv) Construct a rescaling isomorphism (requires conditions on π or c_1):

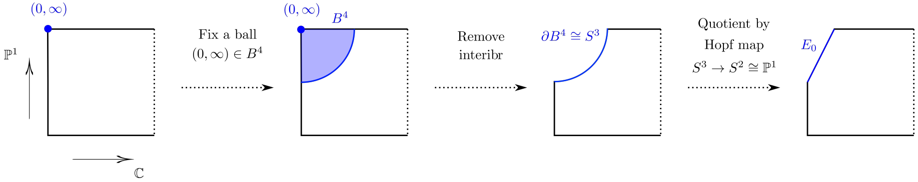
$$\widehat{SH}(M_a \subset M) \to \widehat{SH}(M_\varepsilon \subset M).$$

□

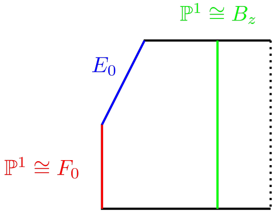
From disks to spheres

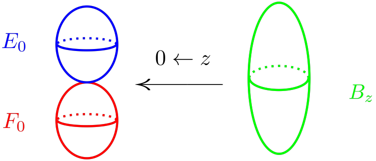
Consider the case of $\overline{M} = \mathbb{P}^1$, $M = \mathbb{C}$, and $\pi =$ Identity.

Want to use holomorphic disks in $\mathbb{C}$ to produce holomorphic spheres in $\mathbb{P}^1$.



From disks to spheres



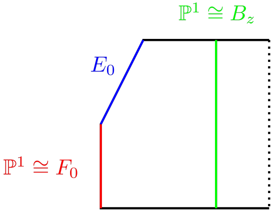


$$Bl_{0\times\pi^{-1}(\infty)}(\mathbb{C}\times\mathbb{P}^1) = B$$

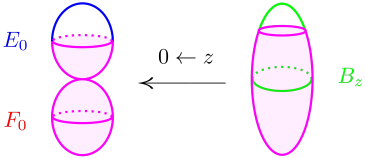
E_0, F_0, B_z are all just $\mathbb{P}^1$s

The B_z Gromov converge to $E_0 \cup F_0$.

From disks to spheres



$$Bl_{0\times\pi^{-1}(\infty)}(\mathbb{C}\times\mathbb{P}^1) = B$$



Fix a sequence of disk

$$\{z \mid |z| \leq 2\} \subset B_z \cong \mathbb{P}^1$$

The Gromov limit of these disks is a nodal disk with

(i) boundary in E_0,

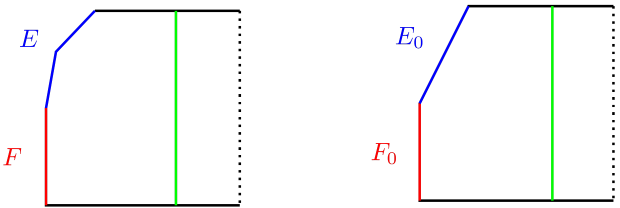
(ii) a component that is all of F_0.

From disks to spheres

For $\overline{M}$ more general, the idea is similar.

Degenerate $\overline{M}$ into two families one over E_0 and one over F_0 and degenerate disks into disks over E_0 and multisections over F_0.

$$P_z \cong \overline{M} \xrightarrow{\ \pi\ } \mathbb{P}^1 \cong B_z$$



$$P = Bl_{0\times\pi^{-1}(\infty)}(\mathbb{C}\times\overline{M}) \xrightarrow{\ \widetilde{\pi}\ } Bl_{0\times\pi^{-1}(\infty)}(\mathbb{C}\times\mathbb{P}^1) = B$$

Questions