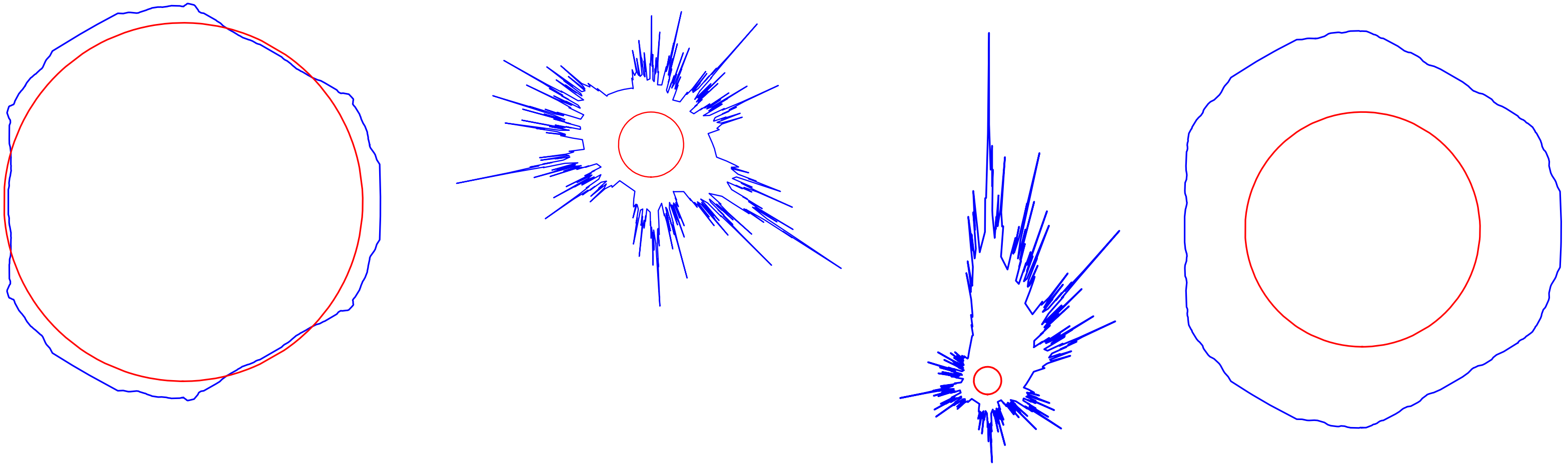
Degenerations of Kähler forms on K3 surfaces, and some dynamics



Simion Filip, University of Chicago

joint with Valentino Tosatti

Setup

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K3 surfaces

X algebraic surface with algebraic symplectic 2-form Ω

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simply connected (to exclude tori)

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work over $\mathbb{C}$

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$\Omega \in H^{2,0}$

simply connected

work over $\mathbb{C}$

$NS(X) = H^2(X;\mathbb{Z}) \cap H^{1,1}(X)$

$N = \mathrm{NS}(X)$ Néron—Severi group

rk $22/\mathbb{Z}$

$\rho = \mathrm{rk} N$

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Standing assumptions (simplifying):

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$N = \mathrm{NS}(X)$ Néron—Severi group

cup-product on NS
signature $(1, \rho - 1)$

$\rho = \mathrm{rk} N$

Standing assumptions (simplifying):

$\rho \geq 3$

$\mathrm{Aut}(X) \to \mathrm{SO}(N) \simeq \mathrm{SO}_{1,\rho-1}(\mathbb{R})$ gives a lattice

(cf. Cone Conjecture in higher dimensions)

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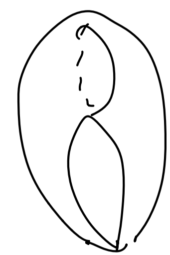

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Singular fibers of elliptic fibrations are reduced and irreducible

Example

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K3 surface

$$X\ :(1+x^2)(1+y^2)(1+z^2)-5xyz=1 \text{ in } \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$$

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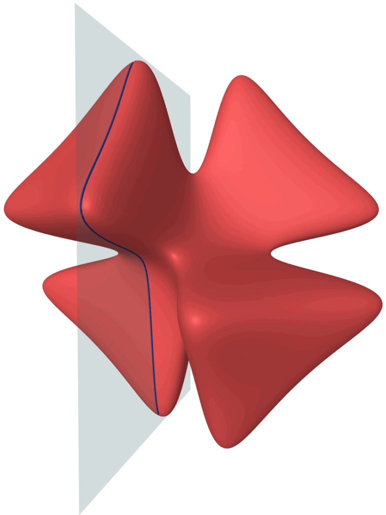

