CAUSTICS OF LAGRANGIAN HOMOTOPY SPHERES WITH STABLY TRIVIAL GAUSS MAP

TODAY: (1) IOU from PhD thesis
(2) UROP with D. Darrow

$L \subset (M^{2n}, \omega)$ Lagrangian htpy sphere
$\gamma \subset TM$ Lagrangian distribution.

Def $\gamma|_L$ is stably trivial if $\gamma \oplus \mathbb{R}$ and $TL \oplus \mathbb{R}$ are homotopic as Lagrangian distributions in $TM|_L \oplus \mathbb{C}$.

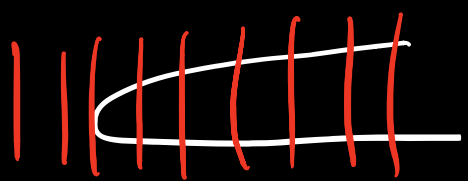
Thm The tangencies of L wrt γ can be simplified to consist only of folds via a Hamiltonian isotopy $\iff$ $\gamma|_L$ is stably trivial.

Remarks:

– A fold is the simplest tangency:

$\{p^2 = q\} \times \mathbb{R}^{n-1} \subset T^*\mathbb{R} \times T^*\mathbb{R}^{n-1} = T^*\mathbb{R}^n$

$\gamma = \ker(d\pi)$, $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$



L

– $T(T^*L)|_L \cong L \times \mathbb{C}^n$ so can view htpy class of $\gamma|_L$ as element of $\pi_n \Lambda_n$, where $\Lambda_n = U_n/O_n$. Then stable triviality means

$[\gamma|_L] \in \ker(\pi_n \Lambda_n \to \pi_n \Lambda_{n+1})$.

– $\pi_n \Lambda_{n+1} \cong \pi_n \Lambda = 0 \quad n \equiv 0, 4, 6, 7 \ (8)$

In my PhD I proved:

<u>Thm</u> The tangencies of L wrt γ can be simplified to consist only of folds via a Hamiltonian isotopy $\Leftrightarrow$ γ is homotopic to a γ' wrt which L only has folds.

Therefore the problem reduces to:

Thm Every $\alpha \in \ker(\pi_n \Lambda_n \to \pi_n \Lambda_{n+1})$ admits a rep $(D^n, \partial D^n) \to (\Lambda_n, i\mathbb{R}^n)$ with only fold tangencies wrt $\mathbb{R}^n$.

Remarks:

— $\ker(\pi_n \Lambda_n \to \pi_n \Lambda_{n+1}) = \begin{cases} \mathbb{Z} & n \equiv 0 \ (2) \\ \mathbb{Z}/2 & n \equiv 1 \ (2) \end{cases}$ $(n > 1)$

we exhibit explicit gen.

§2. (A) Let (W, λ, ϕ) be Weinstein s.t

(1) $\exists \gamma \subset TW$ global field of Lag. planes

(2) ϕ is Morse with 2 critical points.

Corollary (W, λ, ϕ) is homotopic to a Weinstein domain with arboreal skeleton.

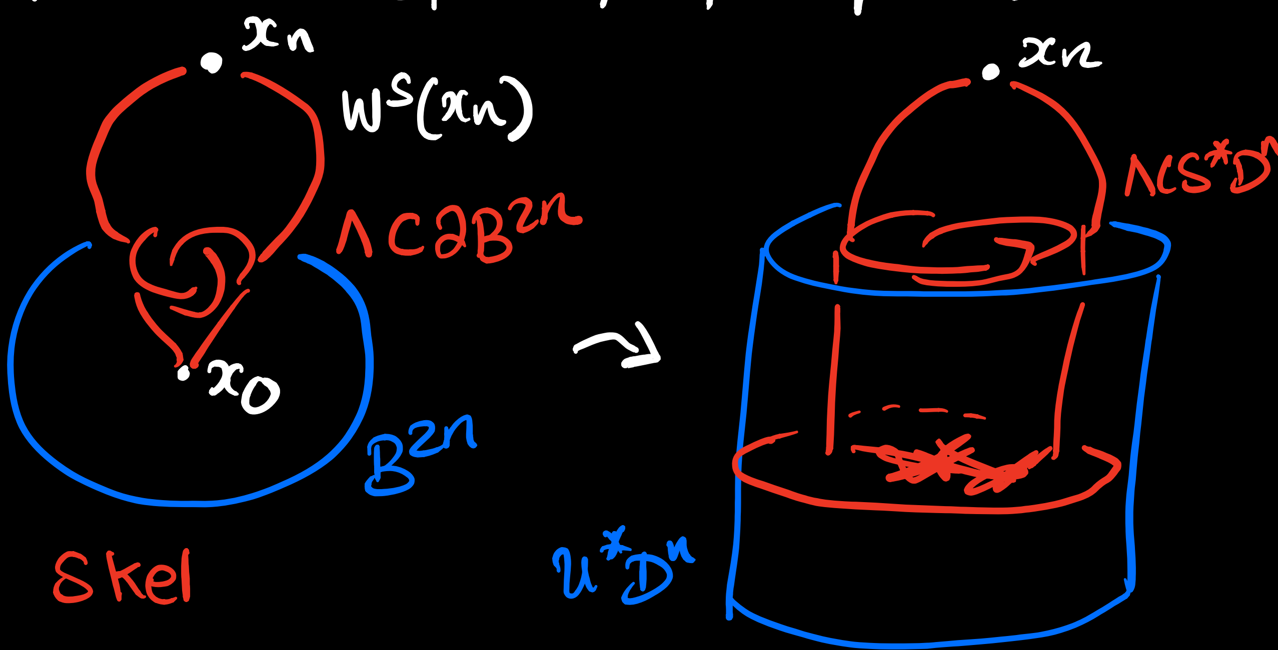
Remarks:

