Detecting non-trivial elements in the homotopy of spaces of Legendrian knots via Algebraic K-theory

Joint with Thomas Kragh

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For instance, the following table gives the list of non-trivial summands in homotopy groups of Legendrian unknots for $n \geq 34$

0	1	2	3	4	5	6	7	8	9	10
0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/8 \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$

We will show how the homotopy groups of stable Cerf's pseudoisotopy space and/or Waldhausen's *h*-cobordism space can be injected into the homotopy groups of Legendrian submanifolds. The computation of these homotopy groups is very difficult, and it is a subject of Algebraic K-theory. However, it is known that they are highly non-trivial even for the case of the point.

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One can define the action of these groups on spaces of Legendrian submanifolds in any contact manifold, but we can prove the non-triviality only in quite restricted situations, where the technique of generating functions can be applied. It would be interesting to expand this list.

Cerf's pseudoisotopy space

A pseudoisotopy of a manifold M is a diffeomorphism $M \times (I = [0,1]) \to M \times I$ which is fixed on $M \times 0$. If M has boundary that the pseudotopy f is required to be identity over the boundary.

The diffeomorphism $f|_{M = M \times 1} : M \to M$ is called pseudoisotopic to the identity.

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We denote by $\mathcal{P}(M)$ the space of pseudoisotopies. As it was observed by Jean Cerf, the space $\mathcal{P}(M)$ is homotopy equivalent to the space $\mathcal{E}(M)$ of functions $M \times \mathbb{R}$ which coincide with the projection $M \times I \to I$ near $\partial M \times \mathbb{R} \cup M \times (\mathbb{R} \setminus (-1,1))$.

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There is an operations of (positive/negative) stabilizations for the isotopies, $\mathrm{st}_\pm : \mathbb{P}(M) \to \mathbb{P}(M \times I)$ which for the model $\mathcal{E}(M)$ amount to adding $\pm t^2$. We denote $\mathrm{st} = \mathrm{st}_+ \circ \mathrm{st}-$. Define a homotopy (co)-limit $\mathcal{P}_\infty(M) = \mathcal{E}_\infty(M)$ under the stabilization operation A theorem of Hatcher-Igusa claims that st induces an isomorphism on homotopy groups in a range of dimension, and hence, so does $\mathrm{st}_\infty = \mathrm{st} \circ \mathrm{st} \circ \cdots : \mathcal{P}(M) \to \mathcal{P}_\infty(M)$.

Waldhausen's h-cobordism space

Define the h-cobordism space (or rather one of the connected components of this space) $\mathcal{H}(M)$ as the space of embedded submanifolds $N \to M \times [-1,1]$ isotopic to $M \times 0 \hookrightarrow M \times [-1,1]$.

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If an inclusion $N \to M$ is k-connected then $\mathcal{H}(N) \to \mathcal{H}(M)$ is $(k-2)$-connected.

Generating function construction

A function $f : M \to \mathbb{R}$ yields a Legendrian

$$\Lambda_f = \{p = \frac{\partial f}{\partial q}, z = f(q)\} \subset (J^1(M) = T^*M \times \mathbb{R}, pdq - dz).$$

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If we have a fibration $p : E \to M$ then under some transversality assumptions it allows to to transport Legendrians from J^1E to J^1M. The pushforward Legendrian is defined via the contact reduction construction: Take the intersection of $\Lambda \subset J^1E$ with the coisotropic submanifold C of 1-forms vanishing along the fibers of $p : E \to M$ and project it to $p_*\Lambda \subset T^*M$. If $\Lambda = \Lambda_G$ for a function $G : E \to \mathbb{R}$ then the intersection $C \cap \Lambda_G$ is the locus of fiberwise critical points of G, and $p_*\Lambda$ consists of horizontal 1-jets of G at these points:

$$p_*\Lambda_G = \{z = G(q,\eta), p = \frac{\partial G}{\partial q}, \frac{\partial G}{\partial \eta} = 0\}.$$

The function G is called a generating function for the Legendrian $\Lambda = p_*\Lambda_G$, and we'll usually denote it just by Λ_G.

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f is a *fibration at infinity*, if there exist a finite segment $[-a, a] \subset \mathbb{R}$ and a compact subset $K \subset f^{-1}[-a, a] \subset X$ such that the restriction of f to the following three subsets *fibers* them over their respective images

(i) $f^{-1}(-\infty, -a] \to (-\infty, -a]$

(ii) $f^{-1}[a, \infty) \to [a, \infty)$

(iii) $(f^{-1}[-a, a]) - K \to [-a, a]$.

