Super-rigidity and bifurcations of embedded curves in Calabi-Yau 3-folds

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Overview

1. Background
   - Embedded curves and super-rigidity
   - Wendl's Theorem
   - BPS invariants and Gopakumar–Vafa formula
2. Results
   - Bifurcations
   - Obstruction bundles
   - An application
3. Further directions

Embedded curves and super-rigidity (I)

- Fix a closed symplectic Calabi–Yau 3-fold (X, ω), i.e., $\dim X = 6$ and $c_1(TX, \omega) = 0$.

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  $\rightsquigarrow$ Gromov–Witten invariant $\mathrm{GW}_{A,g} \in \mathbb{Q}$, independent of J.
- Because of multiple covers, these invariants are not $\mathbb{Z}$-valued and don't directly enumerate curves.

Embedded curves and super-rigidity (II)

Fact

Away from a codimension 2 subset of $\mathcal{J}(X,\omega)$, all **simple** *holomorphic curves are* **embedded** *and have* **pairwise disjoint images***.*

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Fact

Away from a codimension 2 subset of $\mathcal{J}(X,\omega)$, all **simple** *holomorphic curves are* **embedded** *and have* **pairwise disjoint images**.

- Restrict attention to J as in the above fact.
- Any non-constant J-holomorphic stable map $f' : \Sigma' \to X$ can then be factored uniquely as

$$\Sigma' \xrightarrow{\varphi} \Sigma \xrightarrow{f} X$$

where Σ is a smooth closed Riemann surface, f is a J-holomorphic embeddeing and φ is holomorphic.

Embedded curves and super-rigidity (III)

Definition (Super-rigidity)

$J \in \mathcal{J}(X, \omega)$ is called **super-rigid** if, for all stable J-holomorphic maps

$$\Sigma' \xrightarrow{\varphi} \Sigma \subset X$$

we have $\ker(\varphi^* D^N_{\Sigma,J}) = 0$, where $D^N_{\Sigma,J}$ is the **normal Cauchy–Riemann operator** of the embedded J-curve $\Sigma \subset X$.

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- If J is super-rigid, then given any sequence of embedded J_n-curves $\Sigma_n \subset X$ (of bounded genus and area), with $J_n \to J$, we can find a subsequence converging to an embedded J-curve $\Sigma \subset X$.

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- This allows us to separate embedded curves from multiple covers!

Wendl's Theorem

Theorem (Wendl 2019, arXiv:1609.09867)

The subset of $\mathcal{J}(X,\omega)$ *where super-rigidity fails has codimension* ≥ 1*. In particular, the generic* J *is super-rigid.*

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- This provides a strategy to define $\mathbb{Z}$-valued counts of embedded curves using super-rigid J.
- To show symplectic invariance, we must investigate what happens when we cross the codimension 1 strata ("walls").

BPS invariants and GV formula (I)

Conjecture (Gopakumar–Vafa '98)

There exist integers $BPS_{A,h}$ for all $h \geq 0$ and $A \in H_2(X, \mathbb{Z})$ satisfying the following identity

$$\sum_{A \neq 0, g \geq 0} GW_{A,g} t^{2g-2} q^A = \sum_{A \neq 0, h \geq 0} BPS_{A,h} \sum_{k=1}^{\infty} \frac{1}{k} \left(2 \sin \left(\frac{kt}{2} \right) \right)^{2h-2} q^{kA}$$

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Theorem (Ionel–Parker, 2018)

There exist integers $BPS_{A,h}$ for $h \geq 0$ and $A \in H_2(X,\mathbb{Z})$ satisfying the Gopakumar–Vafa formula.

BPS invariants and GV formula (II)

- Recently, Doan–Ionel–Walpuski (arXiv:2103.08221) have also shown that for any $A \in H_2(X, \mathbb{Z})$, we have $\mathrm{BPS}_{A,h} = 0$ for $h \gg 0$.

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- However, neither of these proofs show how to interpret the integers $\mathrm{BPS}_{A,h}$ enumeratively.

Motivating question

How to define $\mathbb{Z}$-valued symplectic invariants by counting embedded curves? How are these counts related to the BPS invariants?

Results

- Our recent paper (arXiv:2106.01206) addresses parts of this question.
- For the first question, we study the bifurcations in the space of embedded curves which occur when we cross one of the walls from Wendl's theorem.
- For the second question, we study how the (Euler numbers of) obstruction bundles change under some simple bifurcations.

Bifurcations (I)

Theorem A (Bai–S., 2021)

Let $\{J_t\}_{t\in[-1,1]}$ be a generic path in $\mathcal{J}(X,\omega)$. Assume that there exists an embedded rigid J_0-curve $\Sigma \subset X$ along with a d-fold genus h branched multiple cover $\varphi : \Sigma' \to \Sigma$ which has non-trivial normal deformations. If this cover determines an **elementary wall type**, then $\mathrm{Aut}(\varphi) \subset \mathbb{Z}/2\mathbb{Z}$ and the change in the signed count of embedded curves of genus h and class $d[\Sigma]$ near φ is given by $\pm 2/|\mathrm{Aut}(\varphi)|$.