— Recovers special case of more general result joint with Eliashberg & Nadler without hypothesis (2).

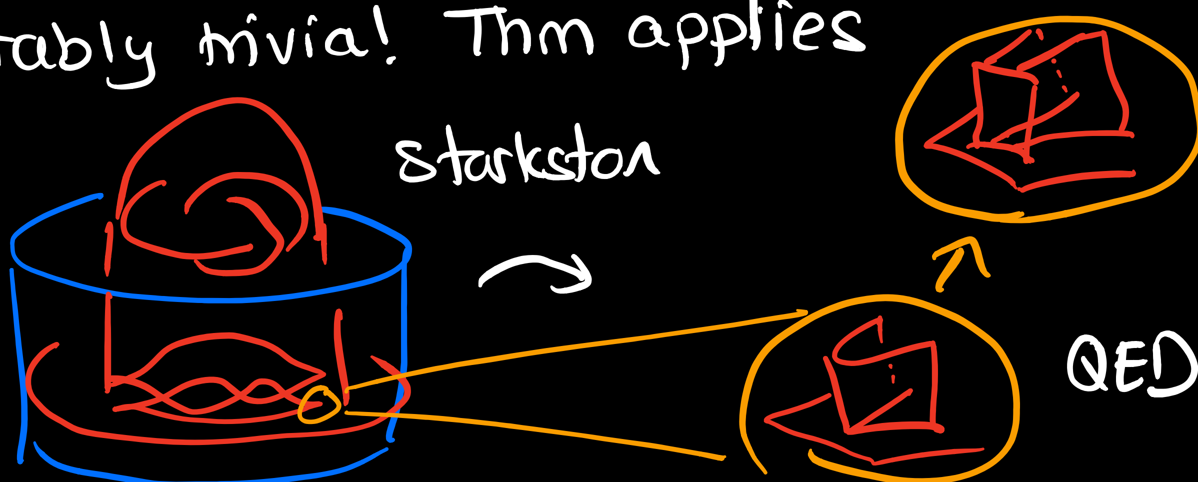
— Difficulties in general case arise from interaction of ≥ 3 strata.

— Here can use Starkston's strategy, used to arborealize 4D Weinstein manifolds.

Proof: $\mathrm{Crit}(\phi) = \{x_0, x_n\}$, $\mathrm{ind}(x_i) = i$.



wlog $\gamma = \ker(d\pi)$ on U^*D^n, where $\pi: U^*D^n \to D^n$ proj. Then $\gamma|_\Lambda = \nu \oplus Z$ for Z Liouville direction. Singularities of $\pi|_\Lambda : \Lambda \to D^n$ $\equiv$ tangencies of Λ wrt ν. But $\nu \oplus Z$ extends to $W^s(x_n) \Rightarrow \nu$ is stably trivial! Thm applies



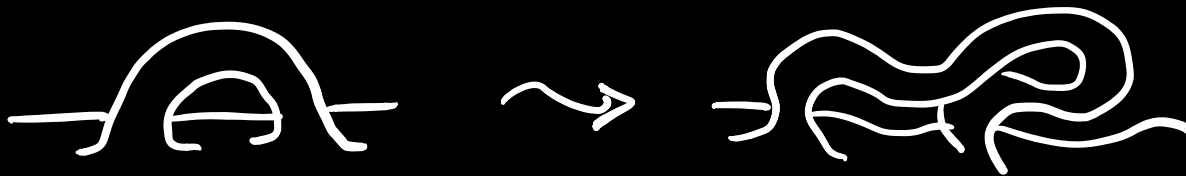
QED

(B) Let $L \subset T^*\Sigma$ Lagrangian htpy sphere.