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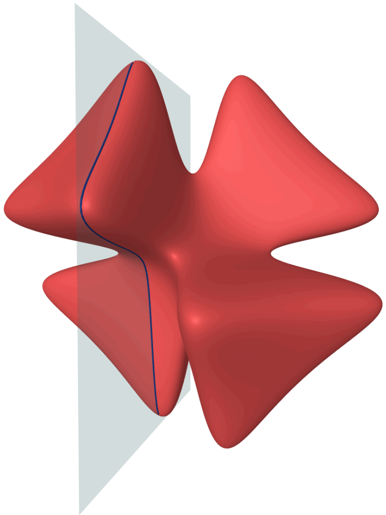

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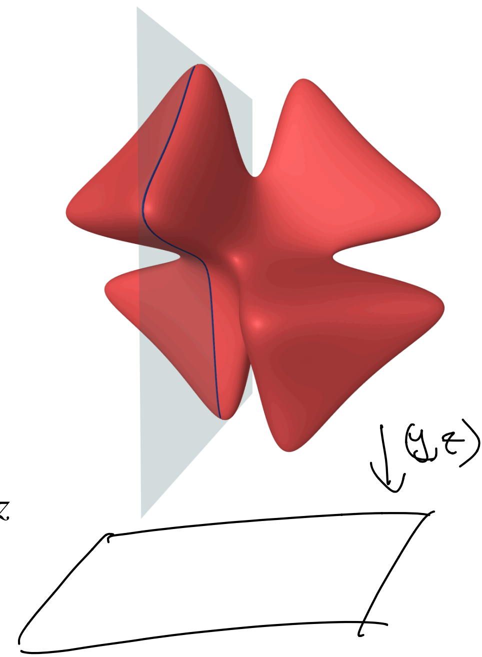
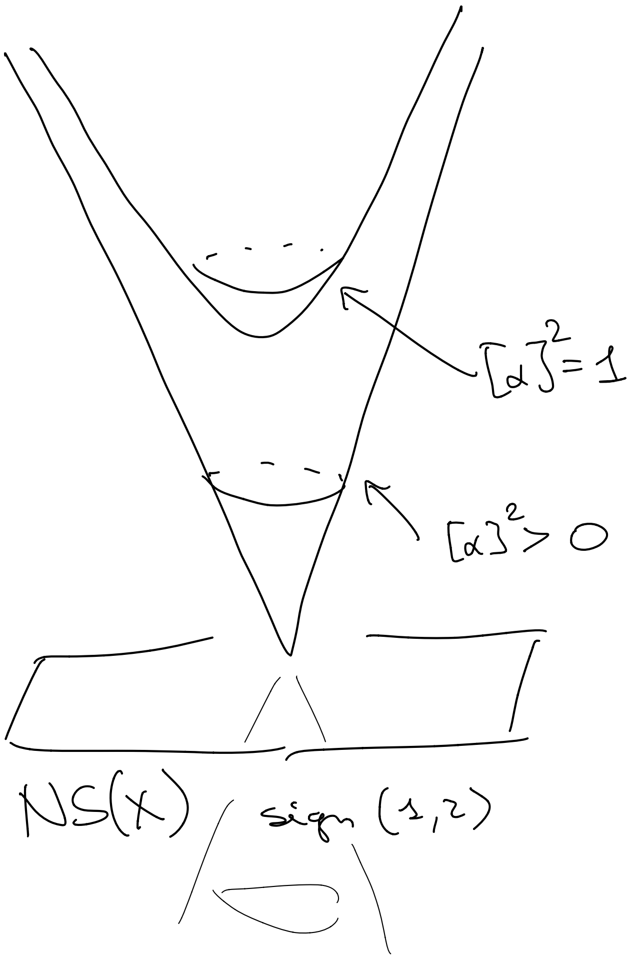
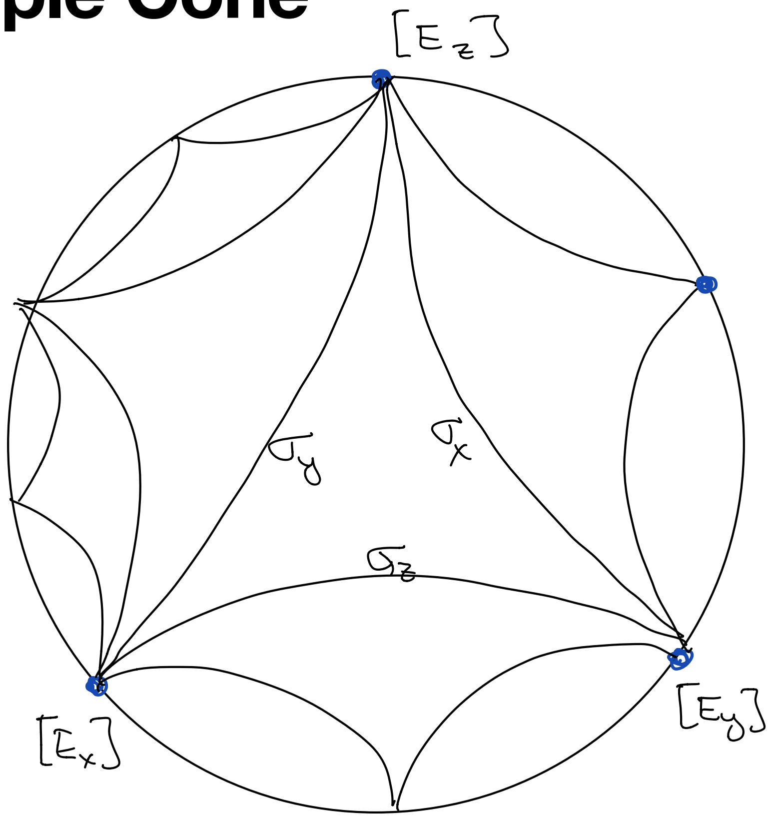


