Intrinsic Mirror Symmetry and Categorical Crepant resolutions

work in progress (based on earlier work with S. Ganatra)

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Zoom
Feb. 19, 2021

Algebro-geometric terminology

Definition

A *pair* (M, D) will consist of a smooth projective variety (over $\mathbb{C}$) with $\mathrm{D} = \cup_i D_i$ a simple normal crossings divisor.

Being simple normal crossings means that each component D_i is smooth and all of the intersections $D_I := \cap_{i \in I} D_i$ are transverse.

Definition

A *positive pair* (M, D) is a pair such that D supports an ample line bundle, i.e. there is some ample $\mathcal{L}$ such that

$$\mathcal{L} \cong \mathcal{O}(\sum_i \kappa_i D_i)$$

where κ_i are positive.

Symplectic structure

- For positive pairs, the complement X is an affine variety. We will be thinking of these affine varieties X as exact symplectic manifolds.
- Equip M with a Kähler form $\omega_{\mathcal{L}}$ associated to (a positive Hermitian metric $||\cdot||$ on) $\mathcal{L}$ and restrict this form to X. $(\omega_L)_X$ has a natural primitive $\theta_{\mathcal{L}}$ $(=-d^c h$ where h is the Kähler potential).

  $h = -\log ||s||$
- The tuple $(X, \omega_{\mathcal{L}}, \theta_{\mathcal{L}})$ equips X with the structure of a (finite-type) convex symplectic manifold. So we can attach several "wrapped Floer theoretic" invariants
  - $SH^*(X)$(Cieliebak-Floer-Hofer, Viterbo)
  - $\mathcal{W}(X)$ (Abouzaid-Seidel) ← Wrapped Floer coh.

Log CY varieties

Ω_M has simple poles on D

Definition

A pair (M, D) is called a Calabi-Yau pair if D is an anti-canonical divisor. The complement $X := M \setminus \mathrm{D}$ is called a log Calabi-Yau variety.

This talk will be concerned with wrapped Floer invariants on affine log Calabi-Yau varieties.

Examples

- The most basic example of an affine log Calabi-Yau variety is $(\mathbb{C}^*)^n = \mathbb{C}P^n \setminus D$ where D is the union of coordinate hyperplanes. The volume form is given in standard local coordinates by

$$\Omega := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \tag{1}$$

- A surprising rich class of examples is given by (affine) cluster varieties —roughly speaking varieties obtain by gluing copies of $(\mathbb{C}^*)^n$ according to specific rules which preserve this holomorphic volume form.

HMS

Kontsevich's homological mirror symmetry (HMS) conjecture in this context predicts that in "nice" cases there is a mirror log Calabi-Yau (not necessarily affine) Y such that:

$$\mathrm{Perf}(\mathcal{W}(X)) \cong D^b Coh(Y) \tag{2}$$

$\mathrm{Perf}(\mathcal{W}(X))$ is the (split-closed) derived wrapped Fukaya category of X and $D^b Coh(Y)$ is the derived category of bounded coherent sheaves on Y.

Remarks

- *One expects HMS to always hold in the above form when* $\dim(X) \leq 3$. *Many cases proven in* $\dim(X) = 2$ *(Pascaleff, Keating, Hacking-Keating).*
- *HMS has many more concrete consequences, for example it implies that the symplectic cohomology of X can be computed in terms of sheaf cohomology groups on Y.*

Semi-affineness

It is a general expectation is that the mirror space Y to an affine log Calabi-Yau X should be semi-affine. This means that the canonical map: $\alpha : Y \longrightarrow \mathrm{Spec}(\Gamma(\mathcal{O}_Y))$ is proper.

Semi-affineness has a number of interesting consequences, the two most important for us being:

- The ring of functions $\Gamma(\mathcal{O}_Y)$ is a finitely generated k-algebra.
- For any $E_0, E_1 \in D^b Coh(Y)$, $\mathrm{RHom}^*_Y(E_0, E_1)$ is a finitely generated module over $\Gamma(\mathcal{O}_Y)$.

Main Theorem

The expectation that mirrors to Y are semi-affine can be turned into the following precise theorem purely on X:

Theorem (P, in progress)

For any affine log Calabi-Yau variety X:

1. *The degree zero symplectic cohomology $SH^0(X)$ is finitely generated and is a filtered deformation of a certain algebra, $\mathcal{SR}(\Delta(\mathrm{D}))$, defined combinatorially in terms of the compactifying divisor D.*
2. *For any L_0, L_1, the wrapped Floer groups $WF^*(L_0, L_1)$ are finitely generated modules over $SH^0(X)$.*

As we will see later, this theorem together with some homological algebra leads to a criterion for HMS to hold "birationally."

