Reeb flows transverse to foliations

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foliations	contact structures
integrable plane field	nowhere integrable plane field
taut	tight
Reeb component	overtwisted disk/Lutz tube

$\exists$ transverse volume preserving flow

M oriented 3-manifold
F a C^2 co-oriented codimension 1 foliation
$M \not\cong S^1 \times S^2$

Theorem (Eliashberg—Thurston): The tangent plane field to F admits C^0 small perturbations to positive and negative contact structures ξ_+, ξ_-. When the foliation is taut, the contact structures are tight, and in fact weakly symplectically fillable.

Why care?
- allows to export genus detection results from foliation theory to Floer theory
- an abundant source of tight contact structures

Question (Colin-Honda): Can you make the Reeb flow of $\xi_\pm$ transverse to F?

Answer (Z): Yes, if and only if F supports no invariant transverse measure.

Corollary: Every orientable 3-manifold with a taut, C^2 foliation has a *hypertight* contact structure (i.e. the Reeb flow has no contractible orbits).

Proof of corollary:

Case 1 ($H_2(M, R) \neq 0$): due to Colin and Honda
Case 2 ($H_2(M, R) = 0$): No invariant transverse measures in this case, so by the main theorem, the approximating contact structure has Reeb flow transverse to the foliation. Loops transverse to a taut foliation are not contractible.

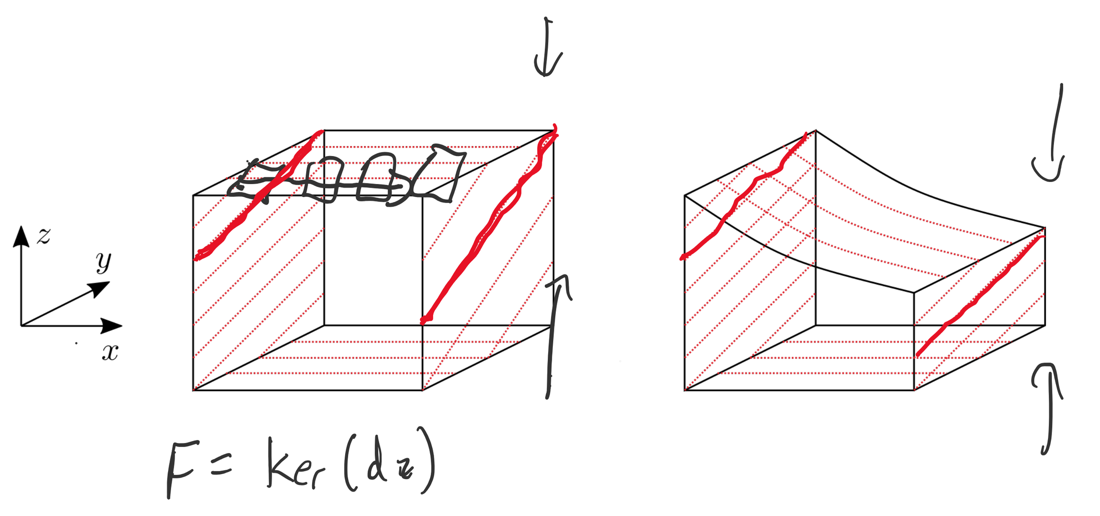
Corollary: Cylindrical contact homology is well defined for $\xi_\pm$.

Corollary: Cylindrical contact homology is an invariant of the taut deformation class of C^2 taut foliations.

Ulterior motive: Explain some tools (harmonic transverse measure) for understanding the holonomy of foliations

Contact perturbations require holonomy

$\Sigma_g \times S^1$



$F = \ker(dz)$

$\xi = \ker(dz + \varepsilon x\, dy)$

Proof sketch of Eliashberg-Thurston theorem:

1. Find sufficiently many closed curves in leaves of F along which F has contracting holonomy
2. Use a model perturbation in neighborhoods of these curves
3. "Spread out" the contactness to the rest of the 3-manifold