$$\sigma_x \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{5yz}{(1+y^2)(1+z^2)} - x \\ y \\ z \end{bmatrix}$$ and similarly σ_y, σ_z

Ample Cone





$$\partial\,\mathrm{Amp}(X) \leftarrow \partial^{\circ}\,\mathrm{Amp}_c(X)$$ (TBE)

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locally

Note: Ricci-flat $\iff$ $\omega \wedge \omega = \Omega \wedge \overline{\Omega}$ $\checkmark$ $f\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$

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K3s are examples of *hyperkähler* manifolds:

ω, Re Ω, Im Ω are Kähler forms for complex structures

$I, J, K \quad I^2 = J^2 = K^2 = -1, \quad IJ = K$ etc.

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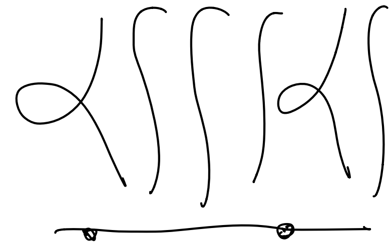
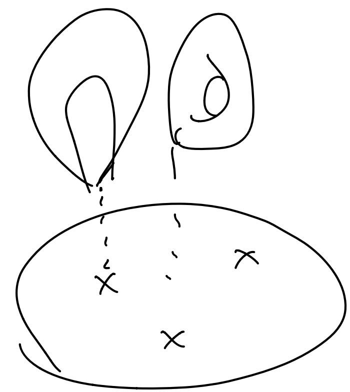
$X \xrightarrow{\pi} B \simeq \mathbb{P}^1(\mathbb{C})$ fibers are elliptic curves





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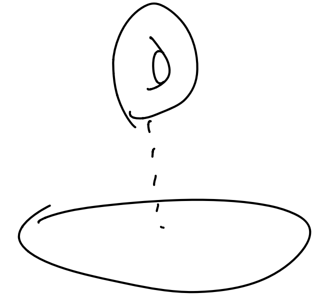

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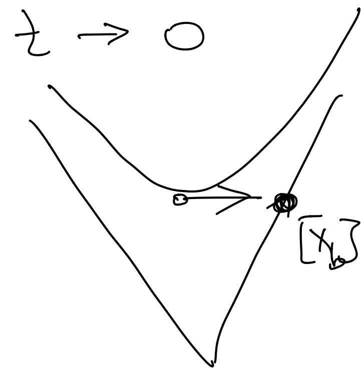
ω_t a Ricci-flat Kähler form on X satisfying:

$$[\omega_t] = t[\omega_0] + [X_b]$$



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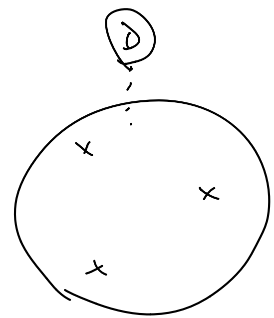

Picture predicted by SYZ view of mirror symmetry

Metric Properties: Gross, Wilson, Tosatti, many others

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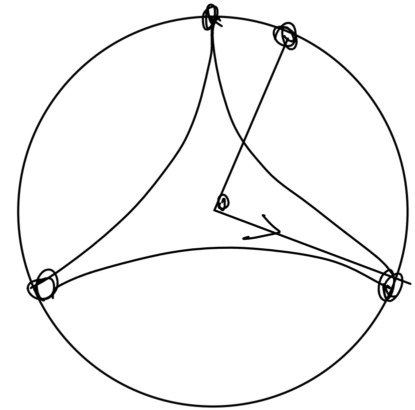

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What about all other points on the boundary?

