Sutured Legendrian homologies and applications to the conormal construction

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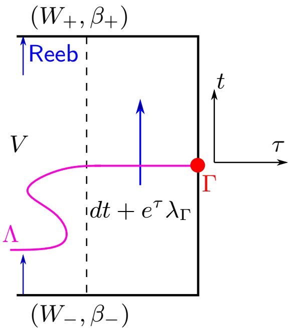
Sutured contact mfd [Colin-Ghiggini-Honda-Hutchings]
(V, λ) with corners.
$(W_\pm, \beta_\pm)$ Liouville domains, of contact boundary (Γ, λ_Γ)

$\simeq$ contact mfd with smooth convex boundary [Giroux]



Sutured Legendrian submanifold
$\Lambda \subset (V, \lambda)$ such that

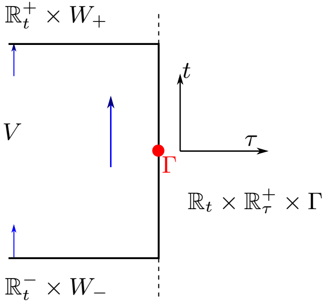
- $\partial\Lambda \subset \{0\} \times \Gamma$;
- near the vertical boundary,
  $\Lambda \simeq (-\epsilon, 0]_\tau \times \{0\} \times \partial\Lambda.$

$\rightsquigarrow \partial\Lambda \subset \Gamma$ Legendrian

Example : Legendrian lift of an exact cylindrical Lagrangian filling

Completion

[CGHH] (V, λ) extended into a non-compact contact mfd V^*.



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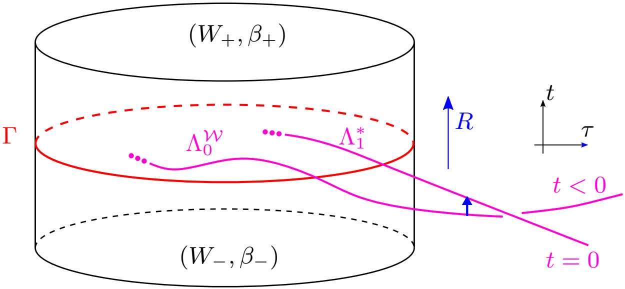


[Abbondandolo-Schwarz]

- *Cylindrical* completion: $\Lambda^* = \Lambda \cup (\mathbb{R}^+_\tau \times \{0\} \times \partial\Lambda)$
- *Wrapped* completion: a Hamiltonian quadratic in τ induces a contact v.f. $\rightsquigarrow \Lambda^{\mathcal{W}} = \phi^1(\Lambda^*)$.

{chords of $\Lambda_0^{\mathcal{W}} \cup \Lambda_1^*$ outside the original mfd} $\leftrightarrow$ {chords of Γ from $\partial\Lambda_0$ to $\partial\Lambda_1$}

Relative sutured Legendrian homologies [Chekanov, Ekholm-Etnyre-Sullivan...]
$\Lambda_0, \Lambda_1 \subset (V, \lambda)$ sutured Legendrians, hypertight

- $LC(\Lambda_0, \Lambda_1, V, \lambda)$ generated by Reeb chords going from Λ_0^* to Λ_1^*
- $WLC(\Lambda_0, \Lambda_1, V, \lambda)$ ... from $\Lambda_0^{\mathcal{W}}$ to Λ_1^*.

Well-defined (maximum principle)

Theorem

$LC(\Lambda_0, \Lambda_1, V)$ is a sub-complex of $\mathcal{W}LC(\Lambda_0, \Lambda_1, V)$, inducing an exact sequence

$$\longrightarrow LH(\Lambda_0, \Lambda_1, V) \longrightarrow \mathcal{W}LH(\Lambda_0, \Lambda_1, V) \longrightarrow LH^{\text{ext}}(\Lambda_0, \Lambda_1, V) \longrightarrow$$

LC^{ext} generated by the exterior chords (ie from $\partial\Lambda_0$ to $\partial\Lambda_1$ in Γ)

Homologies are invariants along Legendrian paths, where the boundary ***can*** move.
[Dimitroglou Rizell] Generalise Lagrangian Floer homology (via Legendrian lift)

Expectations : The *exact sequence* is invariant along paths with ***fixed*** boundary.