Issue: lose control of Reeb flow during step 3

Transverse measures and holonomy

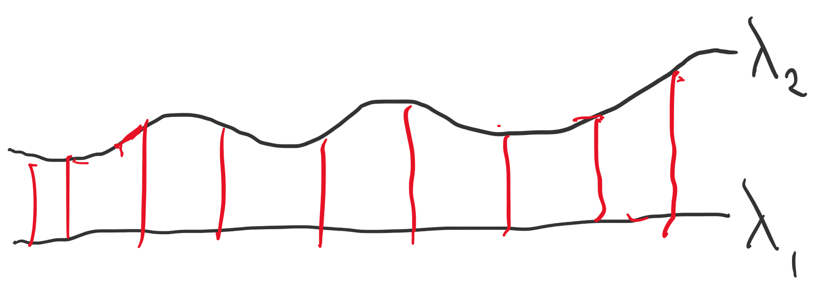
Definition: A **transverse measure** τ is an assignment of a signed length to piecewise smooth arcs in M satisfying

- nonnegative for arcs positively transverse to F
- zero on arcs in leaves of F
- $\tau(\gamma) = -\tau(-\gamma)$
- countably additive wrt concatenation
- continuous wrt C^0 perturbations

Example 1: A 1-form τ (not necessarily closed) with $ker(\tau) = TF$.
Example 2: Given a compact leaf S, can define a hitting measure

$\tau(\gamma) = |\gamma \cap S|$

A transverse measure may or may not have **full support.**
A transverse measure allows us to measure the distance between nearby leaves



Definition: A transverse measure is called **invariant** if the distance to a nearby leaf is locally constant.
Definition: A transverse measure is called **harmonic** if the distance to a nearby leaf is locally a harmonic function (requires a metric on the leaf)
note: only well defined on $\tilde{\lambda}_1 \subset \tilde{M}$.

Lemma: An invariant transverse measure is an obstruction to a transverse Reeb flow.

Proof: Stokes' theorem

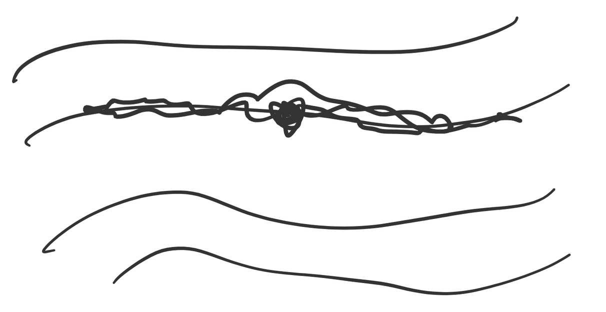
Utility of harmonic transverse measures

- **Leaf pocket theorem (Thurston)**: Holonomy is asymptotically contracting along almost every direction in a leaf. (use the fact that a positive harmonic function on H^2 is bounded in almost every direction).
- **Lemma:** A smooth harmonic transverse measure of full support gives rise to a linear deformation to a contact structure, with Reeb flow transverse to the foliation

  Rough idea: Rotate tangent planes about axis given by direction of contracting holonomy.

Harmonic transverse measures via leafwise Brownian motion

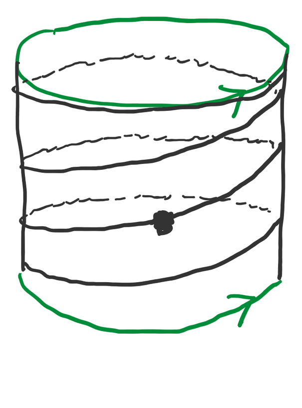
Choose a leafwise Riemannian metric.
Given a measure on M, let it evolve by **leafwise Brownian motion**.



Theorem (Garnett): A stationary measure exists. Divide by the leafwise area form to get a harmonic transverse measure.

Caution: the harmonic transverse measure might not have full support!

Example: 2d cylinder



Mass accumulates on green circles!

A relaxation: log superharmonic transverse measures

f is log superharmonic if $\Delta \log(f) \leq 0$

Examples: Gaussian, harmonic functions

For the purposes of producing contact structures, log superharmonic is as good as harmonic.

Theorem (Z): If F supports no invariant transverse measure, then F has a smooth, log superharmonic transverse measure of full support.

Proof sketch:

1. Define a new diffusion operator D^T acting on transverse measures

2. Show that $D^T\tau$ is log superharmonic for large enough T (Bootstrap results of Deroin and Kleptsyn on standard leafwise Brownian motion)

Future directions

- understand the dynamics of the Reeb flow (e.g. is the flow product covered?)
- growth rate of cylindrical contact homology
- higher dimensional analogues?

Thank you for listening!