Cabling knots in overtwisted contact manifolds

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Motivation

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What about knots in them?

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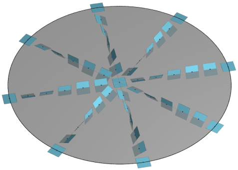

Figure: An overtwisted disk.

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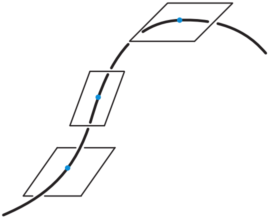

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$$T_pL \subset \xi_p.$$

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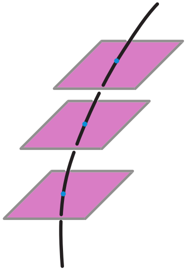

- A knot B is **transverse** if, for each point p in B,

$$T_pB \pitchfork \xi_p.$$

Classical Invariants

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Transverse knots have only one classical invariant

- **self-linking number.**

Legendrian $\leftrightarrow$ Transverse

How are Legendrian and transverse knots related?

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Two Legendrian approximations related by "negative stabilization".

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The only known classification results for non-loose knots are the unknots by Eliashberg-Fraser and a partial classification result for torus knots by Geiges-Onaran which was later extended by Matkovič (for negative torus knots).

Structure theorems of Legendrian knots

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All the results are in tight contact manifolds.

Cabling of Legendrian knots in tight contact manifolds

A (p, q) cable $\mathcal{K}_{p,q}$ of a knot type $\mathcal{K}$, is the isotopy class of a knot of slope $\frac{q}{p}$ on the boundary of a solid torus representing $\mathcal{K}$.

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Etnyre-Honda studied knots in (S^3, ξ_{std}) and showed that the structure theorems for cabled knots are not so simple and rely on the "Uniform thickness property" of the knot.

Cabling of Legendrian knots in tight contact manifolds

Theorem (Etnyre-Honda)

Let $\mathcal{K}$ be a knot type which is Legendrian simple and satisfies the UTP. Then $\mathcal{K}_{p,q}$ is Legendrian simple and admits a classification in terms of the classification of $\mathcal{K}$.

The study has been further extended by Tosun, Etnyre-Tosun-LaFountain and recently by Chakraborty-Etnyre-Min.

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Question: What about cabling knots in overtwisted manifolds?

Cabling of knots in overtwisted manifolds

Cables of loose knots $\rightarrow$ easy!

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Cables of non-loose knots $\rightarrow$ Difficult problem.

Tools and techniques that work for tight manifolds do not necessarily work for overtwisted manifolds.

Positive cables of non-loose knots

Theorem (C-Etnyre-Min-Mukherjee)

Suppose L be a non-loose representative of a knot type $\mathcal{K}$ in (M, ξ). If $\frac{q}{p} > \mathrm{tb}(L)$ for $q, p > 0$, $L_{p,q}$ is non-loose.

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Idea of Proof.

- Realize the cable as a ruling curve on the standard neighborhood of L. Then use convex surface theory. In particular, use state transition technique and bypass attachments.

□

Negative cables

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How to solve this problem?
Use extra conditions.

Theorem (Work in progress, C-Etnyre-Min-Mukherjee)

Suppose L be a non-loose representative of a knot type $\mathcal{K}$ in (M, ξ) such that L has non-loose transverse push off. Then if $\frac{q}{p} > tb(L)$, then $L_{p,q}$ is non-loose in (M, ξ) for $p > 0, q < 0$.

Future work directions

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Question: What about Whitehead double of a non-loose knot? Is it always non-loose?

Thank you for your attention !!!