Main Theorems

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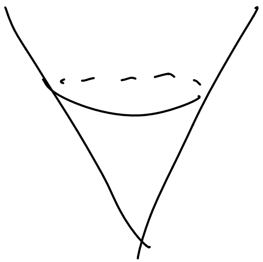


Theorem C $\exists\,!\eta : \partial^\circ \operatorname{Amp}_c(X) \to \mathcal{Z}^{pos}_{1,1}(X)$

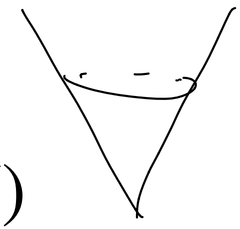
currents =

diff'l. fms.

~~w/ distrib. coeff~~

msr. coeff.

Main Theorems



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- currents have continuous potentials

(locally on X
$\eta([\alpha]) = dd^c \varphi$

Main Theorems

$\sum_{i \in I} T_i$

Theorem C $\exists ! \eta : \partial^\circ \operatorname{Amp}_c(X) \to \mathcal{Z}^{pos}_{1,1}(X)$

$T_i = \int [D(z)] \, d\mu(z)$

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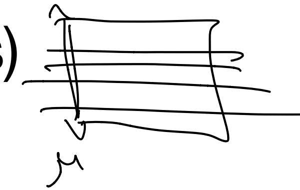
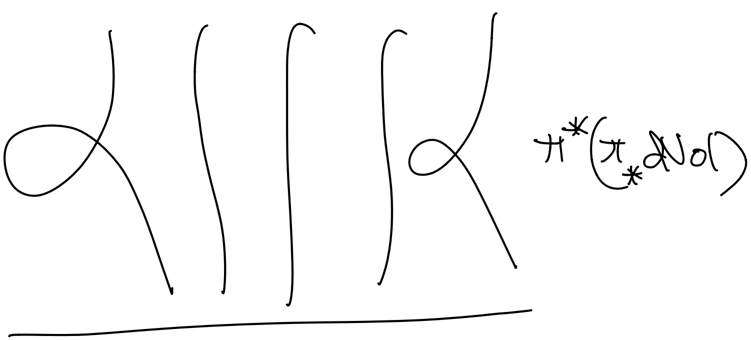
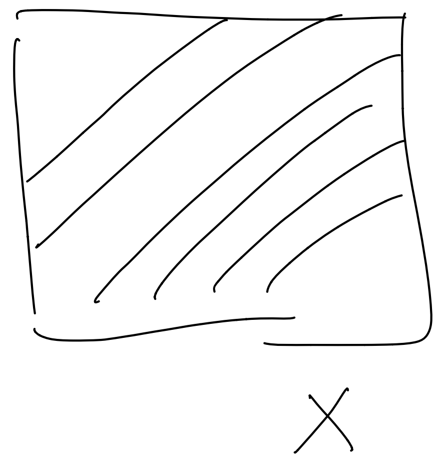
- Unique positive representatives in irrational classes (cf. Verbitsky-Sibony)







