Toward super-rigidity of holomorphic disks in Calabi-Yau threefolds

by

Sara B. Tukachinsky

supervised by

Dr. Jake P. Solomon

A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Einstein Institute of Mathematics
Faculty of Science
The Hebrew University of Jerusalem

April 2011

Contents

0. Introduction

In 1986 Katz [1] introduced a conjecture inspired by Clemens. It says that there is only a finite number of spheres of a given degree in a general quintic threefold. It is now known as Clemens' conjecture, and has some variations with stronger statements. The current work is a step towards using Clemens' conjecture to deduce a similar statement for disks. This amounts to almost proving super-rigidity of disks, which says that no simple disks can "get close" to non-simple (or, in our case, multiply-covered) disks. More details are given in section 1.

For start, some machinery needs to be developed; we work with a surface with boundary, and most standard results are only developed for the closed case. Working with a surface with boundary, we require smooth totally real boundary conditions. We tend to prove results in more generality then necessary for the current write-up, as long as it does not require essential additions.

In section 2 we explore the notion of holomorphic vector bundles over surfaces with boundary. As done in the closed case, we show equivalence of a holomorphic structure on a bundle to a $\bar{\partial}$ operator on it. This operator allows us to gain much knowledge on the bundle. A key lemma is quoted from [2] – a generalized version of the Riemann-Roch theorem.

Section 3 presents the notion of Maslov index of bundles. As a special case, it gives the first Chern class for closed surfaces.

In section 4 we discuss the Birkhoff factorization theorem. The original statement concerns spheres. It says that any holomorphic bundle over a sphere is isomorphic to a sum of line bundles. We follow the ideas of Grothendieck in [3] to prove a similar statement for bundles over disks. In the course of proof we give a full classification of line bundles over the disk, similar to the existent classification over spheres: we show that every line bundle is trivial with boundary conditions of the form

$$(\Lambda_\nu)_z = z^{\nu/2}\mathbb{R}, \quad z \in S^1 \tag{1}$$

for some $\nu \in \mathbb{Z}$. We denote such a bundle by L_ν. Same way as line bundles over the sphere are classified by their Chern class, line bundles over the disk are classified by their Maslov index, where the index of L_ν equals ν. Also, we spell out the relation between meromorphic sections and line subbundles of a given bundle over arbitrary Riemann surface, as well as the relation between the Maslov index of the subbundle and the zeroes and poles of the section generating it. Again, this result is well known in the closed case, but the boundary conditions require more effort to be dealt with.

The gained knowledge is combined in section 5 to deduce that the sheaf of holomorphic sections of a holomorphic bundle is locally free. This is again a standard result when dealing with closed manifolds. In order to treat the boundary, we need the full power of the Birkhoff factorization.

The last preliminary result is the Dolbeault isomorphism. It states that the sheaf cohomology of the sheaf of sections of a bundle is isomorphic to the cohomology defined by the $\bar{\partial}$ operator. This is too a standard result for manifolds with empty boundary. As soon as we know the sheaf of sections is locally free, we can apply same reasoning as in the closed case to deduce it for the nonempty-boundary case.

Finally, section 7 essentially proves infinitesimal super-rigidity:

We work with X a symplectic manifold equipped with a generic integrable complex structure, L a generic Lagrangian submanifold. We show that any simple holomorphic map from the disk

$$u : (D, \partial D) \longrightarrow (X, L) \tag{2}$$

is an immersion, and use this fact to deduce our main result:

Theorem 0.1. *Let*

$$u : (D, \partial D) \longrightarrow (X, L)$$

be a simple holomorphic map. Then its normal bundle is of the form

$$N_u \simeq L_{-1} \oplus L_{-1},$$

Here L_{-1} satisfy boundary conditions as in (1) with $\nu = -1$.

In particular, by Remark 4.8, the normal bundle has no holomorphic sections. As a consequence, we deduce in Proposition 7.2 that for any, even non-simple holomorphic map of the form (2), the normal bundle admits no holomorphic sections.

It is therefore reasonable to expect that a non-simple map cannot be approached by simple maps.

0.0.1. *Notation.* Throughout, we use the notation fixed here:

Σ stands for a compact Riemann surface with (possibly empty) boundary, with fixed complex structure $j = j_\Sigma$.

$\mathbb{H} = \{z \in \mathbb{C} \big| Im(z) \geq 0\}$ is the closed halfplane.

$D = \{z \in \mathbb{C} \big| |z| \leq 1\}$. In some cases D will denote a Cauchy-Riemann operator. However, the meaning in each case should be clear from the context.

$HD = \{z \in \mathbb{C} \big| |z| < 1, Im(z) \geq 0\} = \mathbb{H} \cap \mathring{D}$.

A subset of the Euclidian space will be called a region if it is a connected topological submanifold with boundary, of maximal dimension.

For a region with boundary U (here and elsewhere we will mean its boundary as a submanifold), define $C^k(U)$ for $k < \infty$ as the set of all functions that are C^k in $intU$ and whose partial derivatives of order $\leq k$ can be continuously extended to the boundary.

For $k = \infty$, set $C^\infty(U) = \bigcap_{n=1}^\infty C^n(U)$.

$\mathcal{A}^{p,q}$ is the sheaf of smooth (p,q)-forms on Σ. $\Omega^p \subset \mathcal{A}^{p,0}$ is the subsheaf of holomorphic p-forms on Σ.

If E is a vector bundle over Σ, $F \subset E\big|_{\partial\Sigma}$ is a subbundle, S a type of sections of E, we denote by S_F the elements in S with boundary values in F. E.g., $C^\infty_{\mathbb{R}}(HD, \mathbb{C})$ stands for smooth functions on HD with real values on $\partial HD = HD \cap \mathbb{R}$.

1. Motivation

A complex manifold X is said to be Calabi-Yau if its first Chern class vanishes, i.e., $c_1(X) = c_1(TX) = 0$. See [4] for some benefits of Calabi-Yau manifolds. A quintic threefold is a hypersurface of degree 5 in $\mathbb{C}P^4 = \mathbb{P}^4$. Whenever nonsingular, it is Calabi-Yau. The moduli space of quintic threefolds forms an algebraic variety. We say a property holds for a general threefold if it holds on a Zariski-open set in the moduli space.

We say "holomorphic spheres" for the images of holomorphic maps

$$u : S^2 \longrightarrow X.$$

Similarly, "holomorphic disks" with boundary values in L are images of holomorphic maps

$$u : (D, \partial D) \longrightarrow (X, L).$$

A holomorphic disk or sphere is said to be embedded if there exists such u that is an embedding.

With these conventions, consider the following conjecture (first formulated in [1, Conjecture 1.1], based on [5]):

Conjecture 1.1 (Weak Clemens)**.** *Let $X \subset \mathbb{P}^4$ be a general quintic threefold, $A \in H_2(X)$. Then there are finitely many holomorphic spheres in X representing A.*

We introduce an analogous statement for disks, adding a stronger requirement. First of all it is necessary to specify boundary conditions.

Definition 1.2. Let X be a Calabi-Yau manifold. A Lagrangian submanifold L is called a **Fukaya Lagrangian** if for any map

$$u : (D, \partial D) \longrightarrow (X, L)$$

the Maslov index vanishes: $\mu(u^*TX, u^*TL) = 0$.

See Theorem 3.3 for the definition of the Maslov index. Note that, since Maslov index is homotopy invariant, if ϕ is a symplectomorphism and L is a Fukaya Lagrangian, then $\phi(L)$ is again a Fukaya Lagrangian.

Fukaya Lagrangians appear naturally in the description of the Fukaya category, see [6].

Conjecture 1.3 (Strong Clemens for disks)**.** *Let $X \subset \mathbb{P}^4$ be a general quintic threefold, $L \subset X$ a general Fukaya Lagrangian and $A \in H_2(X, L)$. Then there are finitely many simple holomorphic disks in X with boundary conditions in L representing A. Moreover, each disk is embedded and has normal sheaf $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

We outline an argument showing this statement follows from Conjecture 1.1.

1.1. ***J*-holomorphic curves.** In the sequel Σ is a compact Riemann surface with complex structure j. X is a closed manifold with symplectic structure ω. An almost complex structure J is called ω-tame if

$$\forall v \neq 0, \quad \omega(v, Jv) > 0.$$

Denote by $\mathcal{J}$ the space of smooth ω-tame almost complex structures on X. Take L a compact Lagrangian submanifold of X, $A \in H_2(X, L)$, $J \in \mathcal{J}$.

Given a differentiable map into X, the J-antilinear part of the derivative is defined to be

$$\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) \in C^\infty(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes u^*TX) =: \mathcal{A}^{0,1}(\Sigma, u^*TX).$$

Definition 1.4. A curve

$$u : (\Sigma, \partial\Sigma) \to (X, L)$$

is called ***J*-holomorphic** if it is smooth up to the boundary and

$$\bar{\partial}_J u \equiv 0.$$

In other words, a map is J-holomorphic iff its derivative commutes with the complex structures. Elliptic regularity results imply that it is actually enough to require continuity up to the boundary in the above definition, for smoothness will follow.

For any J-holomorphic map u, its energy satisfies [2, Lemma 2.2.1]:

$$E(u) := \int_\Sigma |du|^2 \,\mathrm{dvol}_\Sigma = \int_\Sigma u^*\omega.$$

Definition 1.5. A J-holomorphic curve u is called **somewhere injective** if there exists a point $z \in \Sigma$ for which

$$u^{-1}\big(u(z)\big) = \{z\}, \quad du(z) \neq 0.$$

Such a point z is called an **injectivity point**.

The map u is called **simple** if the set of its injectivity points is dense in Σ.

The map u is said to be **multiply-covered** if there exist a surface $(\Sigma', \partial\Sigma')$, a simple map $v : (\Sigma', \partial\Sigma') \to (X, L)$ and a surjective map $p : (\Sigma, \partial\Sigma) \to (\Sigma', \partial\Sigma')$ of degree > 1, continuous on Σ, holomorphic on $int(\Sigma)$, satisfying

$$p^{-1}(\partial\Sigma) = \partial\Sigma \quad \text{and} \quad u = v \circ p.$$

In case $\partial\Sigma = \emptyset$, [2, Proposition 2.5.1] states that any J-holomorphic curve is either simple or multiply-covered. For surfaces with boundary the situation is more complicated. For generic almost complex structures, [7, Theorem B] gives a similar result:

Theorem 1.6. *Assume* $\dim X \geq 6$. *Then there exists a set* $\mathcal{J}_0$ *of second category in* $\mathcal{J}$ *such that for any* $J \in \mathcal{J}_0$*, any nonconstant* J*-holomorphic curve* $u : (D, \partial D) \to (X, L)$ *is either simple or multiply-covered.*

We may hope to get a similar result for generic Lagrangians. Namely, that under appropriate restrictions on X, for fixed $J \in \mathcal{J}$ and a Lagrangian submanifold L, there exists a set $\mathcal{L}_0$ of second category in $\{\phi(L) \,|\, \phi \text{ is a Hamiltonian isotopy}\}$ so that for every $L' \in \mathcal{L}_0$, any nonconstant J-holomorphic curve $u : (D, \partial D) \to (X, L')$ is either simple or multiply covered.

1.1.1. *The vertical differential –* D_u*.* The definition here is absolutely general, so we formulate it for any Σ, although, clearly, we only mind about disks. Let

$$\mathcal{B} = \big\{u : (\Sigma, \partial\Sigma) \to (X, L) \big| u \text{ is simple and } [u] = A\big\} \subset C^\infty(\Sigma, X).$$

It has a structure of a Fréchet manifold with tangent space

$$T_u\mathcal{B} = \mathcal{A}^0_{u^*TL}(\Sigma, u^*TX) \tag{3}$$

– the space of smooth vector fields along u with boundary conditions in TL.

Define now the bundle $\mathcal{E} \to \mathcal{B}$ to have a fiber

$$\mathcal{E}_u = \mathcal{A}^{0,1}(\Sigma, u^*TX)$$

(this time without specifying boundary conditions). Then there is a section $\mathcal{S} : \mathcal{B} \to \mathcal{E}$ defined by

$$\mathcal{S}(u) = (u, \bar{\partial}_J(u)).$$

Choose a connection on TX. For a map $u' \in \mathcal{B}$ sufficiently close to a fixed map u, we can uniquely write $u' = \exp_u(\xi)$. One can identify u'^*TX with u^*TX fiber-wise by parallel transport along the geodesic $\exp_u(t\xi)$. Taking a connection that preserves J, this defines an isomorphism

$$\mathcal{A}^{0,1}(u^*TX) \xrightarrow{\sim} \mathcal{A}^{0,1}(u'^*TX).$$

This essentially gives a local trivialization of $\mathcal{E}$, therefore defines a splitting of $T\mathcal{E}$. D_u is defined as the vertical part of $d\mathcal{S}$ with respect to this splitting. More precisely, if

$$\pi_u : T_{\mathcal{S}(u)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u \to \mathcal{E}_u$$

is the projection on the vertical space, then D_u is given by the composition

$$D_u = D_{u,J} : \mathcal{A}^0_{u^*TL}(\Sigma, u^*TX) \xrightarrow{d\mathcal{S}(u)} T_u\mathcal{B} \oplus \mathcal{E}_u \xrightarrow{\pi_u} \mathcal{E}_u.$$

Remark 1.7. When J is integrable, by [2, Remark 3.1.2] D_u is locally given by $\bar{\partial}$ and is therefore a $\mathbb{C}$-linear Cauchy-Riemann operator in the sense specified in Definition 2.4. This remains true with the simplicity condition removed.