The combinatorial ring

Assume for simplicity that all strata D_I are connected. k comp. to D

- For a vector $\mathrm{v} = (v_i)$ in $(\mathbb{Z}^{\geq 0})^k$, we define the support of v, $|\mathrm{v}|$ to be the set of $i \in \{1, \cdots, k\}$ such that $v_i \neq 0$. We let $B(M, \mathrm{D}) \subseteq (\mathbb{Z}^{\geq 0})^k$ to be the set of vectors v such that $D_{|\mathrm{v}|} \neq \emptyset$.
- Let $\mathcal{A}$ denote the vector space:

$$\mathcal{A} := \bigoplus_{\mathrm{v} \in B(M,\mathrm{D})} k \cdot \theta_{\mathrm{v}}. \tag{3}$$

- We can equip $\mathcal{A}$ with a ring structure

$$\theta_{\mathrm{v}_1} * \theta_{\mathrm{v}_2} = \begin{cases} \theta_{\mathrm{v}_1+\mathrm{v}_2} & D_{|\mathrm{v}_1+\mathrm{v}_2|} \neq \emptyset \\ 0 & \textit{otherwise} \end{cases} \tag{4}$$

The ring $\mathcal{SR}(\Delta(\mathrm{D})) := (\mathcal{A}, *)$.

Gross-Siebert construction

- In a recent paper Gross-Siebert have defined a (degree zero) "logarithmic quantum-cohomology" $(\mathcal{A}, *_{GS})$, which is defined on the vector space $\mathcal{A}$ and is a deformation of $\mathcal{SR}(\Delta(\mathrm{D}))$.
- The deformation is given given by counting genus zero curves with tangencies("punctured GW invariants"). It can often be computed essentially combinatorially using methods of tropical geometry (c.f. work of T. Mandel).

The Main Theorem suggests:

Conjecture

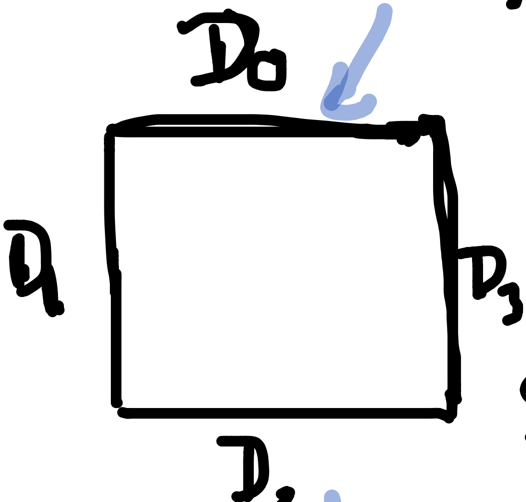
There is an isomorphism of rings $(\mathcal{A}, *_{GS}) \cong SH^0(X, k)$.

Our proof of the Main Theorem is modelled on the usual PSS isomorphism between quantum cohomology and Floer cohomology and is hopefully a good first step in establishing a ring isomorphism.

Illustration of Theorem

- Let $\bar{M} = \mathbb{C}P^1 \times \mathbb{C}P^1$ and $\bar{D}$ be the toric divisors
- Take M to be the blowup of $\bar{M}$ at n points in $\infty \times \mathbb{C}P^1$. Let D be the proper transform of $\bar{D}$.

$$SR(D) \cong \frac{K[\Theta_0, \Theta_1, \Theta_2, \Theta_3]}{(\Theta_0 \cdot \Theta_2 = 0, \Theta_1 \Theta_3 = 0)}$$



$$SH^0 \cong \frac{K[\Theta_0, \Theta_1, \Theta_2, \Theta_3]}{(\Theta_0 \cdot \Theta_2 = 1, \Theta_1 \Theta_3 \ )}$$