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- The technical condition of "elementary wall type" is satisfied by a large class of branched covers. For example, this includes all d-fold covers $\Sigma' \to \Sigma$ with generalized automorphism group S_d.

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- For the proof, we study the local structure near $(J_0, \varphi : \Sigma' \to \Sigma \subset X)$ of the moduli space

$$\overline{\mathcal{M}}_h(X, \{J_t\}, dA)$$

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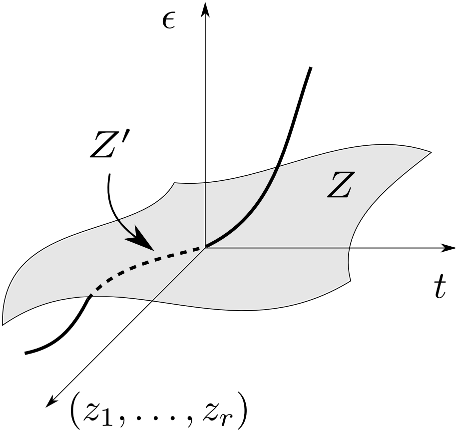
where $A = [\Sigma] \in H_2(X, \mathbb{Z})$.
- We obtain a local Kuranishi model by applying the implicit function theorem. We then analyze the first few terms in the Taylor expansion of the Kuranishi map to complete the proof.

Schematic picture of the local Kuranishi model



$(z_1, \dots, z_r)$ are coordinates on $T_\varphi \mathcal{M}_h(\Sigma, d)$, ϵ is a coordinate $\ker(\varphi^* D^N_{\Sigma,J})$ and Z, Z' are the local irreducible components of the moduli space.

Obstruction bundles (I)

Fix a compact Riemann surface Σ of genus g and a $\mathbb{C}$-vector bundle $N \to \Sigma$ of rank 2 with $\deg(N) = 2g - 2$.

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Definition

A Cauchy–Riemann operator D on N is said to be **super-rigid**, if $\ker(\varphi^* D) = 0$ for all (possibly branched) holomorphic covers $\varphi : \Sigma' \to \Sigma$. For super-rigid D and integers $d \geq 2$ and $h \geq 0$, we define the (canonically oriented) **cokernel bundle** $\mathcal{N}_{\Sigma,D}^{(d,h)} \to \overline{\mathcal{M}}_h(\Sigma, d)$ by

$$[\varphi : \Sigma' \to \Sigma] \mapsto \mathrm{coker}(\varphi^* D).$$

Since, $\mathrm{vdim}\ \overline{\mathcal{M}}_h(\Sigma, d) = \mathrm{rank}\ \mathcal{N}_{\Sigma,D}^{(d,h)}$, this bundle has a well-defined **virtual Euler number** $e_{d,h}(D) \in \mathbb{Q}$.

Obstruction bundles (II)

Theorem B (Bai–S., 2021)

Let $\mathcal{D} = \{D_t\}_{t\in[-1,1]}$ be a generic 1-parameter family of Cauchy–Riemann operators on N. Assume that $([\varphi : \Sigma' \to \Sigma], t) \mapsto \mathrm{coker}(\varphi^* D_t)$ gives a vector bundle of the expected rank on the space

$$\overline{\mathcal{M}}_h(\Sigma, d) \times [-1, 1] \setminus \Delta \times \{0\}$$

where $\Delta \subset \mathcal{M}_h(\Sigma, d)$ is a finite set where super-rigidity fails for D_0. Then,

$$e_{d,h}(D_+) - e_{d,h}(D_-) = \sum_{p\in\Delta} \frac{2 \cdot \mathrm{sgn}(\mathcal{D}, p)}{|\mathrm{Aut}(p)|}$$

with $\mathrm{sgn}(\mathcal{D}, p) \in \{-1, +1\}$ determined by the behavior of $\mathcal{D}$ near p.

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The proof is by local finite dimensional reduction to a model case.

An application

Theorem C (Bai–S., 2021)

Given any primitive homology class $A \in H_2(X, \mathbb{Z})$, the number $\mathrm{BPS}_{2A,0}(X) \in \mathbb{Z}$ is a weighted count of embedded J-holomorphic genus 0 curves (of classes $2A$ and A) when J is super-rigid.

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- To define $\mathrm{BPS}_{2A,0}$ directly, we count the $2A$ curves Σ' with the usual signs, while we count any A curves Σ with a weight which counts the signed number of wall crossings along generic path from $D^N_{\Sigma,J}$ to the standard Cauchy–Riemann operator $\bar{\partial}$ on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

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- Symplectic invariance of this definition follows from Theorem A.
- The verification of the GV formula uses Theorem B and the standard computation of $e_{2,0}(\bar{\partial})$ for $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

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- non-minimal covers,
- branched covers where not all branch points are distinct,
- nodal covers (possibly with ghost components).

We hope to address this in future work.

Thank you!