1.1.2. *The space of simple disks.* Define

$$\begin{aligned}\mathcal{M}^*(A;L;J) &= \mathcal{M}^*(A; D, \partial D; X, L; J)\\ &= \left\{ u \in C^\infty\Big((D,\partial D),(X,L)\Big) \,\middle|\, \begin{array}{c} J \circ du = du \circ j,\ [u] = A \\ u \text{ is simple} \end{array} \right\}\end{aligned}$$

the space of simple J-holomorphic disks representing A. Note that

$$\mathcal{M}^*(A; L; J) = \mathcal{S}^{-1}(0) \subset \mathcal{B}.$$

Define

$$\mathfrak{D} = \{\phi \in \mathrm{Diff}(X) \big| \phi \text{ is a Hamiltonian isotopy}\}.$$

Theorem 1.8 ([8, Theorem 1]). *There exists a dense subset $\mathfrak{D}^L_{reg} \subset \mathfrak{D}$ such that for any $\phi \in \mathfrak{D}^L_{reg}$ and any simple disk*

$$v : (D, \partial D) \longrightarrow (X, \phi(L))$$

D_v is onto.

Call L regular if $L = \phi(L')$ for some Fukaya L' and $\phi \in \mathfrak{D}^{L'}_{reg}$. Note that the horizontal part of $d\mathcal{S}$ is always onto, so D_v being onto for every $v \in \mathcal{S}^{-1}(0)$ means that $\mathcal{S}$ is transverse to the zero section, and therefore $\mathcal{M}^*(A; L; J)$ is a smooth manifold, for any regular L. Its dimension is given by

$$\dim \mathcal{M}^*(A; L; J) = \operatorname{ind} D_v = n\chi(\Sigma) + \mu(u^*TX, u^*TL)$$

(see formula (5)). In our case, where $\Sigma = D$, X is assumed to be 3-dimensional and L is Fukaya, we have

$$\dim \mathcal{M}^*(A; L; J) = 3 \cdot 1 + 0 = 3. \tag{4}$$

1.2. **Gromov compactness for disks.** We follow the approach of [9].

Use $T = (T, E)$ to denote a tree (a connected graph with no cycles), where T stands for the set of vertices and E stands for edges. Specifically, we write $\alpha E \beta$ when there is an edge between α and β, vertices in T.

Definition 1.9. A J-holomorphic **stable map** of genus zero with one boundary component in L modelled over T is a tuple

$$(\mathbf{u}) = \big(\{(\Sigma_\alpha, \Gamma_\alpha, u_\alpha)\}_{\alpha \in T}, \{z_{\alpha\beta}\}_{\alpha E \beta}\big)$$

where Σ_α is either S^2 or D, $\partial\Sigma_\alpha \subset \Gamma_\alpha \subset \Sigma_\alpha$, $u_\alpha : (\Sigma_\alpha, \Gamma_\alpha) \to (X, L)$ is a J-holomorphic map and $z_{\alpha\beta} \in \Sigma_\alpha$. The set of **nodal points** is

$$Z_\alpha = \begin{cases} \{z_{\alpha\beta} \big| \alpha E \beta\} & \text{if } \Sigma_\alpha = D \\ \{z_{\alpha\beta} \big| \alpha E \beta\} \cup \Gamma_\alpha & \text{if } \Sigma_\alpha = S^2. \end{cases}$$

and the **boundary tree** is

$$\partial T = \{\alpha \in T \big| \Gamma_\alpha \neq \emptyset\}.$$

The following conditions are required to hold:

(1) If $\Sigma_\alpha = D$, then $\Gamma_\alpha = \partial D$.
 If $\Sigma_\alpha = S^2$, then Γ_α is either empty or consists of one point.
(2) $\forall \alpha, \beta \in T,\ \alpha E \beta \Rightarrow u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$.
(3) If $\alpha E \beta$ and $\alpha E \gamma$ for $\beta \neq \gamma$ then $z_{\alpha\beta} \neq z_{\alpha\gamma}$.
 If $\Sigma_\alpha = S^2$, then $z_{\alpha\beta} \notin \Gamma_\alpha$ for $\alpha E \beta$.
(4) If $\alpha E \beta$, then $z_{\alpha\beta} \in \partial\Sigma_\alpha \iff z_{\beta\alpha} \in \partial\Sigma_\beta$.
(5) If u_α is constant, then if $\Sigma_\alpha = S^2$, $\#Z_\alpha \geq 3$. If $\Sigma_\alpha = D$, Z_α consists either of at least three elements or of two elements not both in ∂D.
(6) ∂T is a nonempty subtree of T and If $\alpha \in \partial T$ with $\Sigma_\alpha = S^2$, then $\#\partial T = 1$.

Intuitively, one should think of a stable map as a tree of bubbles. Choose one disk as a "root vertex". Some J-holomorphic spheres might bubble off from an interior point of it (second statement of condition (3)), or some J-holomorphic disks from a boundary point (condition (4)). Nodal points are those where the bubbles bubble off from or, if you please, those where bubbles are glued together (condition (2)). All nodal points are distinct (first assertion of condition (3)). Conditions (1) and (6) say that the boundary tree is either a connected branch on which only disks are modeled, or a single vertex for a J-holomorphic sphere – there is a J-holomorphic sphere with a boundary point. This is a degenerate case when a J-holomorphic sphere bubbles out of a J-holomorphic disk that collapses into a point. In this case we denote

$$\Gamma_\alpha = \{z_\alpha^\infty\}.$$

Condition (5) justifies the name "stable". Fixing enough points ensures us that there are only a finite number of automorphisms of a stable map. (We did not give a definition of automorphism. Intuitively it can be thought of as a result of applying an automorphism on the tree, perhaps rescaling the bubbles but respecting the nodal points.)

Definition 1.10. A sequence $\{u^\nu\}$ of J-holomorphic disks **Gromov converges** to a J-holomorphic stable map $(\mathbf{u})$ if there exists a collection $\{\varphi_\alpha^\nu\}_{\alpha\in T}$ of Möbius transformation such that the following holds.

(1) If $\Sigma_\alpha = D$, then ϕ_α^ν preserves D.
(2) If $\Sigma_\alpha = S^2$, then for every compact $K \subset S^2 \setminus \{z_\alpha^\infty\}$, for large enough ν, $\phi_\alpha^\nu(K) \subset D$.
(3) $\forall \alpha \in T$, $u^\nu \circ \varphi_\alpha^\nu$ converges to u_α uniformly on compact subsets of $\Sigma_\alpha \setminus Z_\alpha$.
(4) If $\beta \in T$ is such that $\alpha E \beta$, then

$$\sum_{\gamma \in T_{\alpha\beta}} E(u_\gamma) = \lim_{\epsilon\to 0}\lim_{\nu\to\infty} E\Big(u^\nu, \varphi_\alpha^\nu\big(B_\epsilon(z_{\alpha\beta})\big)\Big).$$

(5) If $\Gamma_\alpha = \{z_\alpha^\infty\}$, then

$$\lim_{\epsilon\to 0}\lim_{\nu\to\infty} E\Big(u^\nu \circ \varphi_\alpha^\nu, B_\epsilon(z_{\alpha\beta}) \cap (\varphi_\alpha^\nu)^{-1}(D)\Big) = 0.$$

(6) $(\varphi_\alpha^\nu)^{-1} \circ \varphi_\beta^\nu$ converges to $z_{\alpha\beta}$ uniformly on compact subsets of $\Sigma_\beta \setminus \{z_{\beta\alpha}\}$.

Theorem 1.11 (Gromov compactness, [9, Theorem 3.3])**.** *Let $u^\nu : (D, \partial D) \to (X, L)$ be a sequence of J-holomorphic disks with bounded energy. Then u^ν has a Gromov convergent subsequence.*

Defining a suitable notion of equivalence relation between stable maps, it is also possible to show uniqueness of the limit, up to equivalence [9, Theorem 3.4].

It is also possible to define the notion of Gromov convergence of stable maps. Then the Gromov compactness theorem holds for sequences of stable maps as well. On the space of stable maps Gromov convergence therefore defines a topology in which the space is compact.

Denote by $\overline{\mathcal{M}}(A;L;J)$ the closure of $\mathcal{M}^*(A;L;J)$ in the space of stable maps with the Gromov topology. Assume the existence of a set $\mathcal{L}_0$ of Lagrangians as described after Theorem 1.6. By definition, for every $L \in \mathcal{L}_0$, any J-holomorphic disk is either simple or multiply-covered. Therefore, the elements of $\overline{\mathcal{M}}(A;L;J) \setminus \mathcal{M}^*(A;L;J)$ can a-priori be of two kinds:

(1) Stable maps modeled over a nontrivial tree,
(2) Maps modeled over a tree with one vertex, that is, multiply covered disks.

The idea of proving Conjecture 1.3 is based on analyzing the compactification, as will be outlined here. Carrying out the idea should require some effort; we will not implement it in the current work. The assumptions are that X is a quintic threefold in $\mathbb{P}^4$, J is integrable and $L \in L_0$ is a regular Fukaya Lagrangian.

1.3. **Strong Clemens for disks.** In section 7 (Lemma 7.1), we prove that any simple disk with generic Lagrangian boundary conditions has normal sheaf $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (for notation, see Definition 4.4). This gives the additional statement in Conjecture 1.3, assuming simple disks are embedded. We only show them to be immersed. Oh and Zhu in [10] show embeddedness of simple holomorphic spheres; a similar idea should work here as well. This form of the normal bundle implies (Proposition 7.2) that the normal bundle of any multiply covered disk has no nontrivial holomorphic sections. On the other hand, it should be possible to show that whenever a multiply-covered map is a limit of a sequence of simple maps, its normal bundle does admit holomorphic sections; a similar result was proved in [11, Theorem 5.1] for the boundaryless case.

Concluding there are no multiply covered elements in the compactification, we are left only with the option of bubbling. In order to avoid bubbling as well, we will use the full power of the weak Clemens' conjecture (Conjecture 1.1). This is where we need X to be specifically a quintic threefold in $\mathbb{P}^4$.

By assumption, there is a finite number of holomorphic spheres in X. Since both spheres and disks are 2-dimensional, and the ambient space is 6 dimensional, generically they don't intersect. This means that no holomorphic spheres can bubble off.

Similar reasoning leads us to the conclusion that no disks can bubble off as well: the automorphism group of the disk has real dimension $\dim PSL_2(\mathbb{R}) = 3$. Therefore $\mathcal{M}^*(A; L; J)$ taken modulo rescaling is a 0-dimensional manifold. In particular, its elements form a discrete set. ∂D is 1-dimensional, and it lives in L that is 3 dimensional, so generically the boundaries don't intersect each other. In order to obtain general position here, we might need to change the original Lagrangian. This is possible due to Theorem 1.8.

It follows that, taking the compactification of $\mathcal{M}^*(A; L; J)$, no elements need to be added. That is, $\mathcal{M}^*(A; L; J)$ is a compact space itself. Being a 0-dimensional manifold, it means that it is finite. This completes the proof of Conjecture 1.3.

2. Holomorphic vector bundles with boundary conditions

2.1. **Smooth extensions.**

In the literature, two definitions for a function being C^m on a closed region exist. One is similar to what we introduced in 0.0.1: the function is C^m in the interior, with derivatives of order $\leq m$ continuously extendable to the boundary. The other says a function is C^m if it is extendable to a C^m function on some open neighbourhood of the region. This is equivalent to being extendable to the whole space, because we could multiply the extended function by a cutoff function that is 1 on the region and 0 outside the neighbourhood.

It is well known (see, e.g., [12, Lemma 6.3.7]) that for $m < \infty$ the two notions are equivalent, given the boundary is C^m. Whitney in [13] extends the notion of differentiability to functions defined on closed sets that are not regions, and proves that any function that is C^m (for m either finite or infinite) in his sense is extendable, imposing no conditions whatsoever on the set or its boundary. In particular, this implies equivalence of the two definitions of smoothness for closed regions, as we explain in the current subsection.

Let A be any closed subset of $\mathbb{R}^n$, and f a function defined on it. In the following, we use multi-index notation.

Definition 2.1. Let $m \in \mathbb{Z}_{\geq 0}$, and f_k be functions defined on A for all multi-indices k such that $|k| \leq m$. We say $f = f_0$ is of class C^m in A in terms of the functions f_k if for all k,

$$f_k(x') = \sum_{|l| \leq m - |k|} \frac{f_{k+l}(x)}{l!}(x' - x)^l + R_k(x'; x)$$

where $R_k(x';x)$ is required to have a uniform boundness property:

$$\forall x_0 \in A\ \forall \epsilon > 0\ \exists \delta > 0 \quad s.t.$$
$$x, x' \in A \text{ and } d(x, x_0), d(x', x_0) < \delta \Rightarrow |R_k(x';x)| \leq d(x, x')^{m-|k|}\epsilon.$$

We say f is of class C^∞ in A if it is of class C^m for all $m \in \mathbb{N}$.

With this definition of differentiability, we have the following result:

Theorem 2.2. *([13, Theorem I]) With the above notation, if $f(x)$ is of class C^m in A in terms of $f_k(x)$, then there is a function $F(x)$ in $\mathbb{R}^n$, C^m in the ordinary sense, such that*

(1) $F|_A = f$,
(2) $\frac{\partial^k}{\partial x^k} F|_A = f_k$.