$(1+\Theta_0)^n$

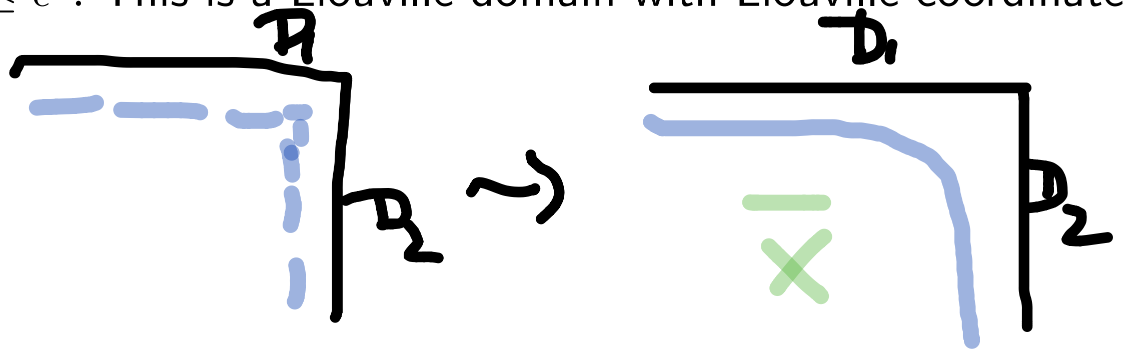
Symplectic normal crossings

- If D is a smooth divisor and (ρ, ∇) is a Hermitian structure on ND, then Weinstein's tubular neighborhood theorem shows that there is an embedding from $\psi : U \subset ND \longrightarrow M$ such that
$$\psi^*(\omega) = \pi^*(\omega_{|D}) + \frac{1}{2} d(\rho\alpha)$$
where α is the connection one-form.
- In the normal crossings case, McLean-Tehrani-Zinger have introduced the notion of a "regularization, " which is essentially a system of compatible Weinstein tubular neighborhoods $\psi_i : U_i \longrightarrow M$ which intersect nicely.

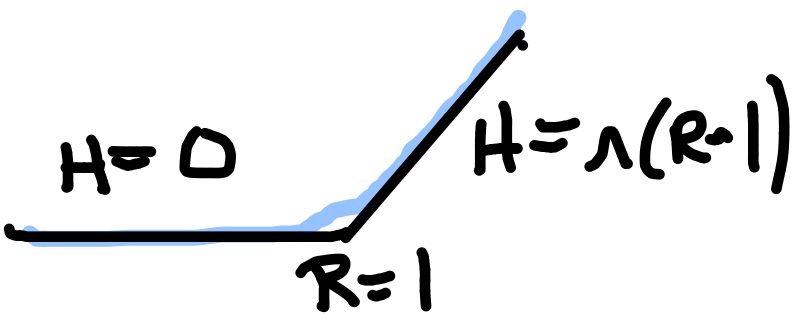
$U_I = \cap U_i$

- MTZ have shown that one can always deform ω (in the same cohomology class, keeping all D_i symplectic) so that a regularization exists. McLean has further demonstrated that one can find a primitive θ for $\omega_{|X}$ which has some nice normal form.

We let $\bar{X}$ be a small rounding of the corners of $M \setminus \cup_i (U_{i,\epsilon})$ where $\epsilon > 0$ is some small real number and $U_{i,\epsilon}$ is the region where $\frac{\rho_i}{\kappa_i/2\pi} \leq \epsilon^2$. This is a Liouville domain with Liouville coordinate R.



We want to take H^λ to be a smoothing of: $\lambda \in (0,\infty)$

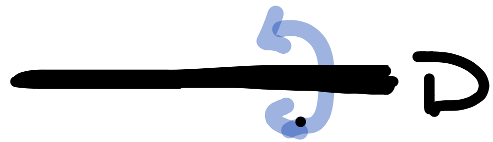


Define

$$SH^*(X) := \varinjlim_{\lambda} HF^*(X; H^\lambda)$$

Having made these choices, it is easy to calculate periodic orbits because the flow preserves the fibers of $U_I := \cap_{i \in I} U_i \longrightarrow D_I$. The orbits come in connected families $\mathcal{F}_{\mathrm{v}}$ which wind around the divisors with multiplicity v. For example, in the smooth case divisor one has

↳ $v \in B(M, D)$



- constant orbits in the interior of $\bar{X}$.
- orbits which wind around the divisor $\mathrm{v} > 1$ times.

Calculation: If $x_0 \in \mathcal{F}_{\mathrm{v}}$, its action can be made arbitrarily close to

$$A_{H^\lambda}(x_0) \approx -w(\mathrm{v})(1 - \epsilon^2/2) \tag{5}$$

where $w(\mathrm{v}) = \sum_i \kappa_i v_i$. Thus if ϵ is small, the filtration by $w(\mathrm{v})$ is essentially the same as the action filtration (up to sign).

from L

Recall that a PSS solution asymptotic to an orbit x_0 is a map $u : \mathbb{C}P^1 \setminus \{0\} \to M$ satisfying a variant of Floer's equation:

$\beta = p(s)dt$

$$(du - X_{H^\lambda} \otimes \beta)^{0,1} = 0 \tag{6}$$

(where $(0, 1)$ is taken with respect to some J_S) such that

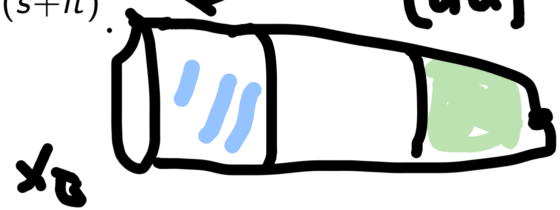
$$\lim_{s \to -\infty} u(\varepsilon(s, t)) = x_0 \tag{7}$$