<u>Corollary</u> $\exists \varphi_t$ Ham isotopy s.t. $\varphi_1(L)$ is generated by a framed function on some tube bundle.

Remarks:

– A tube $T \subset \mathbb{R}^{N+1}$ is, up to a compactly supported isotopy, the result of attaching a standard handle to $\{x_{N+1} \leq 0\}$.



– A framed function $f: W \to \mathbb{R}$ on a fibre bundle $F \to W \to B$ is:

- $f_b : F_b \to \mathbb{R}$ Morse / generalized Morse $\forall b \in B$.
- Negative eigenspaces at critical points of f_b are framed.

Framed functions are the homotopically canonical way of studying smooth fibre bundles via parametrized Morse theory.

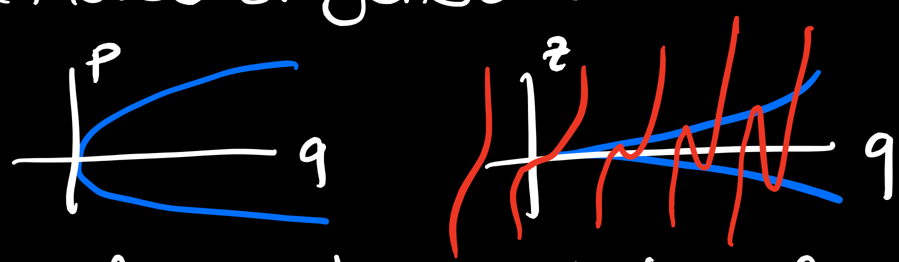
Proof: Heavy lifting is done by:

Thm (Abouzaid, Courte, Guillermou, Kragh)
$L \subset T^*\Sigma$ admits a generating function on some tube bundle $W \to \Sigma$.

This implies $\Sigma \to U/O$ nullhomotopic, hence our theorem applies with $\gamma = \ker(d\pi)$, $\pi: T^*\Sigma \to \Sigma$.

By homotopy lifting property can cover $\varphi_t(L)$ with generating functions f_t on a stabilization of W.

Since $\varphi_1(L)$ & γ only have fold tangencies, f_1 restricts to each fibre as a Morse or generalized Morse function.



Now, f_1 might not admit a framing but this can be corrected by a twisted stabilization of W using Abouzaid: $\varphi_1(L) \to \Sigma$ htpy equiv.

QED

§3 We rely on the fibrations

$$\begin{array}{ccccc} O_n & \to & U_n & \to & \Lambda_n \\ \downarrow & & \downarrow & & \downarrow \\ O_{n+1} & \to & U_{n+1} & \to & \Lambda_{n+1} \end{array}$$

& their LES in homotopy.

Lemma 1 For $n \neq 1,3,7$ we have

$$\ker(\pi_n \Lambda_n \to \pi_n \Lambda_{n+1}) \cap \ker(\pi_n \Lambda_n \to \pi_{n-1} O_n) = 0.$$

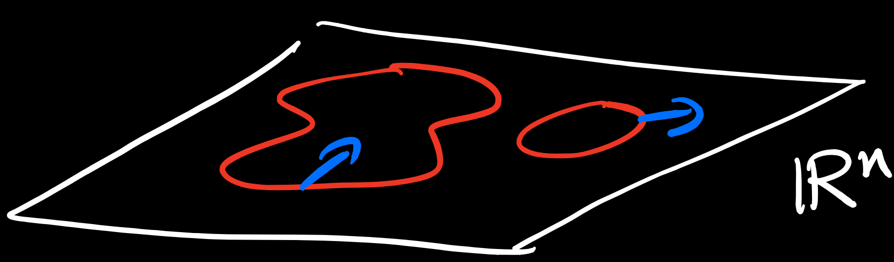
Proof: By cases on n modulo 8, using results in the literature on the homotopy groups of the classical groups in metastable range (Bott, Kervaire, Milnor, Kachi ...)

Definition A formal fold (Σ, v) in $\mathbb{R}^n$ is:

(1) $\Sigma \subset \mathbb{R}^n$ smooth compact hypersurface

(2) v co-orientation of Σ.

