Cylindrical contact homology of links of simple singularities

Leo Digiosia

Rice University

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Outline

1. Overview of simple singularities and their links as Seifert fiber spaces
2. Computation of cylindrical contact homology using Morse functions invariant under symmetry groups
3. Realization of Seifert fiber structure by the homology groups

Links of simple singularities as contact manifolds

A **simple singularity** of a complex 2-dimensional variety may be locally modeled by $\mathbb{C}^2/G$ for a finite subgroup $G \subset \mathrm{SU}(2)$.

- McKay correspondence $\implies \mathbb{C}^2/G \cong V := V(f)$, for some complex polynomial $f \in \mathbb{C}[x, y, z]$
- The **link** of the singularity, L, is defined as $S^5_\epsilon(0) \cap V$ and has a natural contact structure, ξ_L
- There is a contactomorphism $(L, \xi_L) \cong (S^3/G, \xi)$, where ξ is the descent of the standard contact structure on S^3 under the quotient

The group theory of SU(2) and SO(3)

Consider the double cover of Lie groups $P : \mathrm{SU}(2) \to \mathrm{SO}(3)$. Let $H := P(G)$.

By the classification of finite subgroups of SU(2), G is either

- **Cyclic**, and H is also cyclic,
- **Binary dihedral** $\mathbb{D}_{2n}^*$, and $H = \mathbb{D}_{2n}$, or
- **Binary polyhedral** $\mathbb{T}^*$, $\mathbb{O}^*$, or $\mathbb{I}^*$, and H is $\mathbb{T}$, $\mathbb{O}$, or $\mathbb{I}$.

The G-action on S^3 is fixed point free, unlike the H-action on S^2.

The quotient S^2/H is an **orbifold**, homeomorphic to S^2.

$S^3/G \to S^2/H$ as a Seifert fiber space

$$\begin{array}{ccc} S^3 & \xrightarrow{\pi_G} & S^3/G \\ \downarrow{\mathfrak{P}} & & \downarrow{\mathfrak{p}} \\ S^2 & \xrightarrow{\pi_H} & S^2/H \end{array}$$

- An H-invariant Morse function $f\colon S^2 \to \mathbb{R}$ provides a descent to an **orbifold Morse function** $f_H : S^2/H \to \mathbb{R}$
- For $\varepsilon > 0$ small, define $\lambda_\varepsilon := (1 + \varepsilon \mathfrak{p}^* f_H)\lambda$, the **perturbed contact form**
- The fiber over an orbifold point $p \in S^2/H$ is an **exceptional fiber**, denoted γ_p

The perturbed Reeb field

Lemma

The Reeb vector field of the ***perturbed contact form****,* $\lambda_\varepsilon = (1 + \varepsilon \mathfrak{p}^* f_H)\lambda$, *is given by*

$$R_\varepsilon = \frac{R}{1 + \varepsilon \mathfrak{p}^* f_H} - \varepsilon \frac{\widetilde{X}_f}{(1 + \varepsilon \mathfrak{p}^* f_H)^2}.$$

Here, $\widetilde{X}_f$ is a lift of the Hamiltonian vector field for f_H. We see that the γ_p are embedded Reeb orbits of λ_ε

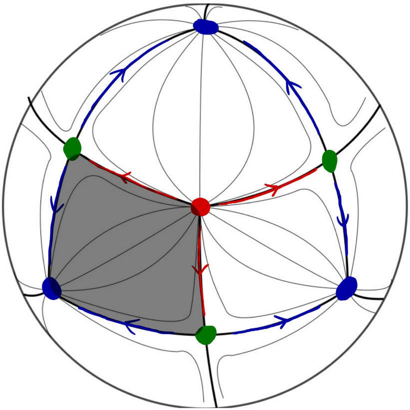
Proposition

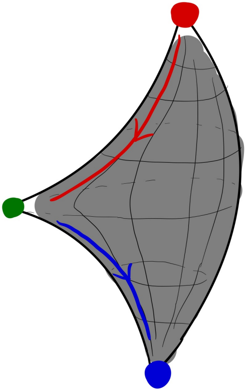
Given $L > 0$, *there exists an* $\varepsilon_0 > 0$ *such that if* $\varepsilon \in (0, \varepsilon_0)$, *then all* $\gamma \in \mathcal{P}^L(\lambda_\varepsilon)$ *are nondegenerate, and project to orbifold critical points of* f_H *under* $\mathfrak{p}$.