We claim that this result implies the following–

Corollary 2.3. *Let A be a closed region, f a smooth function on it. Then f is smoothly extendable to the whole space.*

Here by "smooth" we mean smoothness in the sense specified in 0.0.1. That is, f is smooth in the interior and all its partial derivatives are continuously extendable to the boundary.

Proof. If suffices to show that any smooth function is Whitney-smooth in terms of its partial derivatives. This will be done by direct computation.

Let $f_k = \frac{\partial^k}{\partial x^k} f$ be the k-th partial derivative of f for each multi-index k. Fix $m \in \mathbb{Z}_{\geq 0}$.

Define the remainder term $R_k(x';x)$ by

$$f_k(x') = \sum_{|l| \leq m-|k|} \frac{f_{k+l}(x)}{l!}(x'-x)^l + R_k(x';x).$$

For $x' \in int(A)$, Taylor's theorem gives the expression

$$R_k(x';x) = \sum_{|j|=m-|k|+1} R^j(x';x)(x'-x)^j$$

with the bound

$$R^j(x';x) \leq \sup_{y \in B(x)} \left| \frac{1}{(m-|k|+1)!} \frac{\partial^j f(y)}{\partial x^j} \right|,$$

where $B(x)$ is a ball around x on the closure of which f is defined, and $x' \in B(x)$. Note that $|(x'-x)^j| \leq d(x,x')^{|j|}$.

The uniform bound on the remainder follows from the boundedness of the $(m+1)$-st derivatives. Consider $k=0$. For each x_0 and $\epsilon>0$, take δ so small that $B=B_{\delta/2}(x_0)\subset A$, and for every j with $|j|=m+1$

$$\delta\cdot\sup_B|f_j|\leq(m+1)!\frac{\epsilon}{n^{m+1}}.$$

For any $x,x'\in B$ we have $d(x,x')<\delta$, and

$$\begin{aligned}\left|R_0(x';x)\right|&=\Big|\sum_{|j|=m+1}R^j(x')(x'-x)^j\Big|\\&\leq\Big|\sum_{|j|=m+1}R^j(x')\Big|d(x,x')^{m+1}\\&\leq\sum_{|j|=m+1}\sup_B\left|\frac{1}{(m+1)!}\frac{\partial^jf(y)}{\partial x^j}\right|\cdot\delta\cdot d(x,x')^m\leq\epsilon\cdot d(x,x')^m.\end{aligned}$$

A similar argument works for all R_k.

To see that the conditions hold on the closed set A, it suffices now to verify the boundedness condition of the remainder around boundary points.

Let $x_0\in\partial A$ and fix $\epsilon>0$.

Again, for $k=0$, choose a δ neighbourhood of x_0 in A so that $\delta\cdot\sup|f_{m+1}|\leq(m+1)!\frac{\epsilon}{2\cdot n^{m+1}}$. Fix then $x\neq x'$ in the neighbourhood, and take a sequence $x_j\in int(A)$ converging to x'.

The finite sum $\sum_{|l|\leq m}\frac{f_l(x)}{l!}(x_j-x)^l$ converges to $\sum_{|l|\leq m}\frac{f_l(x)}{l!}(x'-x)^l$, therefore the reminder term converges as well.

As before, we get $|R_0(x_j;x)|\leq\epsilon\cdot\frac{d(x_j,x)^m}{2}$ (whenever x_j is in the chosen neighbourhood). Since $d(x_j,x)$ converges to $d(x',x)$, for large enough j we have $d(x_j,x)^m/2\leq d(x',x)^m$. It follows that $|R_0(x',x)|\leq\epsilon\cdot d(x',x)^m$, as desired.

A similar consideration works for every R_k. Therefore f is of class C^{m+1} in A in terms of its partial derivatives. Since this is true for any m, f is Whitney-smooth on A. □

2.2. **Cauchy-Riemann operators.**

Definition 2.4. A **$\mathbb{C}$-linear smooth Cauchy-Riemann** (CR) **operator** on a bundle $E\to\Sigma$ is a $\mathbb{C}$-linear operator

$$D:\mathcal{A}^0(\Sigma,E)\to\mathcal{A}^{0,1}(\Sigma,E)$$

which satisfies the Leibnitz rule:

$$D(f\xi)=f(D\xi)+(\bar\partial f)\xi$$

for $\xi\in\mathcal{A}^0(\Sigma,E),f\in\mathcal{A}^0(\Sigma)$.

Such an operator D extends uniquely to

$$D : \mathcal{A}^{p,q}(\Sigma, E) \to \mathcal{A}^{p,q+1}(\Sigma, E)$$

that satisfies the Leibniz rule.

Lemma 2.5. *Given a holomorphic structure on a bundle E, there exists a unique CR operator D on E that annihilates local holomorphic sections. Moreover, there exists a connection ∇ on E such that $D = \nabla^{0,1}$.*

Proof. Take a locally finite open cover of the surface with holomorphic trivializations on it, $\{U_\alpha, \varphi_\alpha\}$. Take then a partition of unity subordinate to this cover, $\{\psi_\alpha\}$. Consider $\nabla = \sum \psi_\alpha \cdot \varphi_\alpha^* d$. Then $D = \nabla^{0,1} = \sum \psi_\alpha \cdot \varphi_\alpha^* \bar{\partial}$ is the desired CR operator.

Now, let ξ_j be a local holomorphic frame of E. Let $s = \sum s^j \xi_j$ be a local section. Then

$$Ds = \sum D(s^j \xi_j) = \sum s^j D(\xi_j) + \sum \bar{\partial} s^j \cdot \xi_j = \sum \bar{\partial} s^j \cdot \xi_j.$$

The last expression does not depend on D, therefore D is unique. □

The converse is true as well: any CR operator defines a holomorphic structure on E. This follows from [14, Theorem 2.1.53]:

Lemma 2.6. *A CR operator D on a smooth complex vector bundle over a complex manifold M arises from a holomorphic structure if and only if $D^2 = 0$.*

Since in our discussion M is a surface, the condition $D^2 = 0$ holds trivially ($\mathcal{A}^{0,2} = 0$). This deals explicitly with the case when $\partial\Sigma = \emptyset$. To see that this continues to hold for the case of nonempty boundary, we will use Corollary 2.3.

Proposition 2.7. *Let E be a smooth vector bundle over Σ, and D a CR operator on E. Then around any point of Σ there exists a trivialization where D is given by $\bar{\partial}$.*

Proof. Due to Lemma 2.6, we only need to verify this for $p \in \partial\Sigma$.

Let U be a neighbourhood of p identified with a subset of HD and

$$\varphi : U \times \mathbb{C}^n \longrightarrow E\big|_U$$

a smooth trivialization of E on U. Restrict φ to a subset V of U that is closed in $\mathbb{C}$. Then $\varphi^* D = \bar{\partial} + \alpha$ where α is a matrix of $(0,1)$-forms. By Corollary 2.3, it is possible to extend the coefficients of α to the whole plane. Denote the resulting operator $\hat{\alpha}$. This defines a new, trivial bundle

$$\hat{E} = \mathbb{C} \times \mathbb{C}^n$$

with the smooth CR operator $\bar{\partial} + \hat{\alpha}$ on it. By Proposition 2.6, there exists a neighbourhood W of p in $\mathbb{C}$ – assume $W \cap \mathbb{H} \subset V$ – and a trivialization

$$\psi : W \times \mathbb{C}^n \longrightarrow \hat{E}\big|_W$$

such that $\psi^*(\bar{\partial} + \alpha) = \bar{\partial}$. Then

$$\varphi \circ \psi : V \cap W \times \mathbb{C}^n \longrightarrow E\big|_{V \cap W}$$

is a trivialization around p with $(\varphi \circ \psi)^* D = \psi^*(\bar{\partial} + \alpha) = \bar{\partial}$. $\square$

We would like now to extend a smooth CR operator to a larger space of sections, with the benefit of working in a Banach rather than a Fréchet space.

Definition 2.8. Let E be a bundle over Σ, h a metric on E, ∇ a metric connection and g a Riemannian metric on Σ. The space $W^{l,p}_{\nabla}(\Sigma, E)$ of **Sobolev** (l,p)**-sections** is defined as the completion of $\mathcal{A}^0(\Sigma, E) = C^\infty(\Sigma, E)$ under the norm given by

$$\|\xi\|_{l,p;\nabla} = \left(\sum_{k \le l} \int_\Sigma |\nabla^k \xi|^p \right)^{1/p}.$$

for $p < \infty$, and

$$\|\xi\|_{l,p;\nabla} = \sum_{k \le l} \sup_\Sigma |\nabla^k \xi|$$

when $p = \infty$.

Here $|\nabla^k \xi|$ at $z \in \Sigma$ is the operator norm with respect to g and h of the multilinear operator

$$(\nabla^k \xi)(z) : T_z\Sigma^{\otimes k} \longrightarrow E_z.$$

Similarly, one can consider the completion under (l,p)-norm of the space of smooth E-valued tensors, $\mathcal{A}^0(\Sigma, T^*\Sigma^{\otimes t} \otimes E)$. The metrics g and h induce a metric on $T^*\Sigma^{\otimes t} \otimes E$, and a connection ∇ on E together with the Levi-Civita connection on $T\Sigma$ induce a connection on $T^*\Sigma^{\otimes t} \otimes E$, that we still denote by ∇, which is still metric. Hence we can define an (l,p)-norm on tensors replacing E in the above definition with $T^*\Sigma^{\otimes t} \otimes E$.

Denote the resulting space by $W^{l,p}_{\nabla,t}(\Sigma, E)$. We include the case of sections of E in the notation setting $W^{l,p}_{\nabla,0}(\Sigma, E) = W^{l,p}_{\nabla}(\Sigma, E)$.

Proposition 2.9. *The space* $W^{l,p}_{\nabla,t}(\Sigma, E)$ $(t \ge 0)$ *does not depend on the choice of metrics or connection.*

Lemma 2.10. *Let $A \in C^\infty(\Sigma, \mathrm{End}(E))$, $\xi \in C^\infty(\Sigma, E)$. Then there exists a constant c_0 depending on* $\mathrm{rk}\, E$, *p and l such that*

$$\|A\xi\|_{l,p;\nabla} \leq c_0 \|A\|_{l,\infty;\nabla} \cdot \|\xi\|_{l,p;\nabla}.$$

Proof. Note that, since the connection satisfies Leibnitz rule,

$$\nabla^k(A\xi) = \sum_j \binom{k}{j} (\nabla^j A)(\nabla^{k-j}\xi).$$

Also, note that by the Cauchy-Schwarz inequality we have

$$|\nabla^j A \nabla^{k-j}\xi| \leq |\nabla^j A| \cdot |\nabla^{k-j}\xi|.$$

Therefore

$$\begin{aligned} |\nabla^k(A\xi)|^p &= |\sum_j \binom{k}{j} (\nabla^j A)(\nabla^{k-j}\xi)|^p \\ &\leq k^p \sum_j \binom{k}{j}^p |\nabla^j A|^p \cdot |\nabla^{k-j}\xi|^p \leq c_1 \sum_j |\nabla^j A|^p \cdot |\nabla^{k-j}\xi|^p. \end{aligned}$$

It follows that

$$\begin{aligned} \|A\xi\|^p_{l,p;\nabla} &= \int_U \sum_{k\leq l} |\nabla^k(A\xi)|^p \\ &\leq \int_U \sum_{k\leq l} c_1 \sum_{j\leq k} |\nabla^j A|^p \cdot |\nabla^{k-j}\xi|^p \\ &\leq c_0 \int_U \left(\sum_{j\leq l} \max_{\bar U} |\nabla^j A|^p \right) \left(\sum_{k\leq l} |\nabla^k \xi|^p \right) \\ &= c_0 \|A\|^p_{l,\infty;\nabla} \|\xi\|^p_{l,p;\nabla}. \end{aligned}$$

□

Proof of Proposition 2.9. Assume two connections are given, ∇ and ∇'. We need to show that

$$W^{l,p}_{\nabla,t}(\Sigma, E) = W^{l,p}_{\nabla',t}(\Sigma, E) \quad \forall t \geq 0.$$

We prove by induction on l. For $l = 0$, neither the norm defined by ∇ nor the one defined by ∇' uses the connection. In particular, $(0, p)$-norm is independent of the choice of connection. Take now $l + 1 > 0$, and assume the claim is true for $k \leq l$ (and any t). We need to show

- There exists c such that for any smooth tensor ξ

$$\|\xi\|_{l+1,p;\nabla'} \leq c \cdot \|\xi\|_{l+1,p;\nabla}.$$

- There exists c such that for any smooth tensor ξ

$$\|\xi\|_{l+1,p;\nabla} \leq c \cdot \|\xi\|_{l+1,p;\nabla'}.$$

Step 1. *It is enough to show*

(1) There exists c such that for any smooth tensor ξ

$$\|\nabla'\xi\|^p_{l,p;\nabla'} + \|\xi\|^p_{l,p;\nabla'} \leq c \cdot \left(\|\nabla\xi\|^p_{l,p;\nabla} + \|\xi\|^p_{l,p;\nabla}\right).$$

(2) There exists c such that for any smooth tensor ξ

$$\|\nabla\xi\|^p_{l,p;\nabla} + \|\xi\|^p_{l,p;\nabla} \leq c \cdot \left(\|\nabla'\xi\|^p_{l,p;\nabla'} + \|\xi\|^p_{l,p;\nabla'}\right).$$

Note that

$$\begin{aligned} 2\|\xi\|^p_{l+1,p;\nabla} \geq\ & 2\|\xi\|^p_{l+1,p;\nabla} - \|\nabla^{l+1}\xi\|^p_{L^p} - \|\xi\|^p_{L^p} \\ &= \|\nabla\xi\|^p_{l,p;\nabla} + \|\xi\|^p_{l,p;\nabla} \end{aligned}$$

on the one hand, and on the other,

$$\begin{aligned} \|\nabla\xi\|^p_{l,p;\nabla} + \|\xi\|^p_{l,p;\nabla} &= \|\xi\|^p_{l+1,p;\nabla} + \left(\|\xi\|^p_{l+1,p;\nabla} - \|\nabla^{l+1}\xi\|^p_{L^p} - \|\xi\|^p_{L^p}\right) \\ &\geq \|\xi\|^p_{l+1,p;\nabla}. \end{aligned}$$

Same inequalities hold for the norm defined by ∇'. It follows that

$$\begin{aligned} \|\nabla'\xi\|^p_{l,p;\nabla'} + \|\xi\|^p_{l,p;\nabla'} \leq c\cdot & \left(\|\nabla\xi\|^p_{l,p;\nabla} + \|\xi\|^p_{l,p;\nabla}\right) \\ & \Rightarrow \|\xi\|_{l+1,p;d} \leq c_1 \cdot \|\xi\|_{l+1,p;\nabla} \end{aligned}$$

with $c_1 = (2c)^{1/p}$, and similarly for the second inequality.