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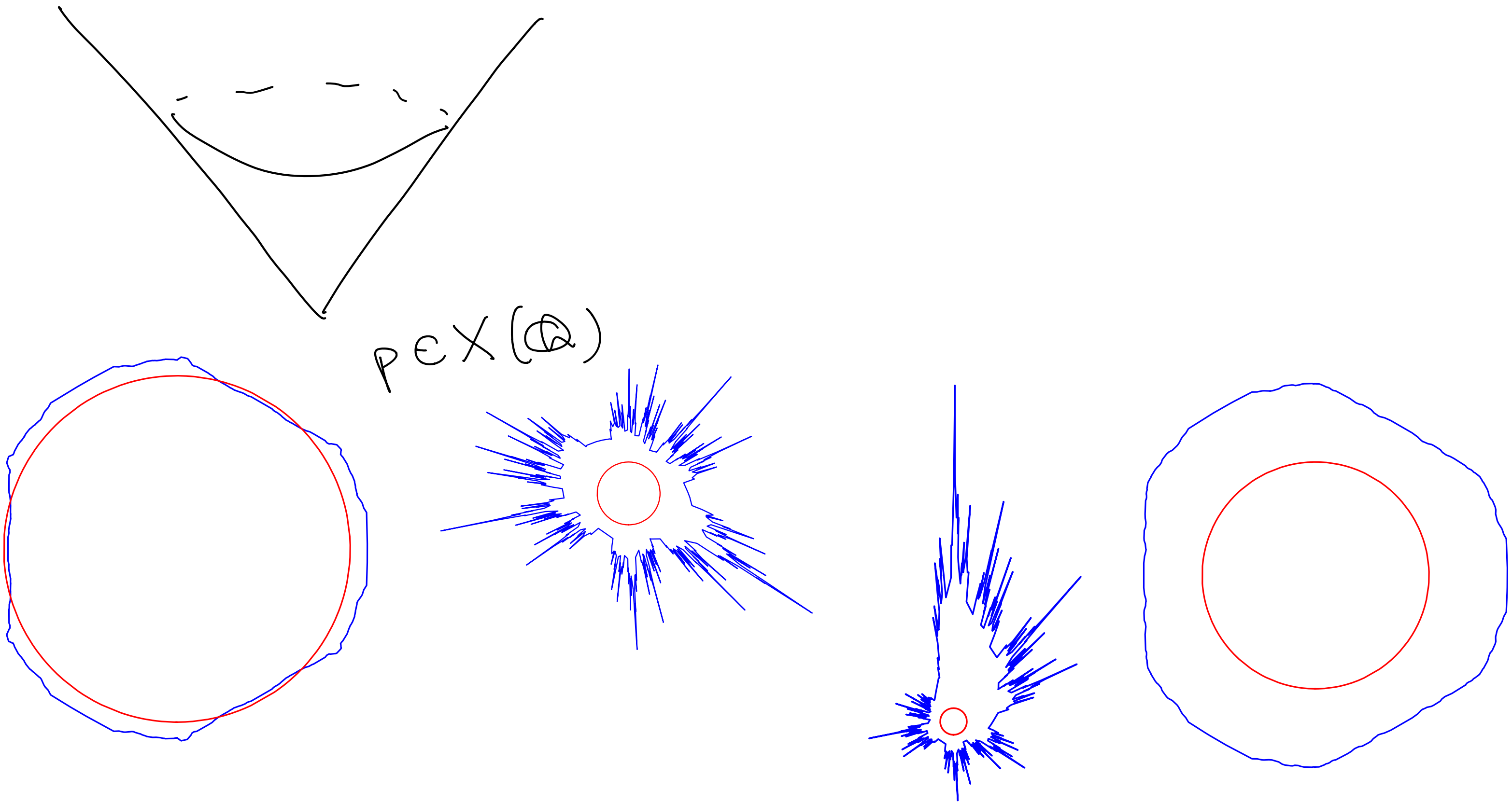
Theorem H $\exists ! h^{can} \colon \partial^\circ \operatorname{Amp}_c(X) \to \mathscr{H}eights(X)$

- equivariant
- $\forall p \in X(\overline{\mathbb{Q}})$ the function $h^{can}_\alpha(p)$ is continuous in α
- …



Projection of region where $h_\alpha^{can}(p) = 1$

Twists in Elliptic Fibrations

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$X \xrightarrow{\pi} B$ elliptic fibration (under standing assumptions)

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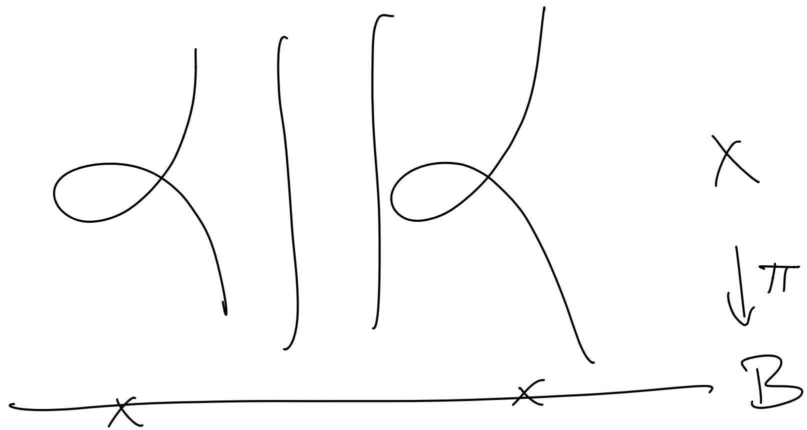