In the last equation we are using the cylindrical coordinates (defined away from $z = \infty$) $\mathbb{R} \times S^1 \longrightarrow S$

$$\varepsilon : (s, t) \longrightarrow e^{2\pi(s+it)}.$$

$(du)^{0,1} = 0$

$(du - X_H \otimes dt)^{0,1} = 0$



Definition

Suppose x_0 is an orbit of H^λ in X. Then a log PSS solution of multiplicity v is a solution such that

- u does not intersect D anywhere except for at $z = \infty$.
- The intersection multiplicity of u with D_i at $z = \infty$ is v_i:

$\lambda \to SH^{\bullet}(X,K)$

The virtual dimension

$$\text{vdim}(\mathcal{M}(\text{v}, x_0)) = \deg(x_0)$$

The (topological) energy is

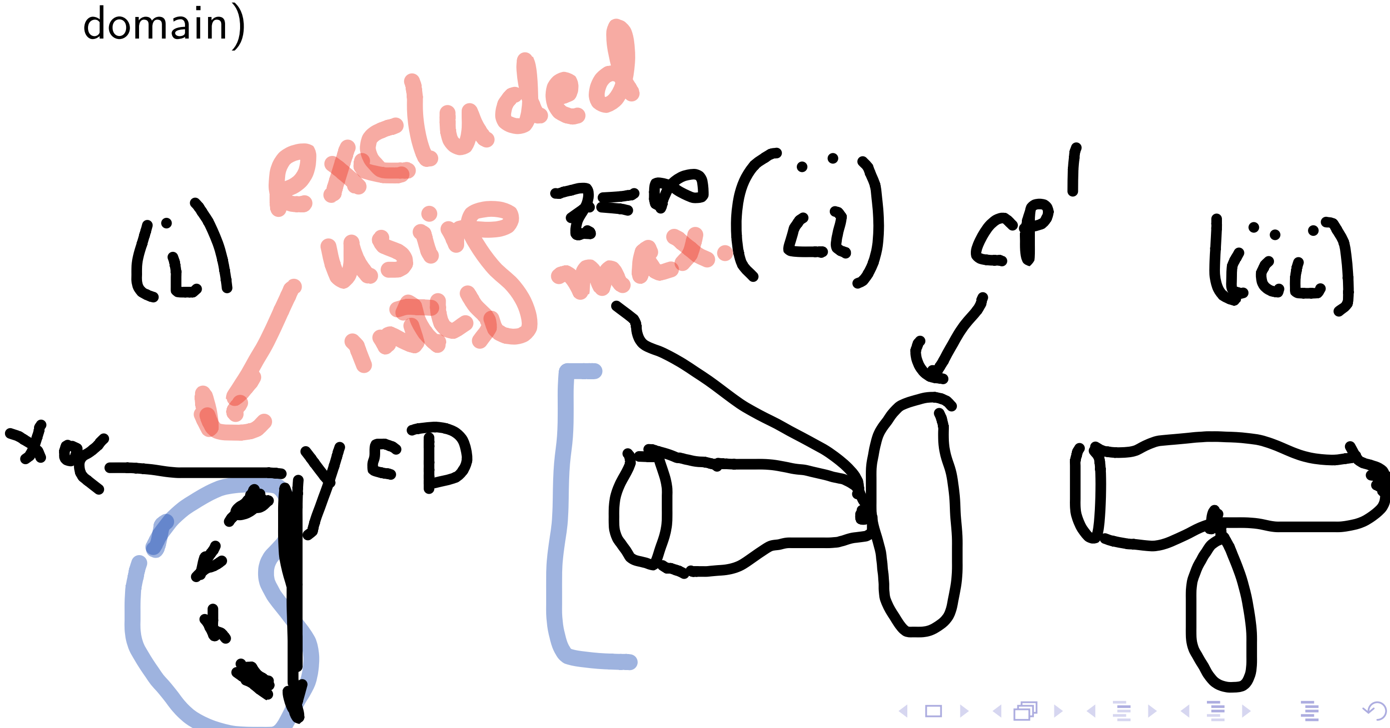
need $J_{s,t}$ to preserve D_i

$$E_{top}(u) = w(\text{v}) + A_{H^\lambda}(x_0)$$

Possible “bad” degenerations

There are two kinds of possible degenerations that we want to avoid

- Breaking along orbits in D.
- Sphere bubbling (both at $z = \infty$ and at other points in the domain)



- It turns out that the breaking along orbits in D can be excluded provided one takes $\lambda > w(\mathrm{v})$.
- If one considers only "low energy" moduli spaces i.e. those where $w(x_0) = w(\mathrm{v})$, then sphere bubbling is excluded by energy constraints.