Step 2. *Proof of* (1).

If we know (1), then (2) follows by symmetry.

By our assumption, there exists $c' \geq 1$ such that, for any smooth η (a tensor of arbitrary degree),

$$\|\eta\|^p_{l,p;\nabla'} \leq c' \cdot \|\eta\|^p_{l,p;\nabla}.$$

Write $\nabla = \nabla' + A$ for A a matrix of 1-forms. In any norm,

$$\|\nabla\xi\| \geq \|\nabla'\xi\| - \|A\xi\|.$$

Therefore

$$2^p(\|\nabla\xi\|^p + \|A\xi\|^p) \geq \|\nabla'\xi\|^p.$$

Choose c_0 from Lemma 2.10 so that $c_0 \geq 1$.

Set $c = 2^p c' c_0(1 + \|A\|^p_{l,\infty;\nabla'})$ to obtain

$$\begin{aligned}
c \cdot \left(\|\nabla\xi\|^p_{l,p;\nabla} + \|\xi\|^p_{l,p;\nabla}\right) &\geq 2^p c' \|\nabla\xi\|_{l,p;\nabla} + c'\|\xi\|^p_{l,p;\nabla} + \\
&+ 2^p c_0 \|A\|^p_{l,\infty;\nabla'} \cdot c'\|\xi\|^p_{l,p;\nabla} + c\|A\|^p_{l,\infty;\nabla'}\|\nabla\xi\|^p_{l,p;\nabla} \\
&\geq 2^p \|\nabla\xi\|^p_{l,p;\nabla'} + \|\xi\|^p_{l,p;\nabla'} + 2^p c_0 \|A\|^p_{l,\infty;\nabla'}\|\xi\|^p_{l,p;\nabla'} \\
&\geq 2^p(\|\nabla\xi\|^p_{l,p;\nabla'} + \|A\xi\|^p_{l,p;\nabla'}) + \|\xi\|^p_{l,p;\nabla'} \\
&\geq \|\nabla'\xi\|^p_{l,p;\nabla'} + \|\xi\|^p_{l,p;\nabla'}.
\end{aligned}$$

□

From now on, we may therefore write $W^{l,p}_t(\Sigma, E)$ and $W^{l,p}(\Sigma, E)$ without referring to a specific connection.

Given a CR operator on E, by Lemma 2.5 there exists a connection ∇ on E such that $D = \nabla^{0,1}$. Then

$$\|D\|_{l,p} = \|\frac{1}{2}(\nabla + J\nabla j)\|_{l,p} \leq \frac{1}{2}2\|\nabla\|_{l,p} = \|\nabla\|_{l,p}.$$

Since ∇ is bounded by definition of the norm, it follows that D is bounded as well. Therefore we can extend it to an operator on the Sobolev space:

$$D : W^{l,p}(\Sigma, E) \longrightarrow W^{l-1,p}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E).$$

2.3. **The Riemann-Roch theorem.**

Definition 2.11. Let J is a complex structure on E. We say $F \subset E|_{\partial\Sigma}$ is a **totally real subbundle** if $F \perp JF$ (on fibers) and F is of maximal (real) rank.

For such F, denote by D_F the restriction of D to the space of sections with boundary values in F:

$$D_F : W^{l,p}_F(\Sigma, E) \to W^{l-1,p}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$$

We will state now the Riemann-Roch theorem in the generality given in [2, Theorem C.1.10]:

Theorem 2.12 (Riemann-Roch)**.** *Let E be a complex vector bundle with $\mathrm{rk}_{\mathbb{C}} E = n$ over a compact Riemann surface with boundary and $F \subset E|_{\partial\Sigma}$ be a totally real subbundle. Let D be a Cauchy-Riemann operator on E.Then the following holds, for every integer k and $q > 1$:*

(1) *The operator*

$$D_F : W^{k,q}_F(\Sigma, E) \to W^{k-1,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$$

is Fredholm. Moreover, its kernel and cokernel are independent of k and q.

(2) *The real Fredholm index of* D_F *is given by*

$$\operatorname{ind}(D_F) = n\chi(\Sigma) + \mu(E, F), \tag{5}$$

where $\chi(\Sigma)$ *is the Euler characteristic of* Σ*, and* $\mu(E, F)$ *is the boundary Maslov index (see sec. 3).*

(3) *If* $n = 1$*, then*

$$\mu(E, F) < 0 \Rightarrow \operatorname{Ker}(D_F) = 0,$$

$$\mu(E, F) + 2\chi(\Sigma) > 0 \Rightarrow \operatorname{Coker}(D_F) = 0.$$

3. The Maslov index

Recall the notion of Maslov index for loops of totally real spaces in $\mathbb{C}^n$ (see, e.g., [15] for definition and properties):

Denote by $\mathcal{T}(n) = GL(\mathbb{C}^n)/GL(\mathbb{R}^n)$ the manifold of totally real subspaces of $\mathbb{C}^n$. Let $\tau \in \Omega\mathcal{T}(n)$ be a continuous loop of totally real spaces. Suppose $\tau(z) = a(z) \cdot GL(\mathbb{R}^n)$. Define

$$\rho : \mathcal{T}(n) \longrightarrow U(n)$$

by

$$a \cdot GL(\mathbb{R}^n) \mapsto \left(\frac{a}{\sqrt{a^H a}}\right)^2,$$

the map giving each matrix the square of its unitary part. Although a is generally a path, $\det(\rho(a(z)))$ depends only on the space $a(z) \cdot GL(\mathbb{R}^n)$, hence $\det(\rho(a))$ represents a loop. Then the Maslov index of the loop is given by

$$\mu(\tau) = \deg(\det(\rho \circ \tau)).$$

Equivalently, if $\alpha : [0, 2\pi] \to \mathbb{R}$ is a lift of $\det(\rho \circ \tau)$ given by

$$\det(\rho \circ \tau(t)) = e^{i \cdot \alpha(t)},$$

then the Maslov index satisfies

$$\mu(\tau) = \frac{\alpha(2\pi) - \alpha(0)}{2\pi}. \tag{6}$$

The index μ classifies homotopy classes of loops.

For a disjoint union of loops, the index is defined as the sum of the indices on each loop.

We refer to [2, app.C.3] for the notions discussed below.

Definition 3.1. A **decomposition of a** compact Riemann **surface** Σ_{02} is a pair of sub-surfaces Σ_{01}, Σ_{12} such that

$$\Sigma_{02} = \Sigma_{01} \cup \Sigma_{12}, \quad \Sigma_{01} \cap \Sigma_{12} = \partial\Sigma_{01} \cap \partial\Sigma_{12}.$$

The boundary of the components is a disjoint union of circles, some of them common, and the rest are boundary components of the original surface.

Definition 3.2. A **decomposition of a bundle pair** (E, F) over Σ_{02} is a pair of bundles $(E_{01}, F_0 \cup F_1)$ over $(\Sigma_{01}, \partial\Sigma_{01})$ and $(E_{12}, F_1 \cup F_2)$ over $(\Sigma_{12}, \partial\Sigma_{12})$ for Σ_{01}, Σ_{12} a decomposition of Σ_{02}.

Here F_1 is the part of the boundary conditions over the common boundary of Σ_{01}, Σ_{12}.

By slight abuse of notation, we write

$$(E_{02}, F_{02}) = (E_{01}, F_0 \cup F_1) \cup (E_{12}, F_1 \cup F_2).$$

Theorem 3.3 ([2, Theorem C.3.5])**.** *There is a unique operation, called **the boundary Maslov index**, that assigns an integer $\mu(E, F)$ to each bundle pair (E, F) and satisfies the following axioms:*

- ***Isomorphism:*** *If $\Phi : E_1 \to E_2$ is a vector bundle isomorphism covering a diffeomorphism $\phi : \Sigma_1 \to \Sigma_2$, then*
$$\mu(E_1, F_1) = \mu(E_2, \Phi(F_1)).$$
- ***Direct sum:***
$$\mu(E_1 \oplus E_2, F_1 \oplus F_2) = \mu(E_1, F_1) + \mu(E_2, F_2).$$
- ***Composition:*** *For a decomposition*
$$(E_{02}, F_{02}) = (E_{01}, F_0 \cup F_1) \cup (E_{12}, F_1 \cup F_2)$$
over $\Sigma_{02} = \Sigma_{01} \cup \Sigma_{12}$, we have
$$\mu(E_{02}, F_{02}) = \mu(E_{01}, F_0 \cup F_1) + \mu(E_{12}, F_1 \cup F_2).$$
- ***Normalization:*** *For $\Sigma = D$ and the trivial bundle $\mathcal{E}$ with boundary conditions $\Lambda_{e^{i\theta}} = e^{ik\theta/2}\mathbb{R}$, we have*
$$\mu(\mathcal{E}, \Lambda) = k.$$

The following holds [2, Theorem C.3.6]:

Theorem 3.4. *The boundary Maslov index satisfies the following:*

- *If $\partial\Sigma \neq \emptyset$ and $E = \mathcal{E} = \Sigma \times \mathbb{C}^n$, then*
$$\mu(\mathcal{E}, \Lambda) = \mu(\Lambda)$$
Here we view the last Λ as a loop of totally real spaces defined by $\Lambda(e^{i\theta}) = \Lambda_{e^{i\theta}}$.
- *If $\partial\Sigma = \emptyset$, then*
$$\mu(E, \emptyset) = 2c_1(E)([\Sigma]).$$

Proposition 3.5. *If $\partial\Sigma \neq \emptyset$, then any complex bundle E over Σ is smoothly trivial.*

We would like to use the following fact [16, Theorem 1.6]:

Lemma 3.6. *Given a vector bundle $p : E \to B$ and homotopic maps $f_0, f_1 : A \to B$, the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if A is paracompact and Hausdorff.*

Two more results are needed.

Lemma 3.7. *Any complex bundle E over S^1 is smoothly trivial.*

Proof. Each copy of S^1 can be covered by exactly two contractible open sets (neighbourhoods of the hemispheres). Being smoothly contractible, the identity map on each of these sets is homotopic to the constant map. Therefore, by Lemma 3.6, any bundle over S^1 restricted to each of these sets is trivial. The original bundle, on all of S^1, is then given by a single transition function, defined on a set homotopic to S^0. That is, it assigns matrix values, say A_1 and A_2 on two points. $GL_k(\mathbb{C})$ is path connected for all k. In particular, there exist smooth paths connecting A_j to $\mathrm{Id} \in GL_d(\mathbb{C})$. Therefore E is trivial. □

Lemma 3.8. *Let E_1 and E_2 be smooth bundles over a manifold with boundary M. Let $f : E_1 \to E_2$ be a continuous isomorphism of the bundles. Then there exists a smooth isomorphism $g : E_1 \to E_2$.*

Proof. Take a locally finite cover of Σ by open sets $\{U_\alpha\}$ on which both E_1 and E_2 are trivial. Then

$$f_\alpha : E_1\big|_{U_\alpha} \longrightarrow E_2\big|_{U_\alpha}$$

can be identified with

$$f_\alpha : U_\alpha \longrightarrow GL_n(\mathbb{C}).$$

For every α, take a smooth approximation g_α of f_α so close that $\mathrm{Im}(g_\alpha) \subset GL_n(\mathbb{C})$ (that is, g_α is a smooth isomorphism between $E_1\big|_{U_\alpha}$ and $E_2\big|_{U_\alpha}$). Take also a smooth partition of unity $\{\psi_\alpha\}$ subordinate to $\{U_\alpha\}$. Set

$$g = \sum_\alpha \psi_\alpha g_\alpha : E_1 \longrightarrow E_2.$$

Since we know $\sum \psi_\alpha f_\alpha = f$ is an isomorphism, we can choose g_α close enough to f_α so that g is an isomorphism as well. □

Proof of Proposition 3.5. Since $\partial\Sigma \neq \emptyset$, Σ is homotopic to a wedge sum of circles; that is, the identity map is continuously homotopic to a map that takes Σ to a wedge sum of copies of S^1. By Lemma 3.7, any bundle over S^1 is trivial. Lemma 3.6 then states that every bundle over Σ is continuously isomorphic to the trivial one. By Lemma 3.8 there exists a smooth trivialization as well. □

Proposition 3.9. *Let* $(E_1, F_1), (E_2, F_2)$, *be bundles over* $(\Sigma, \partial\Sigma)$. *Then*

$$\mu(E_1 \otimes E_2, F_1 \otimes F_2) = \mu(E_1, F_1) rk E_2 + \mu(E_2, F_2) rk E_1.$$

Proof. **Case 1.** $\partial\Sigma \neq \emptyset$.