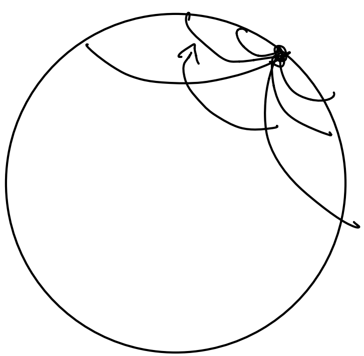

$\rho = 3$

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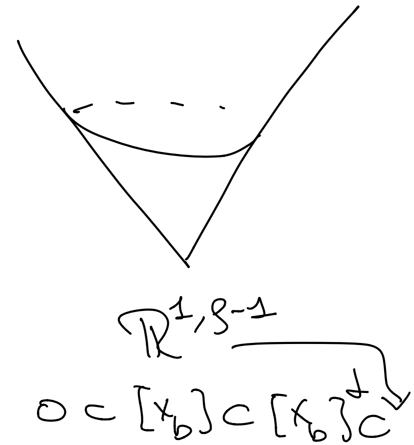

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$Q : \mathbb{Z}^{\rho-2} \to \mathcal{I}_{1,1}^{pos}$

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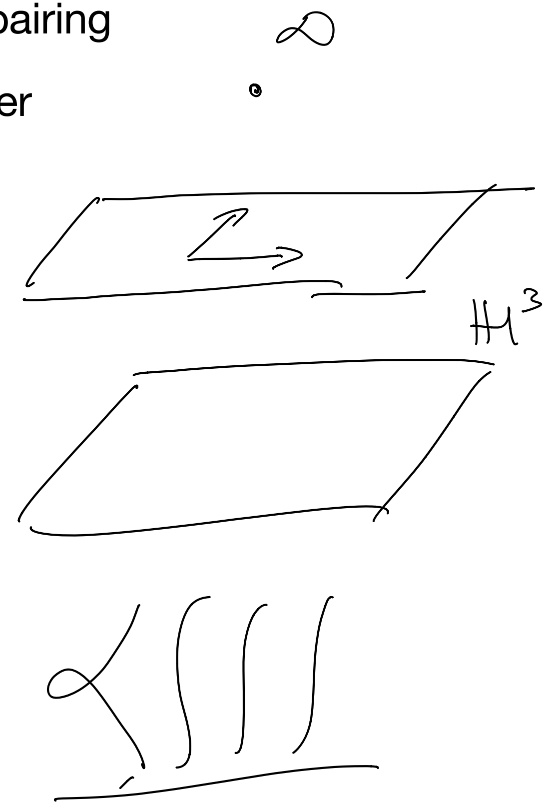

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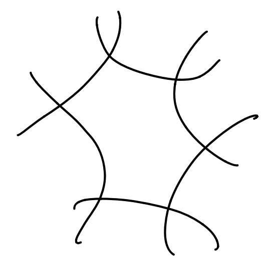

$\exists$**continuous** ϕ s.t. $\omega + dd^c\phi\,|_{X_b} \equiv 0 \quad \forall b \in B$

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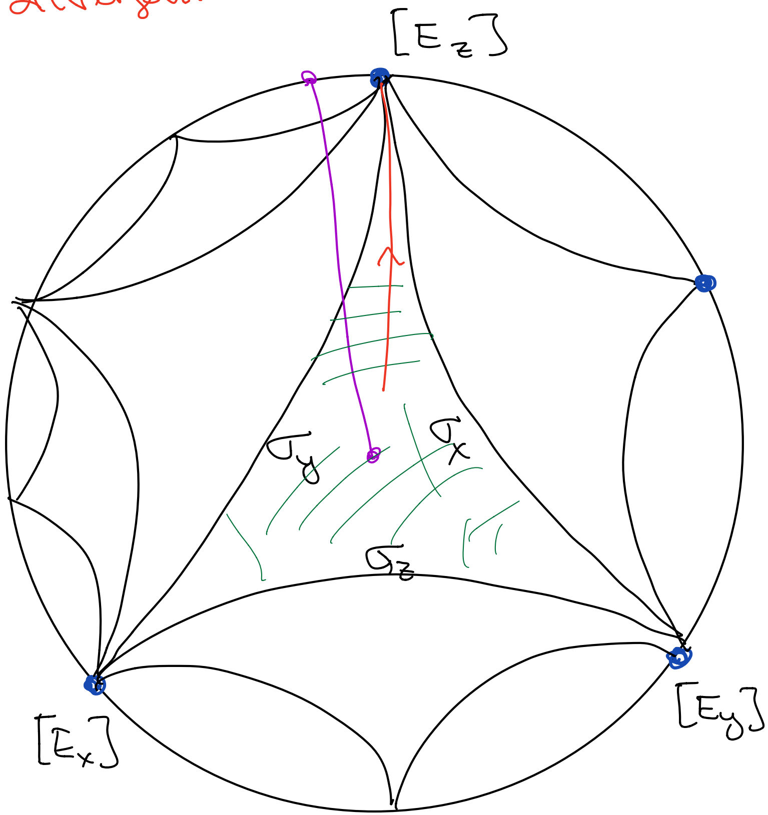