$E_{top}(u) \approx \frac{1}{2} w(v) \epsilon^2$

Using the winding filtration, we have multiplicative spectral sequence

$$E_r^{pq} => SH^*(X)$$

By counting only low energy moduli spaces, Ganatra and I defined a map

$$\mathrm{PSS}_{log}^{low} : \mathcal{SR}(\Delta(D)) \cong E_1^{p,-p}$$

which we proved to be an isomorphism. (of rings)

Stable log moduli spaces

We want to define an an additive isomorphism:

$$\mathrm{PSS}_{log} : \mathcal{A} \cong SH^0(X) \tag{8}$$

by count all moduli spaces with $\deg(x_0) = 0$, not just the low energy ones.

- The sphere bubbles are hard to control if one thinks naively about the usual Deligne-Mumford compactification.
- The main idea is to construct a compatification of stable log PSS solutions following ideas/analysis of M. Tehrani. This has the property that all of the strata with sphere bubbles lie in virtual codimension 2 (hence in our case have negative virtual dimension).
- To "regularize" the boundary strata we adapt Cieliebak-Mohnke's approach of stabilizing divisors.

Basic idea of part (2)

Consider the case of a smooth divisor D and suppose for simplicity we have some cylindrical Lagrangians L_0, L_1 such that

- $\pi(L_0), \pi(L_1)$ are embedded exact $\pi: SD \to D$
- $\pi(L_0) \cap \pi(L_1)$ transversely

For each intersection point $y \in \pi(L_0) \cap \pi(L_1)$, we have chords $x_{y,\mathrm{v}}$ where $x_{y,0}$ is a "short" chord and $x_{y,\mathrm{v}}$ is given by taking that chord and spinning it v times around D. We can arrange our data so that to, to lowest order, we have

$$\theta_{\mathrm{v}} \cdot x_{y,0} = x_{y,\mathrm{v}} + \cdots \tag{9}$$

where $\cdots$ denotes higher action terms.

Maximally degenerate setting

A pair (M, D) is called maximally degenerate if D has a zero dimensional stratum. In this setting, one consequence of our result is:

Proposition

Let (M, D) be a maximally degenerate Calabi-Yau pair of dimension n. Suppose that $char(k) = 0$, $\mathrm{Spec}(SH^0(X))$ is a reduced n-dimensional scheme of finite type which has Gorenstein singularities. Furthermore, it is Calabi-Yau.

- Main idea: use the fact that $SH^0(X)$ is a deformation of $\mathcal{SR}(\Delta(D))$.
- Proving that $\mathcal{SR}(\Delta(D))$ is Gorenstein uses a deep result of Kollar-Xu on the topology of the dual intersection complex of a Calabi-Yau pair.

Resolutions

The above result makes it plausible that $\mathrm{Spec}(SH^0(X))$ is closely related to a mirror variety of X. To be precise, suppose a mirror Y to X existed, then by classical algebraic geometry (baby verison of Zariski's main theorem), one can show that Y is a crepant resolution of singularities of $\mathrm{Spec}(SH^0(X))$ (in particular these varieties are birational). However, there are two problems with this:

- Crepant resolutions are not unique in dimension ≥ 3. So one needs more data to single out a crepant resolution.
- A crepant resolution may not even exist.

Noncommutative resolutions

Definition

Given an affine log Calabi-Yau variety X, a *homological section* is an embedding $\pi^* : \mathrm{Perf}(SH^0(X)) \hookrightarrow \mathcal{W}(X)$.