We begin with a linear algebra remark. Let $A = (a_{jk}) \in GL(\mathbb{C}^n)$ and $B = (b_{jk}) \in GL(\mathbb{C}^m)$ be represented with respect to the bases $v_1, ..., v_n$, and $w_1, ..., w_m$, respectively. Then $A \otimes B \in GL(\mathbb{C}^n \otimes \mathbb{C}^m)$ with respect to the basis $v_j \otimes w_k$ is given by the block matrix $(a_{jk}B)$. Note the following fact:

Lemma 3.10 ([17, Theorem 1])**.** *Let* F *be a field,* R *a commutative subring of* $F^{n\times n}$, *the* $n \times n$ *matrices over* F. *Let* $M \in R$. *Then*

$$\det_F M = \det_F(\det_R M).$$

In our case all blocks commute, therefore

$$\begin{aligned} \det(A \otimes B) &= \det(\sum_{\sigma\in S_n} \prod_{j=1}^{n} (a_{j\sigma(j)}B)) \\ &= \det((\sum \prod a_{j\sigma(j)}) \cdot B^n) \\ &= \det((\det A) \cdot B^n) \\ &= (\det A)^m (\det B)^n. \end{aligned} \tag{7}$$

Now, by Lemma 3.5, both E_1 and E_2 are trivial. This, together with Theorem 3.4, allows us to use (7) to compute the index. Let $U_j \in U(n)$ be paths such that $F_z = U_j(z)\mathbb{R}^n$. Then

$$\begin{aligned} \mu(E_1 \otimes E_2, F_1 \otimes F_2) &= \deg(\det(U_1^2 \otimes U_2^2)) \\ &= \deg(\det U_1^{2\,\mathrm{rk}_{\mathbb{R}} U_2} \cdot \det U_2^{2\,\mathrm{rk}_{\mathbb{R}} U_1}) \\ &= \deg(\det U_1^2) \cdot \mathrm{rk}_{\mathbb{C}} E_2 + \deg(\det U_2^2) \cdot \mathrm{rk}_{\mathbb{C}} E_1 \\ &= \mu(E_1, F_1)\, \mathrm{rk}_{\mathbb{C}} E_2 + \mu(E_2, F_2)\, \mathrm{rk}_{\mathbb{C}} E_1. \end{aligned}$$

Case 2. $\partial\Sigma = \emptyset$.

We need to show that

$$c_1(E_1 \otimes E_2) = c_1(E_1)\, \mathrm{rk}\, E_2 + c_1(E_2)\, \mathrm{rk}\, E_1. \tag{8}$$

By the splitting principle, it is enough to consider line bundles. But then equation (8) reduces to the well know statement

$$c_1(L^1 \otimes L^2) = c_1(L^1) + c_1(L^2).$$

□

The Maslov index classifies bundles, in the following sense:

Theorem 3.11 ([2, Theorem C.3.7])**.** *Two bundle pairs* (E_1, F_1) *and* (E_2, F_2) *over the same manifold* Σ *are isomorphic (over the identity) if and only if* E_1 *and* E_2 *have the same rank, same Maslov index and the restrictions of* $F_j, j = 1, 2$ *to each boundary component are isomorphic.*

The last condition merely reflects orientability of the F_j on boundary components.

4. Birkhoff factorization for disks

Theorem 4.1 (Birkhoff factorization)**.** *Let* (E, F) *be a holomorphic vector bundle over* $(D, \partial D)$ *of rank* k*. Then there exist holomorphic line bundles* $(E_1, F_1) ..., (E_k, F_k)$*, over* D *so that* $(E, F) \simeq \oplus_{j=1}^{k}(E_j, F_j)$*. This factorization is unique up to the order of* (E_j, F_j)*.*

In the proof of existence we will follow Grothendieck's treatment for spheres, as in [3], closely. However, the boundary conditions require additional care.

4.1. **Line bundles over the disk – Classification.**

Lemma 4.2. *Every line bundle over the disk is holomorphically trivial.*

Proof. By Lemma 3.5, any bundle over D is smoothly isomorphic to the trivial bundle. Alternatively, it is an immediate consequence of Lemma 3.6, since D is smoothly contractible.

Now, given a line bundle E with the operator D^{CR} over the disk, take Φ to be a smooth trivialization. That is, $\Phi : \mathcal{E} \tilde{\to} E$ where $\mathcal{E}$ is the trivial bundle. We claim that there exists an isomorphism $\Psi : \mathcal{E} \tilde{\to} \mathcal{E}$ so that $\Psi^*(\Phi^* D^{CR}) = \bar{\partial}$ the standard operator.

Write $\Phi^* D^{CR} = \bar{\partial} + A d\bar{z}$.

Recall the following result (cf. e.g. [18, p. 25]):

Lemma 4.3 (The $\bar{\partial}$-Poincarét lemma)**.** *Let* $f \in C^\infty(\mathring{D})$*. Then there exists* $g \in C^\infty(\mathring{D})$ *such that* $\bar{\partial} g = f$*.*

Using Corollary 2.3, A can be smoothly extended to an open disk containing D. Apply Lemma 4.3 on this extension to conclude the existence of $B : D \to End(\mathbb{C}) = \mathbb{C}$ such that $\bar{\partial} B = -A$ on D. Let Ψ be multiplication by $\exp(B)$.

Note that our case is one-dimensional, therefore everything commutes. Then

$$\begin{aligned}
\Psi^*(\Phi^* D^{CR})\xi &= \Psi^*(\bar\partial + Ad\bar z)\xi \\
&= \exp(-B)(\bar\partial + Ad\bar z)\exp(B)\xi \\
&= \exp(-B)\big((\bar\partial(\exp B))\xi + \exp B\cdot\bar\partial\xi + (\exp B\cdot Ad\bar z)\xi\big) \\
&= \exp(-B)\big(-(\exp B\cdot Ad\bar z)\xi + \exp B\cdot\bar\partial\xi + (\exp B Ad\bar z)\xi\big) \\
&= \bar\partial\xi.
\end{aligned}$$

□

Having this result, classification of line bundles amounts to understanding the boundary conditions. Not very much surprisingly, the Maslov index supplies a complete answer to this problem.

Definition 4.4. For every $\nu \in \mathbb{Z}$, define

$$L_\nu = (\mathcal{E}, \Lambda_\nu)$$

the trivial bundle over the disk with boundary conditions

$$\Lambda_\nu(e^{i\theta}) = e^{i\theta\nu/2}\mathbb{R}.$$

Denote the sheaf of holomorphic sections of L_ν by

$$\mathcal{O}(L_\nu) = \mathcal{O}(\nu)$$

(cf. section 5).

Let us now quote a regularity lemma, [12, Theorem 6.19].

Lemma 4.5. *Let $0 \le \alpha \le 1$, $k \in \mathbb{Z}_{\ge 0}$. Let U be a closed region such that U is a $C^{k+2,\alpha}$ submanifold of $\mathbb{R}^n$. Let L be a strictly elliptic operator with coefficients in $C^{k,\alpha}(U)$, and $\phi \in C^{k+2,\alpha}(U), f \in C^{k,\alpha}(U)$. Suppose $u \in C^0(U) \cap C^2(\mathring{U})$ satisfies*

$$\begin{cases} Lu = f & \text{in } \mathring{U} \\ u = \phi & \text{on } \partial U. \end{cases}$$

Then $u \in C^{k+2,\alpha}(U)$.

Lemma 4.6. *Let $f \in C^\infty(\partial D, \mathbb{C}^\times)$ such that the winding number satisfies $win(f) = 0$. Then there exists $\rho \in C^\infty(\partial D, \mathbb{R}^\times)$ such that $\rho \cdot f$ can be extended to a holomorphic function on D.*

Proof. Since $win(f) = 0$, we may choose a branch of $\arg(f)$, and obtain a well defined function $g(\zeta) := \arg(f(\zeta))$.

Take G the harmonic extension of g to D, and let $-R$ be its harmonic conjugate in the interior. By Lemma 4.5, G is smooth in D up to the boundary. Since the derivatives of R are given in terms of the derivatives of G, we know R is smooth up to the boundary as well. Therefore

$$\tilde{g} = R + iG$$

is a well defined holomorphic function on D. Take

$$\tilde{f} = \exp(\tilde{g}).$$

Observe that for $\zeta \in \partial D$

$$\tilde{f}(\zeta) = e^{R(\zeta)} \cdot e^{i \arg(f(\zeta))} = \rho(\zeta) f(\zeta)$$

with $\rho(\zeta) = \frac{e^{R(\zeta)}}{|f(\zeta)|}$ a nonvanishing real valued function, as desired.

□

Proposition 4.7. *Let (E, F) be a holomorphic line bundle over the disk with $\mu(E, F) = \nu$. Then $(E, F) \simeq L_\nu$.*

Proof. By Lemma 4.2 we may assume (E, F) is of the form $(\mathcal{E}, \Lambda)$ with $\Lambda(z) = f(z)\mathbb{R}$. Then $win(f^2) = \nu$, and therefore $win((z^{-\nu/2} f(z))^2) = 0$. It follows that $win(z^{-\nu/2} f(z)) = 0$. By Lemma 4.6, we can multiply $z^{-\nu/2} f(z)$ by a real valued nonvanishing function so that the result is holomorphically extendable to the disk. The obtained function is nonvanishing in the interior, because the winding number on the boundary is zero. Therefore, $z^{-\nu/2} f(z)$ defines trivial boundary conditions, that is, the bundle L_0. It follows that

$$z^{-\nu/2} f(z)\mathbb{R} = \mathbb{R}$$

or, equivalently,

$$\Lambda(z) = f(z)\mathbb{R} = z^{\nu/2}\mathbb{R}.$$

□

Remark 4.8. Denote by $H^{0,p}_{\bar{\partial}}(L_\nu)$ The **Dolbeault cohomology** of L_ν, given by

$$H^0_{\bar{\partial}}(L_\nu) = H^{0,0}_{\bar{\partial}}(L_\nu) = \mathrm{Ker}\left(\bar{\partial} : W^{l,p}(L_\nu) \to W^{l-1,p}(T^*\Sigma \otimes L_\nu)\right),$$

$$H^{0,1}_{\bar{\partial}}(L_\nu) = \mathrm{Coker}\left(\bar{\partial} : W^{l,p}(L_\nu) \to W^{l-1,p}(T^*\Sigma \otimes L_\nu)\right).$$

Part (3) in Theorem 2.12 states that

$$\nu \leq -1 \Rightarrow H^0_{\bar{\partial}}(L_\nu)) = 0, \quad \nu \geq -1 \Rightarrow H^{0,1}_{\bar{\partial}}(L_\nu)) = 0.$$

4.2. **Line bundles and sections.** Let (E, F) be a rank k holomorphic bundle over $(\Sigma, \partial\Sigma)$ equipped with a CR operator D_F.

We say s is a **meromorphic section** of (E, F) if around any point $p \in \Sigma$ there exist a neighbourhood U and a meromorphic function f on U such that $f \cdot s\big|_U$ is a holomorphic section of (E, F) satisfying $(f \cdot s)(z) \neq 0$.

In other words, given a cover by trivializations

$$\{U_\alpha, \phi_\alpha\} \quad \text{with} \quad g_{\alpha\beta} = \phi_\alpha^{-1} \circ \phi_\beta \quad \text{on} \quad U_\alpha \cap U_\beta,$$

a meromorphic section is expressed as a set of meromorphic functions

$$s = \{s_\alpha : U_\alpha \xrightarrow{\text{mero.}} \mathbb{C}^k\} \quad \text{satisfying} \quad s_\alpha = g_{\alpha\beta} s_\beta \,.$$

Remark 4.9. Note that if s is a section of (E, F) and we require fs to remain a section of (E, F), this restricts f to take real values on the boundary.

Let s be a meromorphic section and let $z \in \Sigma$. In some neighbourhood U of p there exists a meromorphic function f so that $f \cdot s$ is a holomorphic section with a nonzero value at z. We define **the order of s at z** as

$$ord_z(s) := -ord_z(f).$$

At this point, we don't know the order to be finite for boundary points. To see the definition is independent of the choice of f, let V be another neighbourhood of z and $\hat{f}$ a meromorphic function on V such that $\hat{f}s$ is holomorphic nonzero at z. Let $\tilde{s} = fs, \hat{s} = \hat{f}s$ be the resulting holomorphic sections on $U \cap V$. Then $\hat{s} = (\hat{f}/f)\tilde{s}$. Therefore

$$ord_z(\hat{f}) - ord_z(f) = ord_z(\hat{f}/f) = -ord_z(\tilde{s}) = 0.$$

In order to avoid possible confusion, we call all zeroes and poles of a meromorphic section **special points**, although it would have been more natural to call them simply poles (given zero is nothing but a south pole).

Proposition 4.10. *Let s be a meromorphic section of (E, F). Then s defines a line subbundle (L, Λ) of (E, F) whose index is given by the formula*

$$\mu(L, \Lambda) = 2 \cdot \sum_{\substack{z \in int(D) \\ special}} ord_z(s) + \sum_{\substack{z \in \partial D \\ special}} ord_z(s). \tag{9}$$

This proposition will occupy our attention for the rest of the subsection.