We take $L_N \to \infty$, the corresponding $\varepsilon_N \to 0$, and we define $\lambda_N := \lambda_{\varepsilon_N}$, this is L_N-dynamically convex.

Example: $\mathbb{T}$-invariant Morse function on S^2

Let $f : S^2 \to \mathbb{R}$ be a $\mathbb{T}$-invariant Morse function on S^2 with $\mathrm{Crit}(f) = \mathrm{Fix}(\mathbb{T})$, its descent is denoted $f_{\mathbb{T}} : S^2/\mathbb{T} \to \mathbb{R}$.





Graded generators in the tetrahedral setting

We have three embedded Reeb orbits $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ in $S^3/\mathbb{T}^*$ which project to the three orbifold points $\mathfrak{v}$, $\mathfrak{e}$, and $\mathfrak{f}$ of $S^2/\mathbb{T}$.

$\mathrm{CZ} - 1$	Orbits
0	$\mathcal{V}, \mathcal{V}^2, \mathcal{V}^3, \mathcal{E}, \mathcal{F}, \mathcal{F}^2$
1	$\mathcal{E}^2$
2	$\mathcal{V}^4, \mathcal{V}^5, \mathcal{V}^6, \mathcal{E}^3, \mathcal{F}^3, \mathcal{F}^4, \mathcal{F}^5$
3	$\mathcal{E}^4$
4	$\mathcal{V}^7, \mathcal{V}^8, \mathcal{V}^9, \mathcal{E}^5, \mathcal{F}^6, \mathcal{F}^7, \mathcal{F}^8$
$\vdots$	$\vdots$

These orbits and their gradings are used to compute the **action filtered cylindrical contact homology** of $S^3/\mathbb{T}^*$.

The cylindrical contact homology

Theorem (D, 2021)

Let $G \subset \mathrm{SU}(2)$ *be a finite nontrivial group, and let* $m \in \mathbb{N}$ *denote* $|\mathrm{Conj}(G)|$, *the number of conjugacy classes of* G. *Then*

$$\varinjlim_{N} CH_*^{L_N}(S^3/G, \lambda_N, J_N) \cong \bigoplus_{i\geq 0} \mathbb{Q}^{m-2}[2i] \oplus \bigoplus_{i\geq 0} H_*(S^2)[2i].$$

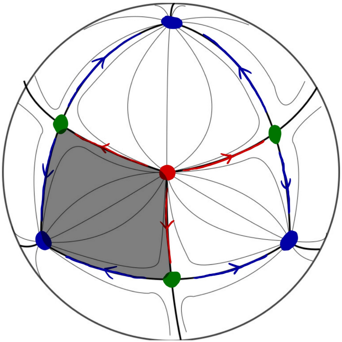
- Here, $L_N \to \infty$, λ_N is L_N-dynamically convex, J_N is a generic λ_N-compatible almost complex structure.
- Copies of $H_*(S^2)$ are to be interpreted as the **orbifold Morse homology** of the orbit space, which is an orbifold S^2.
- The dimension of the $\mathbb{Q}^{m-2}$ term is the total isotropy of the orbit space.

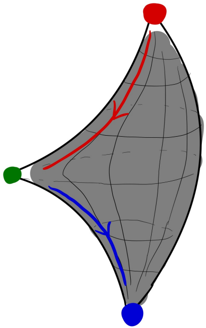
Tetrahedral example

Theorem (D, 2021)

$$\varinjlim_{N} CH_*^{L_N}(S^3/\mathbb{T}^*, \lambda_N, J_N) \cong \bigoplus_{i\geq 0} \mathbb{Q}^5[2i] \oplus \bigoplus_{i\geq 0} H_*(S^2)[2i].$$

$G = \mathbb{T}^*,\qquad m = 7,\qquad m - 2 = 5$





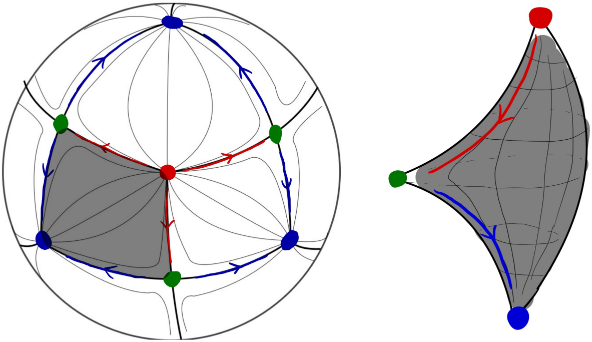
Realizing a contact McKay correspondence