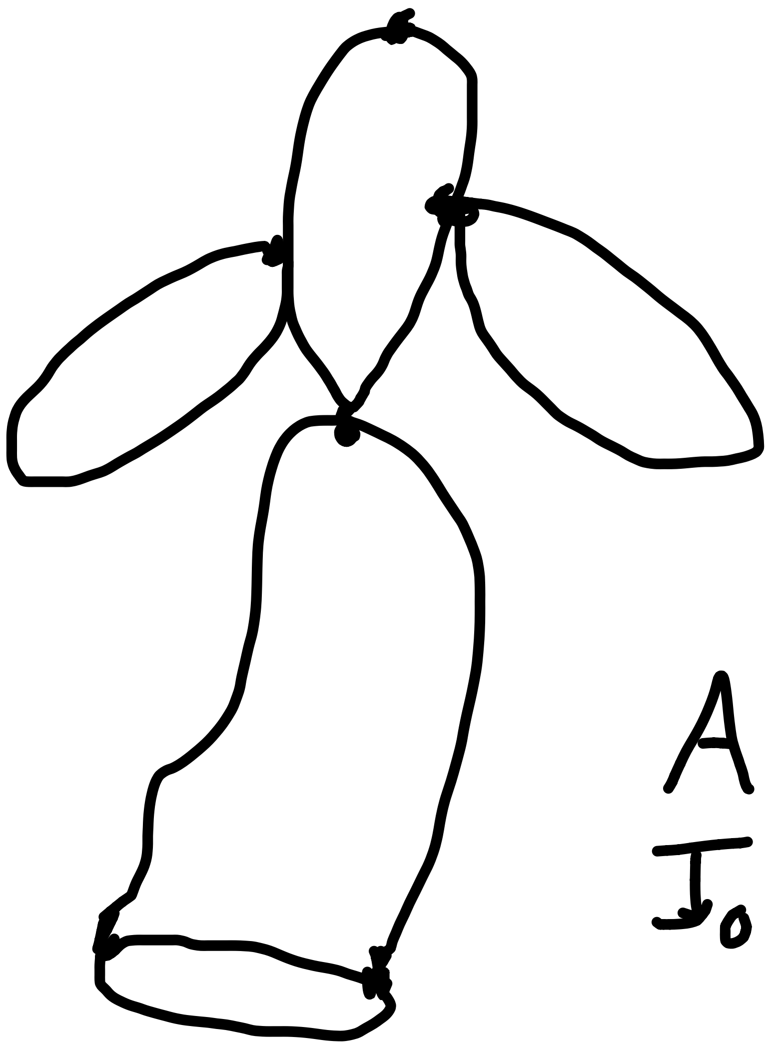
Remark

There are fairly explicit geometric criteria for when a Lagrangian L_0 determines a homological section.

Lemma

Suppose X is equipped with a homological section. Then $(\mathcal{W}(X), \pi^*)$ is a categorical crepant resolution of $\mathrm{Spec}(SH^0(X))$.

A little more on log compactification



Assume
J_0 is "split"
near $z=\infty$

modelled on rooted
trees T w one leg

- If $u_1(\mathbb{CP}^1) \subset D_i$

then that component is decorated with a meromorphic section

$$[\zeta_i] \in \Gamma_{mer}(u^*(ND_i))/\mathbb{C}^*$$

with zeroes/poles only at nodes.

- For each marked point $z_{1,1}$, get an order function.

- Require

$$\boxed{\mathrm{ord}(z_{\lambda,\lambda'}) = -\mathrm{ord}(z_{\lambda',\lambda})}$$