A converse statement holds as well. We prove it for disks only, since the general case would require more sophisticated tools, and we will not need it in the current work.

Lemma 4.11. *Given a line bundle $L = L_\nu$ over the disk, it is generated by the meromorphic section $s(z) = (z+1)^\nu$.*

Note that we use Proposition 4.7 to say that $L = L_\nu$ for some $\nu \in \mathbb{Z}$, necessarily.

Proof. It suffices to exhibit a meromorphic section of L. Obviously, $s(z) = z + 1$ is a holomorphic section of the trivial bundle, and for $z = e^{i\theta} \in \partial D$, we have

$$\arg(z+1) = \arg(e^{i\theta} + e^{i\cdot 0}) = \theta/2 \Longrightarrow z + 1 \in z^{1/2}\mathbb{R},$$

so $z + 1$ generates L_1.

Since all other boundary conditions are given by integer powers of the conditions of L_1, it follows that L_ν is generated by $(z+1)^\nu$. □

Before moving to the proof of Proposition 4.10, we develop an auxiliary result, cf. [19, Proposition 3.1].

Lemma 4.12. *Let $f \in C^\infty(HD, \mathbb{C}^n)$ such that $f(0) = 0$ and $f(\partial HD) \subset \mathbb{R}^n$. Assume $f^{-1}(0)$ is discrete in int(HD) and there is a constant c such that*

$$|\bar\partial f| \le c \cdot |f|.$$

Then there exist $k \in \mathbb{Z}$ and $a \in \mathbb{R}^n \setminus \{0\}$ such that

$$f(z) = az^k + o(|z|^k).$$

Remark 4.13. The equality $f(z) = az^k + o(|z|^k)$ shows that on a small enough neighbourhood of 0, the zeroes of f are precisely those of z^k. It follows that $f^{-1}(0)$ is discrete in all of HD.

Corollary 4.14. *Let s be a meromorphic section of (E, F). Let $z \in \partial\Sigma$ be a special point of s. Then*

$$|ord_z(s)| < \infty.$$

Proof. Take a coordinate neighbourhood $U \subset HD$ of z on which exists a smooth trivialization of (E, F) that identifies the fibers of F with $\mathbb{R}^n \subset \mathbb{C}^n$:

$$\Phi : (\mathbb{C}^n, \mathbb{R}^n) \times (HD, \partial HD) \longrightarrow (E\big|_U, F\big|_{\partial U})$$

Let D denote the CR operator on E defining its holomorphic structure. Write $\Phi^* D = \bar\partial + A$ where A is a matrix of $(0,1)$-forms.

$$Ds = 0 \Longrightarrow \bar\partial s = -As \Longrightarrow |\bar\partial s| \le \|A\| \cdot |s|$$

so s satisfies the conditions of Lemma 4.12. The Lemma then states that $ord_z(s) = k < \infty$. If z is a zero of s, this completes the proof. In case z is a pole, let f be a meromorphic function such that fs is a holomorphic section that does not vanish at z. Since $ord_z(f) > 0$, we know that in a small neighbourhood of z, f is holomorphic. By Remark 4.9, f takes real values on the boundary. Therefore f satisfies the conditions of Lemma 4.12, and we conclude

$$|ord_z(s)| = ord_z(f) < \infty.$$

□

We are now ready to prove Proposition 4.10.

Denote by Z the set of special points of s and let

$$Z_0 = Z \cap int(\Sigma), \quad Z_1 = Z \cap \partial\Sigma.$$

It follows from Remark 4.13 that the elements of Z are isolated. The surface Σ being compact it means Z is finite, and we write

$$Z = \{z_1, ..., z_l\}.$$

The proof has two parts.

4.2.1. *Construction of the generated bundle.* Around any point $z \notin Z$, it is possible to take a trivialization of E

$$\varphi : U \times \mathbb{C}^k \longrightarrow E\big|_U$$

so that s is a well defined, nonvanishing holomorphic function on U. Then

$$L_w = s(w) \cdot \mathbb{C}, \quad w \in U$$

defines a line bundle whose trivialization (over all of U) is given by multiplication by s^{-1}. It is necessary now to specify fibers over the elements of Z.

Given $z = z_j \in Z$, isolate it from the rest of Z in a coordinate neighbourhood U with coordinate w such that $w\big|_{\partial U} : \partial U \to \mathbb{R}$. Define a corrector $c_j : U \to \mathbb{C}$ by

$$c_j(w) = (w - z_j)^{n_j}, \quad n_j = ord_{z_j} s.$$

Again, set

$$L_w = c_j(w) \cdot s(w) \cdot \mathbb{C}, \quad w \in U,$$

and $L\big|_U$ is trivialized by multiplication by $(c \cdot s)^{-1}$.

To make sure L is well defined, take U as above around some $z_j \in Z$, and fix $w_0 \in U \setminus Z$. Then c_j is a nonvanishing holomorphic function around w_0, and so

$$c_j(w_0) \cdot \mathbb{C} = \mathbb{C} \Longrightarrow c_j(w_0) \cdot s(w_0) \cdot \mathbb{C} = s(w_0) \cdot \mathbb{C}.$$

Therefore the fibers of L agree at w_0.

The boundary conditions are constructed in a similar manner: for any $z \in \partial\Sigma$ set

$$\Lambda(z) = \begin{cases} s(z) \cdot \mathbb{R} & z \notin Z \\ c_j(z) \cdot s(z) \cdot \mathbb{R} & z = z_j \in Z. \end{cases}$$

To see that Λ is well defined and contained in F, look at a coordinate neighbourhood U around z_j as before. Since $z_j \in \mathbb{R}$, $c_j\big|_{\partial U}$ is a real-valued function. Therefore $c_j s\big|_{\partial U}\mathbb{R} = s\big|_{\partial U}\mathbb{R}$ and, $c_j s$ being continuous at z_j,

$$s\big|_{\partial U}\mathbb{R} \subset F\big|_{\partial U} \Longrightarrow c_j s\big|_{\partial U}\mathbb{R} \subset F\big|_{\partial U}.$$

It is left to verify that our construction was independent of choice of coordinate. Let v be a different coordinate on V around $z_j \in Z$ and c^j a corrector corresponding to v. Then $c_j s / c^j s$ on $V \cap U$ is a quotient of holomorphic nonvanishing functions, therefore holomorphic itself. Therefore the holomorphic structure indeed does not depend on the choice of w.

4.2.2. *Computing the index.* Since the index depends only on the smooth structure, we may use Proposition 3.5 to assume L is trivial.

If s has no special points, then s itself gives a global trivialization of (L, Λ) and therefore the index is zero. In particular, this fits with the required formula. Assume now $Z \neq \emptyset$.

The surface Σ is a obtained by removing open disks from a closed Riemann surface $\widehat{\Sigma}$ of genus g. If $g \neq 0$, $\widehat{\Sigma}$ can be represented as a $4g$-gon Ξ with appropriate identifications of the edges. In case $g = 0$ we take Ξ to be a disk with all of its boundary identified as a point. Thus Σ can be thought of as Ξ with some open disks removed from its interior and corresponding edges identified.

For $j = 1, ..., l$, let U_j be the closure of a neighbourhood of z_j such that $U_j \cap U_i = \emptyset$ whenever $j \neq i$.

Let λ be a smooth closed curve that incloses $\partial\Sigma \cup \bigcup_j U_j$ and does not intersect the boundary of Ξ. Then λ decomposes Σ into two components, R_0 and R_1, where R_0 includes the boundary of Ξ (see Figure 1 below).

Let Λ_j be the loop of totally real subspaces defined by s on the boundary of U_j. When $z_j \in Z_0$, $s\big|_{\partial U_j}$ is a nonvanishing holomorphic section, therefore Λ_j is simply given by

$$\Lambda_j(z) = s(z)\mathbb{R}^n. \tag{10}$$

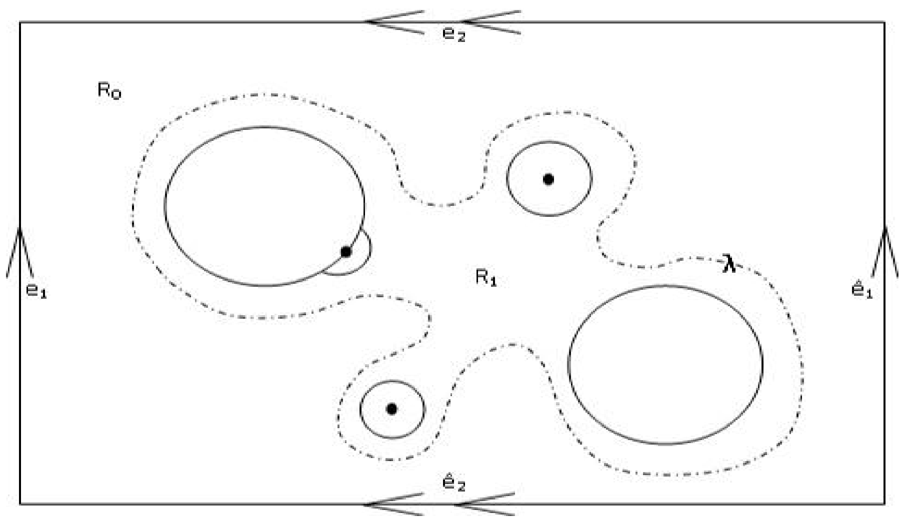


FIGURE 1.

For $z_j \in Z_1$, Λ_j agrees with the boundary conditions Λ of L described in subsection 4.2.1 on $\partial U_j \cap \partial\Sigma$ and is defined by s via (10) on $\partial U_j \setminus \partial\Sigma$.

Denote by Λ_0 and Λ^0 the loops defined via (10) over λ and $\partial\Xi$ respectively.

Note that λ is homotopic to $\partial\Xi$ through R_0. Denote the homotopy by f:

$$f : S^1 \times [0,1] \longrightarrow \mathbb{R}_0,$$

$$f : (S^1 \times \{0\}) = \lambda, \quad f(S^1 \times \{1\}) = \partial\Xi.$$

Since s is holomorphic nonvanishing on R_0, this yields a homotopy between Λ_0 and Λ^0 as well:

$$F : S^1 \times [0,1] \longrightarrow \mathcal{T}(n),$$

$$F(z,t) = s((f(z,t)))\mathbb{R}^n. \tag{11}$$

Since Maslov index for loops is a homotopy invariant, we have

$$\mu(\Lambda_0) = \mu(\Lambda^0).$$

If Ξ is a disk, then $\partial\Xi$ is identified as a point in Σ. Hence Λ^0 is a constant loop, and so its index equals zero.

If Ξ is a $4g$-gon, then any edge of its boundary is identified with a corresponding edge with opposite orientation. The values of Λ^0 are equal on these edges. Denote by e_k, $1 \leq k \leq 2g$, distinct edges of $\partial\Xi$

and by $\hat{e}_k$, $1 \le k \le 2g$, the edges identified with e_k in Σ (see Figure 1 above), so that

$$\partial \Xi = \bigcup_k e_k \cup \bigcup_k \hat{e}_k.$$

Then the contribution to the index of Λ^0 restricted to $\bigcup e_k$ equals the contribution on $\bigcup \hat{e}_k$ but with opposite sign. Therefore we have

$$\mu(\Lambda_0) = \mu(\Lambda^0) = 0. \tag{12}$$

Consider now

$$R = R_0 \setminus \big(\bigcup_j int(U_j) \cup \bigcup_{\substack{1 \le j \le l \\ z_j \in Z_0}} int\,(\partial U_j \cap \partial \Sigma) \big).$$

The interior of $\partial U_j \cap \partial \Sigma$ is meant to be the interior of the 1-dimensional manifold with boundary, in our case – a curve minus its endpoints.

The boundary of R can have four types of components:

(1) λ
(2) ∂U_j for any j with $z_j \in Z_1$
(3) Curves enclosing boundary components of Σ together with U_j's for z_j's lying on the boundary component.
    Denote by Λ^k, $k = 1, ..., q$, the loops generated by s via (10) on such boundary components
(4) C_k, $k = 1, ..., m$, boundary components of Σ on which no special points occur

Since s is holomorphic nonvanishing on R, it defines the trivial bundle with trivial boundary conditions (see also Remark 4.15 below). Therefore

$$\sum_{k=1}^{m} \mu(\Lambda\big|_{C_k}) + \sum_{\substack{1 \le j \le l \\ z_j \in Z_0}} \mu(\Lambda_j) + \sum_{k=1}^{q} \mu(\Lambda^k) + \mu(\Lambda_0) = 0. \tag{13}$$

By (12), $\mu(\Lambda_0) = 0$, which yields

$$\sum_k \mu(\Lambda\big|_{C_k}) + \sum_k \mu(\Lambda^k) = - \sum_{\substack{1 \le j \le l \\ z_j \in Z_0}} \mu(\Lambda_j). \tag{14}$$

Take a boundary component C of Σ on which a special point z_j occurs. Take a concatenation of Λ_j with $\Lambda\big|_C$. On the boundary portion $\partial U_j \cap \partial \Sigma$ the two loops take the same value. In the concatenation, they are taken twice, with opposite orientations. Therefore this path contributes nothing to the index computation, and we may consider the loop over $(C \cup \partial U_j) \setminus int(\partial U_j \cap \partial \Sigma)$ without changing the index.