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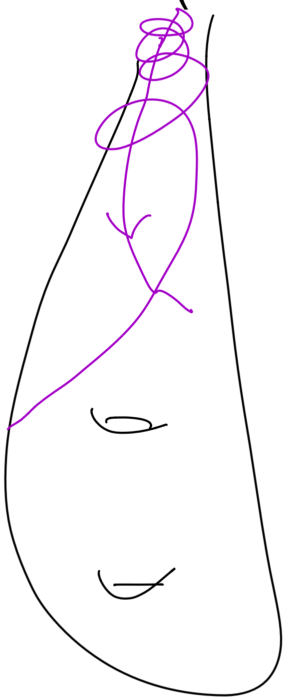

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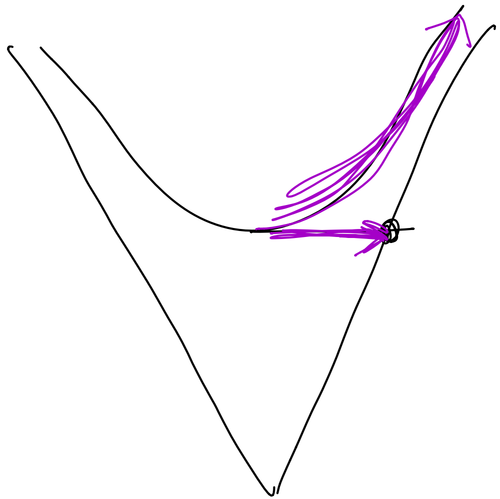

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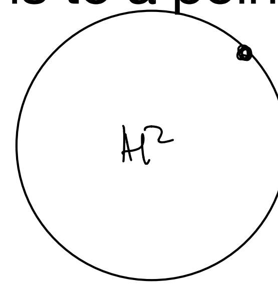

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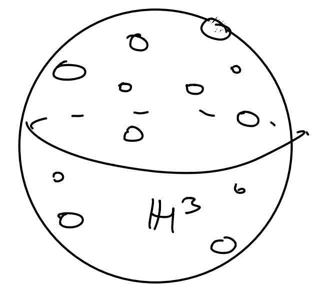

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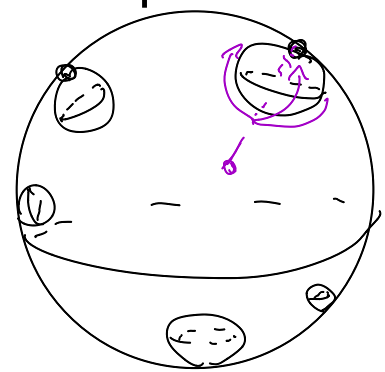

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- Blow up $\partial \mathrm{Amp}(X)$ at the rational rays. i.e. add in $\mathbb{P}\left([X_b]^{\perp}/[X_b]\right)$
- Alternatively, $\text{C}_{\text{artan}}\text{A}_{\text{lexandrov}}\text{T}_{\text{oponogov}}(0)$-compactification of hyperbolic space with horospheres removed (Hadamard)

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$$\prod_{v \in \Sigma_{\mathbb{Q}}} |a|_v = 1 \quad \forall a \in \mathbb{Q}^\times$$

$$h(a) := \sum_{v \in \Sigma_{\mathbb{Q}}} h_v(a) \qquad \Sigma_{\mathbb{Q}} = \text{primes} \cup \infty$$

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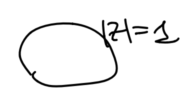
$$h(a) := \sum_{v \in \Sigma_{\mathbb{Q}}} h_v(a) \qquad \Sigma_{\mathbb{Q}} = \text{primes} \cup \infty$$

$$h_v(a) := \log \max(|p|_v, |q|_v)$$

Any "compatible" system of metrics on $\mathcal{O}_{\mathbf{P}^1}(-1)$ works.