this is the "pre-log" space

$ob_T : M^{\mathrm{prelog}}(v, x_0, T) \to G(T)$

complex affine torus

$M^{\log}(v, x_0, T)$

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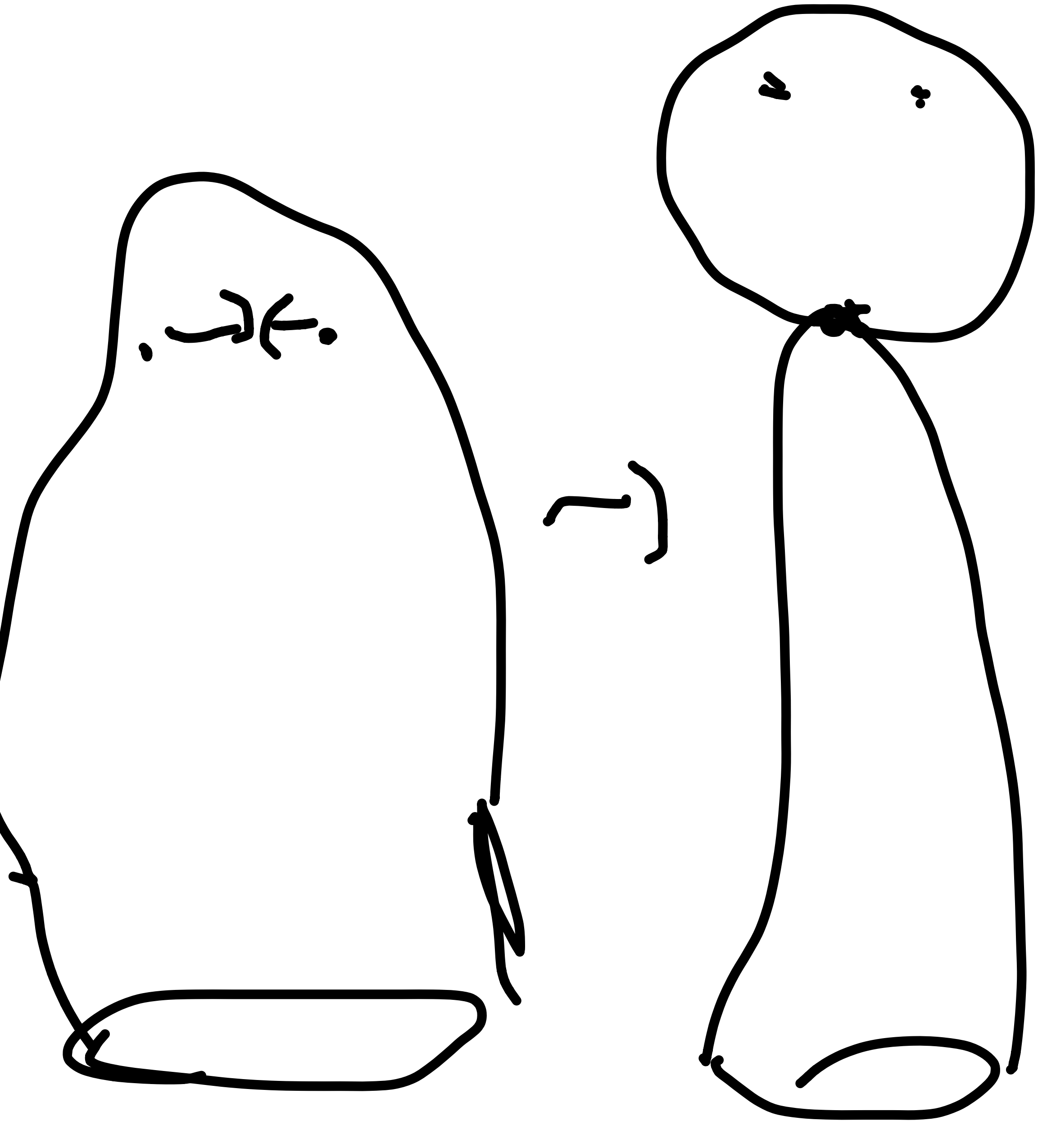
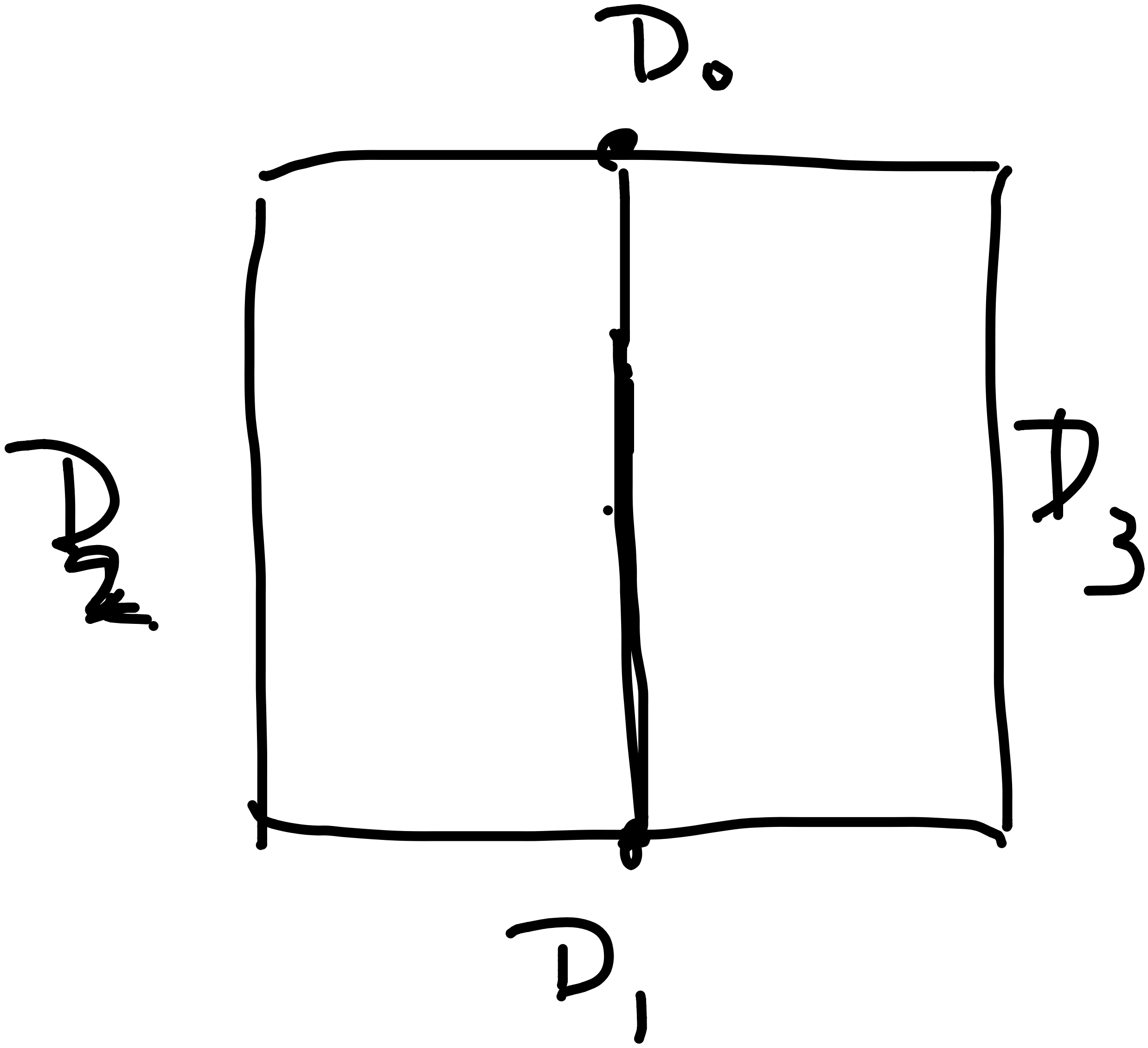