Repeating this process for all special point on C, adding each at a time, we conclude that for the appropriate k

$$\mu(\Lambda^k) = \mu(\Lambda\big|_C) + \sum_{z_j \in C} \mu(\Lambda_j).$$

Combining this with formula (14) we have

$$\mu(\Lambda) = \sum_{\substack{C \text{ boundary} \\ \text{component} \\ \text{of } \Sigma}} \mu(\Lambda\big|_C) = -\sum_{j=1}^{l} \mu(\Lambda_j).$$

It is now left to compute the indices $\mu(\Lambda_j)$.

From the proof of Lemma 4.7 we see that for $z_j \in Z_0$, $\mu(\Lambda_j) = 2ord_{z_j}(s)$ when taken with the orientation of ∂U_j as a boundary of U_j. In our computation we took it as the boundary of R, which has opposite orientation. Therefore in our discussion

$$-\mu(\Lambda_j) = 2ord_{z_j}(s)\,.$$

Take now $z_j \in Z_1$. Identify U_j with a subset of $\mathbb{H}$ so that $U_j \cap \partial\Sigma \hookrightarrow \mathbb{R} = \partial\mathbb{H}$ and z_j corresponds to $0 \in \mathbb{R}$.

Given any two curves in Σ that are homotophic so that the homotopy does not pass through a point of Z, we can construct a homotopy between the loops defined on these curves, using similar formula as in (11). Again, homotopic loops will have the same index.

In particular, we may assume U_j is as small as we please. Choose U_j small such that there exists a smooth trivialization

$$\Phi : (\mathbb{C}, \mathbb{R}) \times (U_j, \partial U_j \cap \mathbb{R}) \longrightarrow (L\big|_{U_j}, \Lambda\big|_{\partial U_j \cap \mathbb{R}}).$$

In other words, on $\partial U_j \cap \partial\Sigma$, Λ_j is given by the trivial, constant path. It follows that Λ_j restricted to $\partial U_j \backslash \partial\Sigma$ is a smooth loop, parameterized by $[0, \pi]$.

By Lemma 4.12, $s\big|_{U_j}$ can be written as

$$s(z) = az^k + \vartheta(z), \quad \vartheta(z) \in o(|z|^k).$$

For $0 \le t \le 1$ define

$$s(z,t) = az^k + t \cdot \vartheta(z).$$

For U_j small enough, $s(z,t)$ is nonvanishing on $U_j \setminus \{z_j\}$, as in Remark 4.13. Therefore $s(z,t)$ defines a homotopy of Λ_j to the loop given by $z^k\mathbb{R} = e^{ik\theta}$ for $\theta \in [0, \pi]$. By formula (6), we therefore have

$$\mu(\Lambda_j) = \frac{k\pi - 0}{\pi} = k = ord_{z_j}(s).$$

Again, we need to take the loop with opposite orientation, which changes the sign.

Remark 4.15. Concluding equation (13), we used Maslov index of bundles. Strictly speaking, it is only defined for smooth bundles, but the boundary components on which Λ^k are defined are not smooth. So, formally we should have taken a smooth curve enclosing each such boundary component, and consider the smooth bundle over the appropriate smooth surface. Then use homotopy (defined as in (11)) to see that the index of the loop on the new boundary component equals the index of Λ^k.

4.3. **Proof of existence.** Given (E, F) a holomorphic bundle, denote by $H^p_{\bar\partial}(E, F)$ the Dolbeault cohomology (cf. Remark 4.8):

$$H^0_{\bar\partial}(E, F) = H^{0,0}_{\bar\partial}(E, F) = \operatorname{Ker} D_F,$$

$$H^{0,1}_{\bar\partial}(E, F) = \operatorname{Coker} D_F.$$

The notation makes sense due to uniqueness of D (Lemma 2.5). Note that $H^0_{\bar\partial}(E, F)$ consists by definition of global holomorphic sections.

Lemma 4.16. *Any holomorphic bundle (E, F) over the disk admits a nonzero meromorphic section.*

Proof. Denote $k = \operatorname{rk} E$. By Theorem 2.12,

$$\operatorname{ind}(D_F) = k\chi(n) + \mu(E, F) = k + \mu(E, F).$$

Tensoring E with L_ν, by Proposition 3.9, $\mu((E, F) \otimes L_\nu) = \mu(E, F) + k\nu$. Hence

$$\operatorname{ind}(D_{F\otimes\Lambda_\nu}) = k + \mu(E, F) + k\nu.$$

Taking ν so large that $\operatorname{ind}(D_{F\otimes\Lambda_\nu}) > 0$, we conclude that

$$\operatorname{Ker}(D_{F\otimes\Lambda_\nu}) \neq 0.$$

That is, $(E, F) \otimes L_\nu$ admits a nonzero holomorphic section.

Note that this implies that (E, F) has a meromorphic section. For if s is a holomorphic section of $(E, F) \otimes L_\nu$, then, by Lemma 4.11, $s \cdot (z + 1)^{-\nu}$ is a meromorphic section of (E, F) with pole of order ν. □

In view of this, given any holomorphic bundle (E, F), by Proposition 4.10, one can find some line subbundle (E_1, F_1). It is possible then to take, again, a line subbundle of $(E/E_1, F/F_1)$; denote by (E_2, F_2) the corresponding rank 2 subbundle of (E, F): on each fiber it is given by the preimage of the line subbundle under the projection

$(E, F) \to (E/E_1, F/F_1)$. Continuing in the same fashion, we construct a filtration

$$0 = (E_0, F_0) \subset (E_1, F_1) \subset \cdots \subset (E_k, F_k) = (E, F)$$

where $(E_j/E_{j-1}, F_j/F_{j-1})$ are line bundles. Denote

$$d_j = \mu(E_j/E_{j-1}, F_j/F_{j-1}).$$

Lemma 4.17. *For any line subbundle (L, Λ) of (E, F),*

$$\mu(L, \Lambda) \le \max\{d_j\}.$$

Proof. Let j be the first index such that $(L, \Lambda) \subset (E_j, F_j)$. Take s to be a section generating (L, Λ), and let $\hat{s}$ be its projection on $(E_j/E_{j-1}, F_j/F_{j-1})$. Then at every $z \in D$ we have $ord_z(s) \le ord_z(\hat{s})$. In particular, by formula (9), $\mu(L, \Lambda) \le \mu(E_j/E_{j-1}, F_j/F_{j-1})$. □

Lemma 4.18. *Assume* $\operatorname{rk} E = 2$. *Then there exists a line subbundle (E_1, F_1) so that $\mu(E_1, F_1) \ge \mu(E/E_1, F/F_1)$.*

Proof. By Lemma 4.17, the set of all possible indices of line subbundles is bounded. The values being integers, it admits a maximum. Choose (E_1, F_1) to be a bundle on which this maximum is obtained. Now, tensoring with a power of L_ν if necessary (as in the proof of Lemma 4.16), we may assume that $d_1 = -1$ and $d_2 \ge 0$, and try to get to a contradiction.

Consider the short exact sequence of bundles:

$$0 \longrightarrow (E_1, F_1) \longrightarrow (E, F) \longrightarrow (E/E_1, F/F_1) \longrightarrow 0.$$

It gives rise to a long exact sequence of cohomology groups, part of which is

$$H^0_{\bar\partial}(E, F) \xrightarrow{\alpha} H^0_{\bar\partial}(E/E_1, F/F_1) \longrightarrow H^{0,1}_{\bar\partial}(E_1, F_1)$$

Note that, by Remark 4.8,

$$H^{0,1}_{\bar\partial}(E_1, F_1) \simeq H^{0,1}_{\bar\partial}(L_{-1}) = 0,$$

so α is onto. Besides, since $d_2 \ge 0$,

$$H^0_{\bar\partial}(E/E_1, F/F_1) \simeq H^0_{\bar\partial}(L_{d_2}) \ne 0.$$

Therefore there exists a nonzero section $s \in H^0_{\bar\partial}(E/E_1, F/F_1)$, and it comes from some nonzero holomorphic section $s' \in H^0_{\bar\partial}(E, F)$. By Proposition 4.10 it follows that there exists a line subbundle (L, Λ) of (E, F) with $\mu(L, \Lambda) \ge 0$, contradicting the maximality of d_1. □

Lemma 4.19. *There exists a filtration of (E, F) such that $\{d_j\}$ form a non-increasing sequence.*

Proof. Choose (E_1, F_1) to be a line subbundle of maximal index. Choose now (E_2, F_2) so that its projection in $(E/E_1, F/F_1)$ is of maximal index. Note that (E_1, F_1) has to be a line subbundle of maximal index in (E_2, F_2). The proof of Lemma 4.18 then shows that

$$d_1 = \mu(E_1, F_1) \geq \mu(E_2/E_1, F_2/F_1) = d_2.$$

Choose (E_3, F_3) so that its projection in $(E/E_2, F/F_2)$ is of maximal index. Lemma 4.18 again gives

$$\begin{aligned} d_2 = \mu(E_2/E_1, F_2/F_1) &\geq \mu\big((E_3/E_1)/(E_2/E_1), (F_3/F_1)/(F_2/F_1)\big) \\ &= \mu(E_3/E_2, F_3/F_2) = d_3. \end{aligned}$$

Continuing in the same fashion, we obtain the required filtration. □

Fix filtration as in Lemma 4.19. We prove by induction on $k = \operatorname{rk}_{\mathbb{C}} E$ that

$$(E, F) \simeq \bigoplus_{j=1}^{k} (E_j/E_{j-1}, F_j/F_{j-1}).$$

For $k = 1$ the claim is trivial. Assume now it is true for bundles with rank at most $k-1$, and take again $\operatorname{rk} E = k$.

By assumption, we know $(E_{k-1}, F_{k-1}) \simeq \oplus_{j=1}^{k-1}(E_j/E_{j-1}, F_j/F_{j-1})$. So, all we have left to show is that the following short exact sequence splits:

$$0 \longrightarrow (E_{k-1}, F_{k-1}) \longrightarrow (E, F) \xrightarrow{\pi} (E/E_{k-1}, F/F_{k-1}) \longrightarrow 0. \tag{15}$$

We will do this by showing that there exists a homomorphism

$$r : (E/E_{k-1}, F/F_{k-1}) \longrightarrow (E, F)$$

such that $\pi \circ r = Id$.

Tensor the sequence 15 with the (locally trivial) dual bundle

$$(E/E_{k-1}, F/F_{k-1})^*.$$

Note that for all finite dimensional vector spaces V, W, one can canonically identify $Hom(V, W)$ with $V^* \otimes W$: take v_j, w_i bases for V, W, respectively, and v_j^* the basis dual to v_j. The correspondences are then given by $T \mapsto \sum v_j^* \otimes (Tv_j)$ and $v^* \otimes w \mapsto T$ s.t. $T(u) = v^*(u) \cdot w$. Tensoring vector bundles, we obtain isomorphism on each fiber that is compatible with transition functions (that are nothing but linear transformations at each point). Hence there is the short exact sequence

$$\begin{aligned} 0 \longrightarrow\ & Hom((E/E_{k-1}, F/F_{k-1}), (E_{k-1}, F_{k-1})) \to \\ &\to Hom((E/E_{k-1}, F/F_{k-1}), (E, F)) \xrightarrow{\pi} \\ &\to Hom((E/E_{k-1}, F/F_{k-1}), (E/E_{k-1}, F/F_{k-1})) \longrightarrow 0. \end{aligned}$$

Taking the sheaf of holomorphic sections and moving to the long exact sequence of cohomology:

$$H^0_{\bar\partial}(Hom((E/E_{k-1},F/F_{k-1}),(E,F))) \xrightarrow{\pi^*}$$
$$\longrightarrow H^0_{\bar\partial}(Hom((E/E_{k-1},F/F_{k-1}),(E/E_{k-1},F/F_{k-1}))) \longrightarrow$$
$$\longrightarrow H^{0,1}_{\bar\partial}((E/E_{k-1},F/F_{k-1})^*\otimes(E_{k-1},F_{k-1})).$$

By induction hypothesis,

$$(E/E_{k-1},F/F_{k-1})^*\otimes(E_{k-1},F_{k-1}) \simeq$$
$$(E/E_{k-1},F/F_{k-1})^*\otimes\bigoplus_{j=1}^{k-1}(E_j/E_{j-1},F_j/F_{j-1}) \simeq$$
$$\bigoplus_{j=1}^{k-1}((E/E_{k-1},F/F_{k-1})^*\otimes(E_j/E_{j-1},F_j/F_{j-1})) \simeq \bigoplus_{j=1}^{k-1}L_{d_j-d_k}.$$

Now, by our choice of the filtration, $d_j \geq d_k\ \forall j$, i.e. $d_j - d_k \geq 0 > -1$. This implies that $H^{0,1}_{\bar\partial}(\bigoplus_{j=1}^{k-1}L_{d_j-d_k}) = 0$. That is, π^* is onto. Therefore the section $\mathrm{Id} \in Hom((E/E_{k-1},F/F_{k-1}),(E/E_{k-1},F/F_{k-1}))$ comes from some section $r \in Hom((E/E_{k-1},F/F_{k-1}),(E,F))$. So, we found a homomorphism r such that $\pi\circ r = \mathrm{Id}$, as desired.

4.4. **Proof of uniqueness.** We prove by induction on $k = \mathrm{rk}\,E$.

For $k=1$ the statement is obvious.