Remark (Seidel isomorphism [Ekholm, Dimitroglou Rizell]) :
$LH^{\text{ext}}(\Lambda_0, \Lambda_1, V)$ should be a bilinearized version of the dga $\mathcal{L}C(\partial\Lambda, \Gamma)$

Conormal construction

Theorem

$N \subset M$, such that $\partial N \subset \partial M$
$\Rightarrow U_N M \subset UM$ sutured Legendrian

$\rightsquigarrow$ invariants of smooth submanifolds with boundary

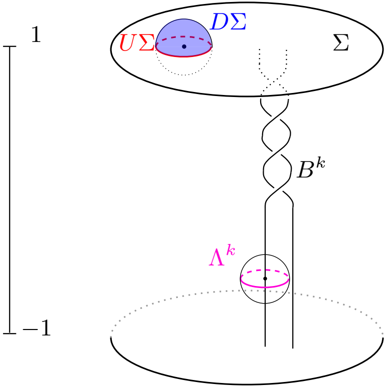
Theorem

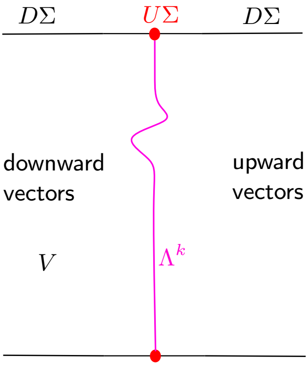
The sutured exact sequence is a complete invariant for local 2-braids.

$\rightsquigarrow$ If the conormals of two local 2-braids are Legendrian isotopic with ***fixed boundary***, the braids are equivalent.

Remark [Shende, Ekholm-Ng-Shende] : For a knot $K \subset S^3$, the torus $U_K S^3$ is a complete knot invariant.

Apply the conormal construction to a pure local 2-braid $B^k \subset [-1,1]_u \times \Sigma$





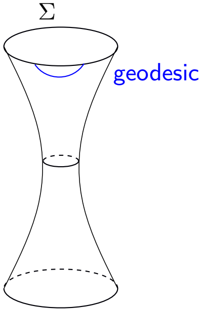
$$V = U\big([-1,1]\times\Sigma\big) \simeq [-1,1]\times\big(D\Sigma \underset{U\Sigma}{\cup} D\Sigma\big)$$ contact manifold with smooth convex boundary

$\Lambda^k = U_{B^k}\big([-1,1]\times\Sigma\big)$ two Legendrian cylinders

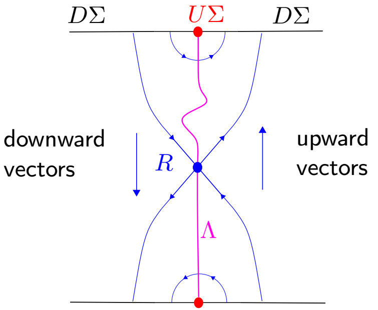
The sutured manifold

Metric $\frac{du^2+g_\Sigma}{1+u^2}$ on $[-1,1]_u \times \Sigma$

$\rightsquigarrow$ contact form on the unit bundle.
Reeb trajectories project to geodesics.



"hyperbolic" metric



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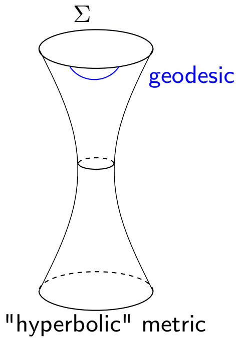
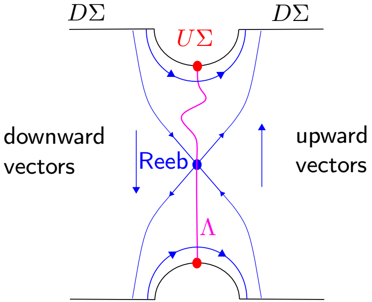




$$W_+ = W_- = D\Sigma \sqcup D\Sigma$$

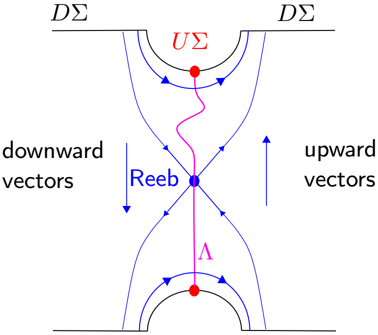
The sutured complex

Reeb trajectories $\leftrightarrow$ geodesics
$\Rightarrow$ hypertight

$\partial\Lambda =$ fibers $\subset (U\Sigma, \lambda_\Sigma)$

$\forall \gamma \in \pi_1(\Sigma)$ we get :