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Addition property

If f_i on X_i are fibrations at infinity for $i = 1, 2$, then so is $f_1 \oplus f_2$ on $X_1 \times X_2$.

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In particular, if $f : X \to \mathbb{R}$ is a fibration at infinity and $Q : \mathbb{R}^N \times \mathbb{R}$ is a non-degenerate quadratic form, then the *quadratic stabilization* $f \oplus Q : X \times \mathbb{R}^N \to \mathbb{R}$ is a fibration at infinity.

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We will require the generating functions for Legendrians in $J^1(M)$ defined on the total space of a fibration $E \to M$ to be fiberwise fibrations at infinity. If M is non compact then $f : E \to \mathbb{R}$ is also required to be a fibration at infinity.

Serre fibration property for generating functions

Suppose we are given a fibration $p : E \to M$ and a (fiberwise) fibration at infinity G. Let Gen_G be the space of functions on $M \times \mathbb{R}^{2N}$ (N is not fixed) which coincide with G_{st} at infinity and generate embedded Legendrian submanifold. Let Leg_G be the space of Legendrian submanifolds in J^1M which properly project to M, and at infinity generated by G.

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Theorem

The generating map $\mathrm{Gen}_G \to \mathrm{Leg}_G$ is a Serre fibration over connected components which intersect the image.

Consider the space $\mathcal{L}(N)$ of Legendrian submanifolds in $J^1(N \times \mathbb{R}) \setminus (N \times \mathbb{R})$ which at infinity coincide with the graph of the 1-jet section $j^1(t)$ of the function t, and are Legendrian isotopic (in $J^1(N \times \mathbb{R})$) to the graph.

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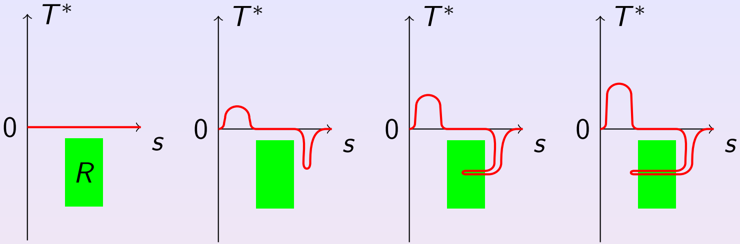
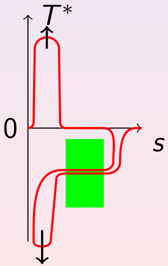
Theorem

The homotopy groups of the stable pseudoisotopy space $\mathcal{P}(N)$ inject into $\mathcal{L}(N)$ in the range of dimensions $<\sim \frac{n}{3}$.



$J_0^\epsilon = I$



$J_{1-\delta}^\epsilon$

Let $\mathrm{Leg}(M \times \mathbb{R})$ be the connected component of the 0-section in space of Legendrian embedding $M \times \mathbb{R} \to J^1(M \times \mathbb{R})$ which coincide with the inclusion (of the 0-section) at infinity.

Theorem

There exist maps $F : \mathcal{P}(M) \to \Omega(\mathrm{Leg}(M \times \mathbb{R})$ and $G : \mathrm{Leg}(M \times \mathbb{R}) \to \mathcal{P}_\infty(M)$ such $(\Omega G) \circ F$ is homotopic to the stabilization map $\mathrm{st}_\infty : \mathcal{P}(M) \to \mathcal{P}_\infty(M)$. In particular, F induces isomorphism on homotopy groups $\pi_j(\mathcal{P}(M))$ for $j <\sim \frac{n}{3}$.

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We showed that $\mathcal{P}(M) \approx \Omega\mathcal{H}(M)$ and that there is a map F of $\mathcal{P}(M) \to \Omega(\mathrm{Leg}(M \times \mathbb{R}))$.
It turns out that the delooping $\mathcal{H}(M) \to \mathrm{Leg}(M \times \mathbb{R})$ also exists.

Modifying a Legendrian submanifold in T^*M with an h-cobordism

Consider a Legendrian $\Lambda \subset J^1(M)$. Let $\Sigma^1(\Lambda) \subset \Lambda$ be the fold locus of the projection $\pi|_\Lambda : \Lambda \to M$ and $A \subset \Sigma$ a codimension 0 submanifold and set $\overline{A} := \pi(A)$.

There is a splitting $N := \overline{A} \times [-1, 1]$ of the tubular neighborhood $N \supset \overline{A}$ such that near A the Legendrian Λ is generated by the function $z^3 - tz$, where t is the coordinate corresponding to the second factor.