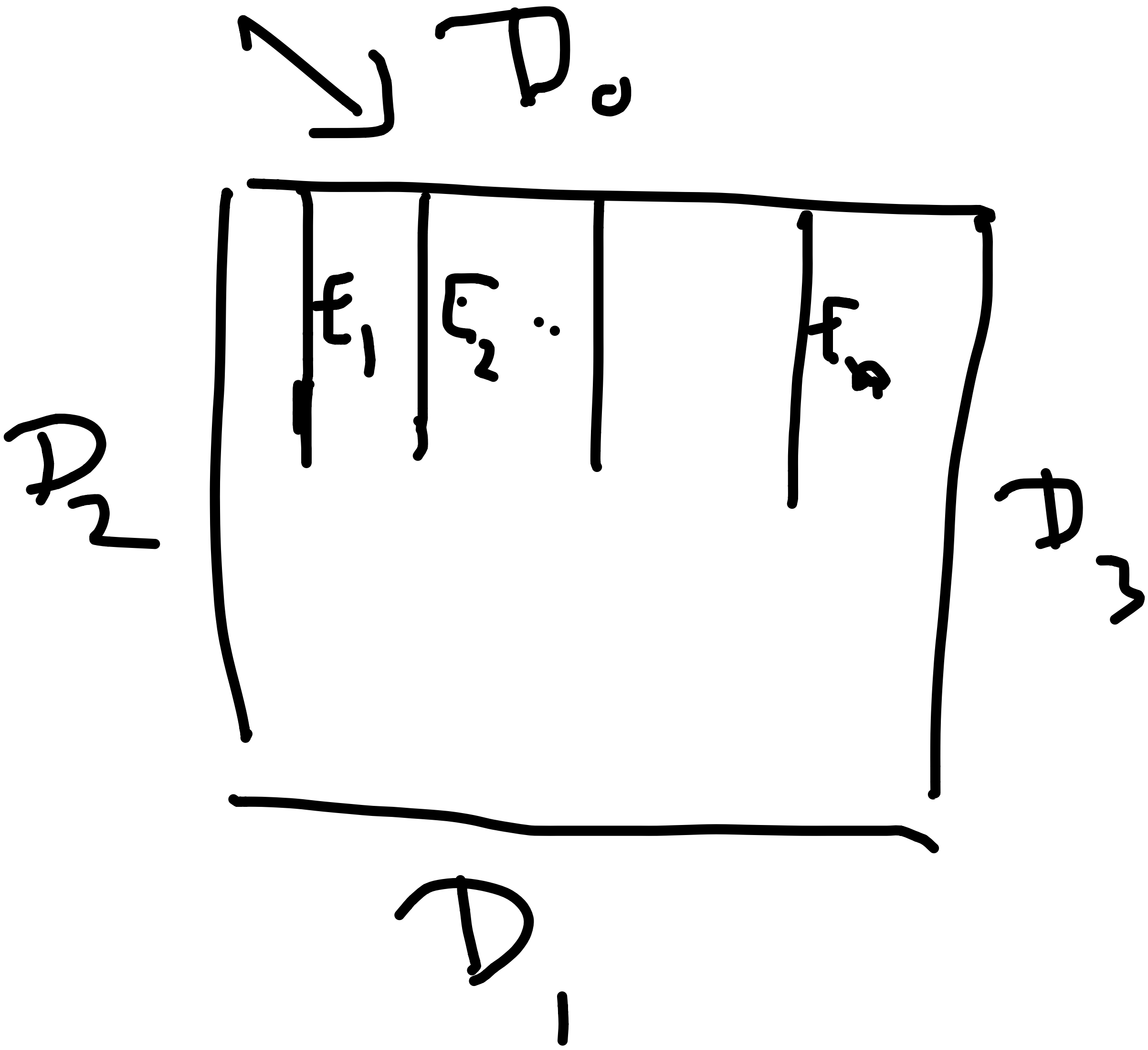
a) $ob_T(u) = 1$

b) "a tropical condition"





$$\Theta_2 \cdot \Theta_3 = (1+\Theta_0)^n$$



D_0
E_1
E_2
E_n
D_2
D_3
D_1

[BSV]

QH^* is a deformation of $SH^*(X)$

the deformation is described by

$$\sum_i PSS_{1,g}(\theta_i) = B$$