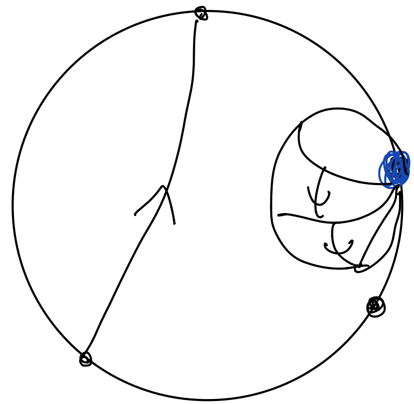
Recall: What is $dd^c \log \max(|z_1|, |z_2|)$?

$dd^c \log \max(|z|, 1)$ on $\mathbb{C}$

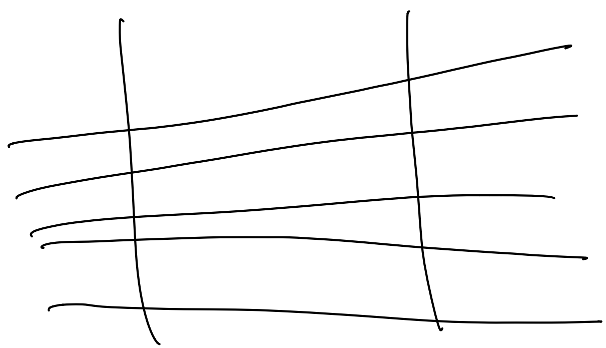


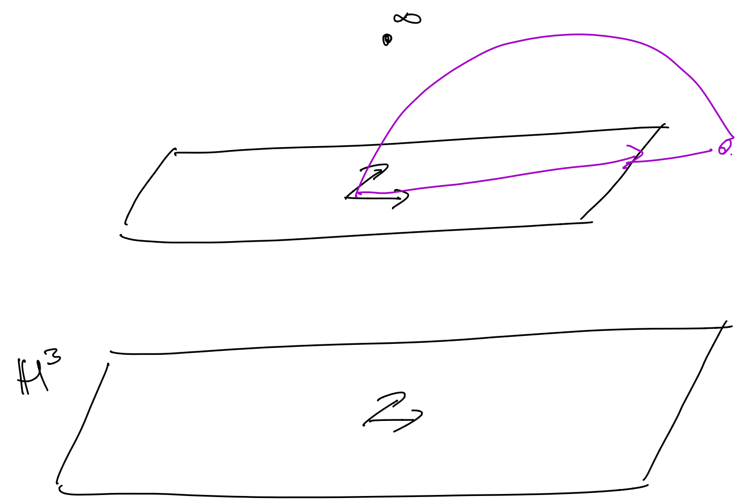
$X \xrightarrow{f} X$ holom

$\Rightarrow \quad \eta_\pm \quad f^* \eta_\pm = e^{\pm\lambda} \eta_\pm$



Thank you!



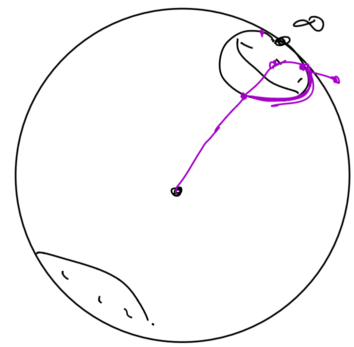


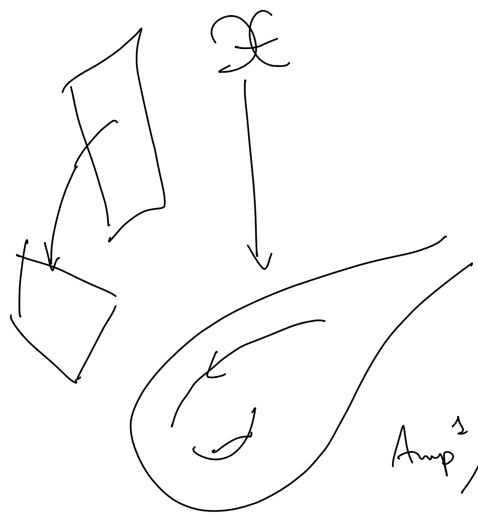
$\mathrm{Isom}(\mathbb{H}^3, \infty) \simeq \mathbb{R}^2$

$\mathrm{Aut}_\pi(X) \simeq \mathbb{Z}^2 \subset \mathbb{R}^2$

$\overset{\shortparallel}{\langle T_1, T_2 \rangle}$

$\frac{1}{n^2} T^n_* \omega \longrightarrow Q_\pi(T)$





g_t

$SO_{1,g-1}(\mathbb{R}) \supset P = \langle g_t, \text{unipt} \rangle$

X

$\downarrow$

$\mathrm{Amp}^1 / \mathrm{Aut}(X)$