Remark: special case of Entov's notion of chain



To (Σ, v) associate a Lagrangian dist in $T^*\mathbb{R}^n$ with folds along Σ & Maslov co-orientation v. This determines $\alpha(\Sigma, \nu) \in \pi_n \Lambda_n$.

<u>Lemma 2</u> The images of the $\alpha(\Sigma, v)$ under $\pi_n \Lambda_n \to \pi_{n-1} O_n$ generate the subgroup $\ker(\pi_{n-1} O_n \to \pi_n O_n)$.

Proof: $\ker(\pi_{n-1} O_n \to \pi_n O_n)$
$$= \mathrm{im}(\pi_n S^n \to \pi_{n-1} O_n) = \langle TS^n \rangle$$
+ Poincaré-Hopf. QED

<u>Corollary</u>: For $n \neq 1, 3, 7$ there is an iso:
$$\ker(\pi_n \Lambda_n \to \pi_n \Lambda_{n+1}) \to \ker(\pi_{n-1} O_n \to \pi_{n-1} O_{n+1}).$$
Proof: Lemma 1 $\Rightarrow$ inj
Lemma 2 $\Rightarrow$ surj. QED

Proof of Thm for $n \neq 1, 3, 7$:
Let $\gamma \in \ker(\pi_n \Lambda_n \to \pi_n \Lambda_{n+1})$. By Lemma 2, $\exists (\Sigma, v)$ s.t γ & $\alpha(\Sigma, v)$ are equal in $\pi_{n-1} O_n$, hence also equal in $\pi_n \Lambda_n$. QED

§4. $n=1$ is trivial for $n=3,7$ the above doesn't work ($\pi_2 O_3 = \pi_6 O_7 = 0$).
Alternative approach: we construct
$F: T(T^* S^n)|_{S^n} \xrightarrow{\sim} S^n \times \mathbb{C}^n$ s.t.
(1) $F^{-1}(i\mathbb{R}^n)$ has folds wrt S^n.
(2) $\hat{F}: T(T^*S^n)|_{S^n} \times \mathbb{C} \to S^n \times \mathbb{C}^{n+1}$

$$\begin{array}{ccc} T(T^*S^n)|_{S^n} \times \mathbb{C} & \to & S^n \times \mathbb{C}^{n+1} \\ \text{is} & & \| \\ S^n \times \mathbb{C}^{n+1} & \to & S^n \times \mathbb{C}^{n+1} \end{array}$$

is trivial in $\pi_n U_{n+1}$. Then:

$(n=3)$ $\ker(\pi_3 \Lambda_3 \to \pi_3 \Lambda_4) \to \pi_3 \Lambda_3 \to \pi_3 \Lambda \to 0$

$$0 \to 2 \cdot \mathbb{Z}/4 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

Quaternions yield a trivialization $TS^3 \xrightarrow{\sim} S^3 \times \mathbb{R}^3$ which complexifies to $G: T(T^*S^3)|_{S^3} \to S^3 \times \mathbb{C}^3$. Put:
$\beta = F \circ G^{-1} \in \pi_3 U_3$. By (2),
$\downarrow$
$\alpha \in \pi_3 \Lambda_3$ is just $\alpha(S^2, n_{B3})$. So, we need to show:

<u>Claim</u>: $\alpha \in \pi_3 \Lambda_3$ is equal to twice a generator.

Proof: $\pi_3 U_3 \to \pi_3 \Lambda_3 \to \pi_2 O_3 = 0$, so enough to show $\beta \in \pi_3 U_3$ is twice a generator. Now, $\pi_3 U_3 \xrightarrow{\sim} \pi_3 U_4$, so enough to show $\hat{\beta} \in \pi_3 U_4$ is twice a generator. But:

$$\hat{\beta} = \hat{F} \circ \hat{G}^{-1} = \hat{G}^{-1} \in \pi_3 U_4$$ is the complexification of quaternion mult, which is $\eta \in \pi_3 O_4$.

<u>Known</u>: η maps to a generator in $\pi_3 O$.

Consider:

$$\begin{array}{ccccccc}
\pi_3 O_4 & \longrightarrow & \pi_3 U_4 & & & & \\
\downarrow & & \downarrow & & & & \\
\pi_3 O_5 & \longrightarrow & \pi_3 U_5 & \longrightarrow & \pi_3 \Lambda_5 & \longrightarrow & \pi_2 O_5 \\
\| & & \| & & \| & & \| \\
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0
\end{array}$$

It follows that the image $\hat{\beta}$ of η in $\pi_3 U_4$ is twice a generator, as claimed.

QED

Similar for octonions.