- The McKay correspondence provides that $|H^*(Y_G;\mathbb{Q})| = m$, where Y_G is the minimal resolution of the singularity $\mathbb{C}^2/G$.
- McLean and Ritter generalize this result for $G \subset \mathrm{SU}(n)$, Y_G a **crepant** resolution of $\mathbb{C}^n/G$.
- They prove this by relating the rank of $SH^*_+(Y_G)$ to the number of conjugacy classes of G.

Consider that our direct limit may be rewritten as

$$\varinjlim_N CH_*^{L_N}(S^3/G, \lambda_N, J_N) \cong \begin{cases} \mathbb{Q}^{m-1} & * = 0, \\ \mathbb{Q}^m & * \geq 2 \text{ and even}, \\ 0 & \text{else}. \end{cases}$$

Thank you for your time.



Comparing differentials

Definition (Eliashberg, Givental, Hofer)

Suppose $\mathcal{M}_1^J(\gamma_+, \gamma_-)/\mathbb{R}$ is a finite set for all $\gamma_\pm \in \mathcal{P}_{\text{good}}(\lambda)$.
Define $\partial : CC_*(Y, \lambda, J) \to CC_{*-1}(Y, \lambda, J)$ by

$$\langle \partial \gamma_+, \gamma_- \rangle := \sum_{u \in \mathcal{M}_1^J(\gamma_+, \gamma_-)/\mathbb{R}} \epsilon(u) \frac{m(\gamma_+)}{m(u)}$$

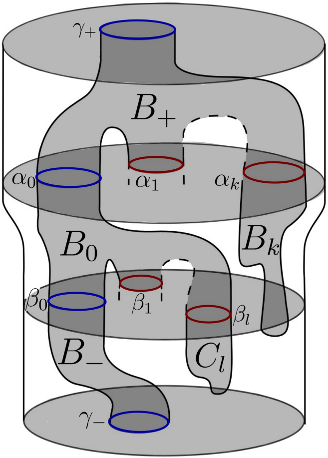
- Here, $\epsilon(u) \in \{\pm 1\}$ is determined by a coherent choice of orientations, defined for cylinders between good Reeb orbits,
- $m(\gamma_+)$ and $m(u)$ are the covering multiplicities of γ_+ and u.

Definition (Cho, Hong)

The differential ∂^{orb} of orbifold Morse homology is

$$\langle \partial^{\text{orb}} p, q \rangle := \sum_{x \in \mathcal{M}(p,q)} \epsilon(x) \frac{|\Gamma_p|}{|\Gamma_x|}$$

Technical considerations (1)



- Direct limits are over homomorphisms between filtered groups induced by **symplectic cobordisms**.
- The Fredholm index 0 cylinders under consideration are regular, by **automatic transversality**.
- **Compactness** of the moduli spaces holds due to strong relationships between action, CZ index, and free homotopy classes of Reeb orbits.

Technical considerations (2)

Proposition (D, 2021)

Suppose γ_+ and γ_- represent the same free homotopy class, where $\gamma_+ \in \mathcal{P}^{L_N}(\lambda_N)$ and $\gamma_- \in \mathcal{P}^{L_M}(\lambda_M)$.

1. *If $\mu_{CZ}(\gamma_+) = \mu_{CZ}(\gamma_-)$, then $m(\gamma_+) = m(\gamma_-)$ and $\gamma_\pm$ project to the same orbifold point of S^2/H.*
2. *If $\mu_{CZ}(\gamma_+) < \mu_{CZ}(\gamma_-)$, then $\mathcal{A}(\gamma_+) < \mathcal{A}(\gamma_-)$.*

Importantly,

1. The first bullet indicates that many of the moduli spaces $\mathcal{M}_0^J(\gamma_+, \gamma_-)$ are **empty**.
2. The second bullet indicates that **there do not exist cylinders of negative Fredholm index** in our cobordisms.