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Take an element $H \in \mathcal{H}(A)$, which can be viewed as a codimension 1 submanifold in N. There is a homotopically canonical function $h : N \to [-1, 1]$ which coincides with t near $\overline{A} \times 0 \cup \overline{A} \times 1$ and such that $h^{-1}(0) = H$ a regular level set. We then replace a neighborhood of A in Λ by a Legendrian generated by the function $z^3 - h(x, t)z$.

Note that if $\Lambda \in \mathrm{Leg}(N \times \mathbb{R})$ then one can canonically create a fold parallel to $N \times 0$, and hence one gets a map $J : \mathcal{H}(N) \times \mathrm{Leg}(N \times \mathbb{R}) \to \mathrm{Leg}(N \times \mathbb{R})$.

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Recall the Serre fibration $\mathrm{Gen}(N \times \mathbb{R}) \to \mathrm{Leg}(N \times \mathbb{R})$. The map J lifts to the map $\widehat{J} : \mathcal{H}(N) \times \mathrm{Gen}(N \times \mathbb{R}) \to \mathrm{Gen}(N \times \mathbb{R})$.

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Our next goal is to construct the map $\delta : \mathrm{Gen}(N \times \mathbb{R}) \to \mathcal{H}_\infty(N)$. Given $F \in \mathrm{Gen}(N \times \mathbb{R})$, $F : E \to \mathbb{R}$ consider its double $DF : E \underset{N \times \mathbb{R}}{\oplus} E \to \mathbb{R}$, $DF((q, \eta_1), (q, \eta_2)) = F(q, \eta_1) - F(q, \eta_2)$.
Note that $\Delta := DF^{-1}(0)$ is the diagonal, and we can assume that DF has no critical values in $(0, 1]$.

Denote $F_\varepsilon = \{F = \varepsilon\}$. Note that when F is the quadratic form Q then $Q_\varepsilon = (N \times \mathbb{R}) \times S^{K-1} \times \mathbb{R}^K$ for any $\varepsilon > 0$.

Hence, F_ε is a submanifold of a trivial cobordism bounded by Q_{ε_0} and Q_{ε_1} for $\varepsilon_0 \ll \varepsilon \ll \varepsilon_1$, and if its inclusion is a homotopy equivalence, can be considered as an element of $\mathcal{H}_\infty(N \times \mathbb{R}) = \mathcal{H}_\infty(N)$.

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On homotopy groups the map δ descends to the map $\underline{\delta} : \mathrm{Leg}(N \times \mathbb{R}) \to \mathcal{H}_\infty(N)$.

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The map $G : \Omega(\mathrm{Leg}(N \times \mathbb{R})) \to \mathcal{P}(N) = \Omega\mathcal{H}(N)$ which we discussed above is just the map $\Omega\underline{\delta}$.

The key observation that

$$\overline{\delta} \circ j : \mathcal{H}(N) \to \mathcal{H}(N)$$

is homotopic to the stabilization map $\mathrm{st}_\infty : \mathcal{H}(N) \to \mathcal{H}(N)$, and hence, induces an isomorphism on homotopy groups up to $\sim \frac{n}{3}$.

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Applications to the topology of the space of unknots in $\mathbb{R}^{2n+1}$

Theorem

The homotopy groups of $\mathcal{H}_{\rm st}(\mathrm{pt})$ split inject into the space of Legendrian unknots in $\sim< \frac{n}{3}$ (in fact $< \min(\frac{n-8}{2}, \frac{n-5}{3})$).

We note that all the non-trivial elements provided by this and other theorems in this talk are trivial on the formal level.

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We note that all the non-trivial elements provided by this and other theorems in this talk are trivial on the formal level.
This contrasts with a recent theorem of Fernandez, Martinez-Aguinaga and Presas which states that all higher homotopy grops of Legendrian unknots in $\mathbb{R}^3$ are detectable on the formal level.

The above results have non-trivial corollaries also on the level of $\pi_0(\mathrm{Leg}(N \times \mathbb{R})$.

Theorem

The Whitehead group $\mathrm{Wh}(\pi_1(N))$ injects into $\pi_0(\mathrm{Leg}(N \times \mathbb{R})$ if dim $N \geq 4$.

Recall that $\mathrm{Wh}(\mathbb{Z}/5) = \mathbb{Z}$. Hence, for $N = L(5,1) \times S^2$ the construction yields infinitely many pairwise non-isotopic Legendrian submanifolds.

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The above construction can be implemented for much more general spaces of Legendrian submanifolds.

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THANK YOU!