- 1 interior chord $(u = 0)$
- 2 exterior chords $(u > 1$ and $u < -1)$



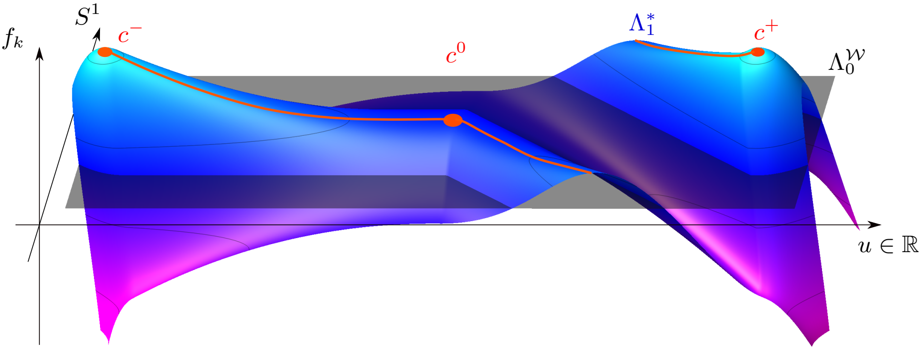
Split by homotopy class.

The $\mathbb{Z}_2[H_1(\Lambda_0)]$-$\mathbb{Z}_2[H_1(\Lambda_1)]$-bimodule with one generator is $C = \bigoplus_{i,j} \mathbb{Z}_2.\mu_0^i c \mu_1^j$

$$LC(\Lambda) = C^0 \qquad \mathcal{W}LC(\Lambda) = C^-[1] \oplus C^0 \oplus C^+[1]$$

Lifting to 1-jets space

Lift Σ to $\mathbb{R}^2$, and use $U\mathbb{R}^2 \simeq J^1(S^1_\theta)$.
After completion, Λ^k lifts to $J^1(\mathbb{R}_u \times S^1_\theta)$ (similar to [Pan-Rutherford])



Reeb chords correspond to positive critical points of f_k.

[Floer, Ekholm] Holomorphic curves degenerate to gradient trajectories
$\rightsquigarrow$ $\partial c^- = c^0$ and $\partial c^+ = \mu_0^{-k} c^0 \mu_1^k$

Sutured exact sequence : In homology we obtain

$$C^0 \xrightarrow{0} H(C^+ \oplus C^0 \oplus C^-) \xhookrightarrow{f_k} C^- \oplus C^+ \overset{\mathsf{Id}\oplus\delta_k}{\twoheadrightarrow} C^0 \xrightarrow{0}$$

where $\delta_k c = \mu_0^{-k} c \mu_1^k$.

By contradiction : Assume $\Lambda^0 \sim \Lambda^k$ as Legendrians with fixed boundaries $(k \neq 0)$

Homotopic and grading restrictions $\Rightarrow$

$$\begin{array}{ccccccccc}
C^0 & \xrightarrow{0} & C & \xrightarrow{f_0} & C^- \oplus C^+ & \xrightarrow{\mathsf{Id}\oplus\delta_0} & C^0 & \xrightarrow{0} & \\
\downarrow & & \downarrow & & \downarrow \mathsf{Id} & & \downarrow & & \\
C^0 & \xrightarrow{0} & C & \xrightarrow{f_k} & C^- \oplus C^+ & \xrightarrow{\mathsf{Id}\oplus\delta_k} & C^0 & \xrightarrow{0} &
\end{array}$$

Impossible : $f_0(c) = (c, c) \notin \ker(\mathsf{Id} \oplus \delta_k)$.

□

Conclusion : $\Lambda^k \sim \Lambda^l \Rightarrow k = l$

The end.

Thank you for your attention