Assume the claim is true for bundles of rank at most $k-1$, and take (E,F) with $\mathrm{rk}\,E = k$. Suppose two factorizations are given:

$$(E,F)\simeq\bigoplus_{j=1}^{k}L_{d_j}\simeq\bigoplus_{j=1}^{k}L_{d'_j}.$$

Assume without loss of generality that $d_1 = max\{d_j\}$ and $d'_1 = max\{d'_j\}$. If $d_1 = d'_1$, then taking the quotient we're done by the induction hypothesis. Otherwise assume $d_1 > d'_1$, and without loss of generality, by tensoring E with L_{-d_1}, assume $d_1 = 0$. But then

$$H^0_{\bar\partial}(L_0)\neq 0 \Longrightarrow H^0_{\bar\partial}(E,F)\neq 0$$

on the one hand, and on the other,

$$H^0_{\bar\partial}(E,F)\simeq\bigoplus_j H^0_{\bar\partial}(L_{d'_j}) = 0$$

since for all j we assumed $d'_j \leq -1$. Contradiction.

5. The sheaf of sections of a bundle

Define the structure sheaf of $(\Sigma, \partial\Sigma)$:

$$\mathcal{O}(U, \partial U) = \mathcal{O}_{(\Sigma,\partial\Sigma)}(U, \partial U)$$
$$= \{f : U \to \mathbb{C} | f|_{\overset{\circ}{U}} \text{ } holomorphic, f|_{\partial U} \in C^\infty(\partial U, \mathbb{R})\},$$

where $\partial U := U \cap \partial\Sigma$. Then

$$\mathcal{O}^n(U, \partial U) = \mathcal{O}^{\oplus n}(U, \partial U)$$
$$\simeq \{f : U \to \mathbb{C}^n \big| f|_{\overset{\circ}{U}} \text{ } holomorphic, f|_{\partial U} \in C^\infty(\partial U, \mathbb{R}^n)\}.$$

Let $\mathcal{O}(E, F)$ denote the sheaf of D_F-holomorphic sections of (E, F).

Our objective in this section is to prove that $\mathcal{O}(E, F)$ is locally free:

Theorem 5.1. *For any $p \in \Sigma$ there exists a neighborhood $U \subset \Sigma$ such that*

$$\mathcal{O}(E, F)(U, \partial U) \simeq \mathcal{O}(U, \partial U)^{\oplus n}.$$

Proof. By Proposition 2.7, around any point there exists a holomorphic trivialization of E. For a point in the interior this is enough. For a point on the boundary, we need to verify that it is possible to find such a trivialization that takes F precisely to $\mathbb{R}^n \subset \mathbb{C}^n$ on each fiber (here $n = \mathrm{rk}_{\mathbb{C}} E$).

Let $p \in \partial\Sigma$. Let V be a coordinate neighbourhood around p on which a holomorphic trivialization of E exists. Identify V with a subset of the disk D. Then $(E, F)\big|_V$ is identified with the trivial bundle over a subset of D with smooth boundary conditions $F_z = A(z)\mathbb{R}^n$. If necessary, take $W \subset V$ so that $p \in W$ and A is bounded on ∂W.

Take the trivial bundle $\mathbb{C}^n \times D$ over D with smooth boundary conditions that extend $F\big|_W$. Denote the resulting bundle by (G, H). By Theorem 4.1,

$$(G, H) \simeq \bigoplus_j L_{k_j}.$$

Note that for every k, $\mathcal{O}(k)$ is locally free: near the boundary, a holomorphic trivialization of $L_k = (\mathcal{E}, \Lambda_k)$ is given by multiplication by $f_k(z) = z^{-k/2}$. This trivialization identifies the boundary conditions with $\mathbb{R} \subset \mathbb{C}$.

Choose a neighbourhood of p, $U \subset W$, and a trivialization of $(G, H)\big|_U$ by multiplication by $\oplus f_{k_j}$. Since $(E, F)\big|_U \simeq (G, H)\big|_U$, $\oplus f_{k_j}$ trivializes $E\big|_U$ so that $F\big|_U \simeq \oplus \Lambda_{k_j}\big|_U$ corresponds to $\mathbb{R}^n$. □

We now introduce a result which will be useful in the last section.

Definition 5.2. A sheaf Tor over a surface Σ is said to be **a torsion sheaf** if its support is a zero dimensional submanifold of Σ.

That is, there exists a discrete set of points $z_1, z_2, ... \in \Sigma$ such that $Tor(U) \neq 0$ if and only if $z_j \in U$ for some j.

Lemma 5.3. *Let $(L, \Lambda), (E, F)$, be holomorphic vector bundles over $(\Sigma, \partial\Sigma)$ such that $\mathrm{rk}_{\mathbb{C}} L = 1$. Let $f : (L, \Lambda) \to (E, F)$ be a morphism of vector bundles. Let $\mathcal{N}$ be the cokernel sheaf of the induced map of $\mathcal{O}$-modules:*

$$\mathcal{O}(L, \Lambda) \xrightarrow{f} \mathcal{O}(E, F) \longrightarrow \mathcal{N} \longrightarrow 0.$$

Then there exists a holomorphic subbundle $(N, M) \subset (E, F)$ and a torsion sheaf Tor such that the following sequence is exact

$$0 \longrightarrow Tor \longrightarrow \mathcal{N} \longrightarrow \mathcal{O}(N, M) \longrightarrow 0.$$

Proof. Cover Σ by coordinate neighbourhoods $\{U_\alpha\}$ on which (L, Λ) is trivial, with trivializations given by holomorphic sections $\{\varphi_\alpha\}$. Consider the line subbundles of (E, F) generated by $f(\varphi_\alpha)$ over the U_α's. These trivializations define a line subbundle (G, H) of (E, F). Note that f factors through a map

$$\tilde{f} : (L, \Lambda) \longrightarrow (G, H).$$

Denote by (N, M) the quotient bundle

$$(N, M) = (E, F)/\,(G, H).$$

Move now to the induced maps on the sheaves of holomorphic sections. Then

$$\tilde{f} : \mathcal{O}(L, \Lambda) \to \mathcal{O}(G, H)$$

is injective: let $U \subset \Sigma$ be an open set over which both (L, Λ) and (G, H) are trivial. Then $\tilde{f}\big|_U$ is given by multiplication by a holomorphic function g. Take an open subset $W \subset U$ on which g doesn't vanish. If there exist holomorphic sections s and s' such that $f(s) = f(s')$, then on U we have

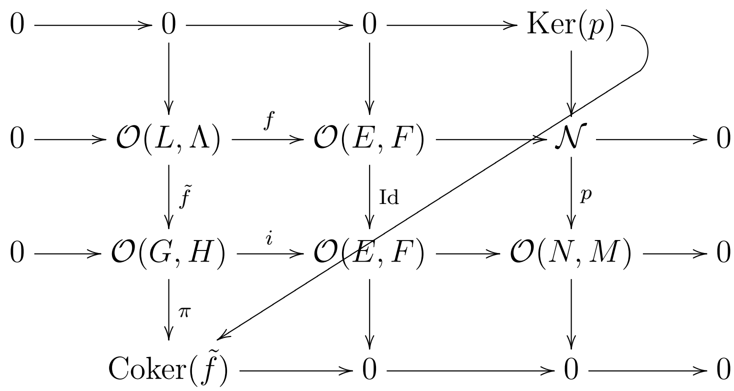
$$gs = \tilde{f}(s) = \tilde{f}(s') = gs'.$$

Since g^{-1} is well defined on W, it follows that $s\big|_W = s'\big|_W$, therefore $s = s'$.

As seen from the diagram below, there exists a map

$$d : \mathrm{Ker}(p) \longrightarrow \mathrm{Coker}(\tilde{f})$$

obtained by the snake lemma:



In particular, by the exactness statement of the snake lemma we see that d is an isomorphism.

This gives the exact sequence

$$0 \longrightarrow \mathrm{Coker}(\tilde{f}) \longrightarrow \mathcal{N} \longrightarrow \mathcal{O}(N, M) \longrightarrow 0.$$

It remains to show that $\mathrm{Coker}(\tilde{f})$ is a torsion sheaf. However, this is clear:

Around any point $p \in \Sigma$ such that $\tilde{f}(p) \neq 0$ exists a neighbourhood U on which $\tilde{f}$ gives an isomorphism of bundles. Therefore $\mathrm{Coker}(\tilde{f})(U) = 0$. It follows that the support of $\mathrm{Coker}(\tilde{f})$ is contained in the set of zeroes of $\tilde{f}$, which is discrete and even finite, since Σ is compact.

□

6. The Dolbeault isomorphism

Denote by $H^*(\mathcal{O}(E, F))$ the sheaf cohomology of $\mathcal{O}(E, F)$. So far in our computations we used $H^{0,*}_{\bar\partial}(E, F)$, the Dolbeault cohomology of the bundle (E, F). In this section we prove that these cohomologies are identical.

For closed manifolds, this result is a slight generalization of the standard Dolbeault isomorphism. See, e.g., [20, Theorem 3.20]:

Theorem 6.1. *Let X be a closed complex manifold. Let E be a holomorphic vector bundle over X. Then*

$$H^q(X, \Omega^p(E)) \simeq H^{p,q}_{\bar\partial}(E).$$

6.1. The $\bar\partial$-Poincaré lemma.

Consider again the standard $\bar\partial$-Poincaré lemma (Lemma 4.3). We prove a similar result for the boundary case:

Lemma 6.2. *Let $f \in C^\infty(HD)$. Then there exists some $g \in C^\infty_{\mathbb{R}}(HD)$ such that $\bar\partial g = f$.*

As a preliminary result, we give a slight variation on Lemma 4.6. Define

$$HD_{1-\epsilon} = HD \cap \left\{ z \middle| |z| \le 1-\epsilon \right\}.$$

Lemma 6.3. *Let $v \in C^\infty(\partial HD, \mathbb{R})$. Then for any $\epsilon > 0$ there exists a holomorphic function*

$$h : \partial HD_{1-\epsilon} \longrightarrow \mathbb{C}$$

such that on $\partial HD_{1-\epsilon}$, $v = Im(h)$.

Proof. Fix $\epsilon > 0$.

v is bounded on $\partial HD_{1-\epsilon}$, therefore it can be smoothly extended to a bounded function on $\mathbb{R}$. Extend it harmonically to $\mathbb{H}$. Denote by $\hat{v}$ the resulting function. Let $-\hat{u}$ be the harmonic conjugate of $\hat{v}$.

By Lemma 4.5, we obtain a holomorphic function

$$h = \hat{u} + i\hat{v}$$

on $\mathbb{H}$. Restricting it to $HD_{1-\epsilon}$ yields the required result. □

Proof of Lemma 6.2. **Step 1.** *A solution exists on a smaller halfdisk.*

Restrict f to some $HD_{1-\epsilon}$.

By Theorem 2.3, we may extend f smoothly outside $\overline{HD}_{1-\epsilon}$. Restricting the resulting function yields $\tilde{f} \in C^\infty(D)$. We may now apply Lemma 4.3. Hence there is some $\tilde{g} \in C^\infty(D)$ with $\bar{\partial}\tilde{g} = \tilde{f}$ on D.

Take $v = Im(\tilde{g})$ on ∂HD. By Lemma 6.3, there exists a holomorphic function h on $HD_{1-\epsilon}$ such that $v = Im(h)$ on $\partial HD_{1-\epsilon}$. Define $g = \tilde{g} - h$.

Then $\bar{\partial} g = \bar{\partial}\tilde{g} - \bar{\partial} h = \bar{\partial}\tilde{g} = f$, and, by construction of h, $g \in C^\infty_{\mathbb{R}}(HD_{1-\epsilon})$.

Step 2. *A solution exists on $HD = HD_1$.*

Define

$$HD_r = HD_{1-\epsilon_r} \text{ with } \epsilon_r \overset{r\to\infty}{\longrightarrow} 0.$$

We will construct a sequence of functions $g_r \in C^\infty_{\mathbb{R}}(\mathbb{H})$ satisfying the following conditions:

(1) $\bar{\partial} g_r = f$ on HD_r
(2) $\sup |g_r(z) - g_{r-1}(z)| \le \frac{1}{2^r}$ on HD_{r-2}

Then there will exist $g = \lim g_r \in C^\infty_{\mathbb{R}}(HD)$, and it will satisfy $\bar{\partial} g = f$.

The process is as follows (cf. [18] for a similar construction on the disk):

g_1 exists by step 1. Assume we constructed $g_1, ...g_r$ that satisfy the requirements. Again by step 1 there exists $h \in C^\infty_{\mathbb{R}}(\mathbb{H})$ so that $\bar{\partial} h = f$ on HD_{r+1}. Then on HD_r the function $h - g_r$ is holomorphic, and has real boundary values. Therefore it can be reflected (by conjugation

– Schwartz's reflection principle), and hence written as a Taylor series. Cutting the series after a finite number of terms, we can take a sufficiently good polynomial approximation p of it, so that

$$\sup_{HD_{r-1}} |(h - g_r - p)(z)| \leq \frac{1}{2^{r+1}}.$$

Set $g_{r+1} = h - p$, and indeed we see that it satisfies both requirements. □

Remark 6.4. In fact, we do not need Step 2 of the current proof for our purposes. We would like to use the $\bar{\partial}$-lemma to show exactness of a short exact sequence of sheaves in the next subsection. But, checking exactness on stalks, it is sufficient to have a solution on a smaller neighbourhood. However, we prove the lemma as is for the sake of completeness.

6.2. **The Dolbeault isomorphism.**

Theorem 6.5 (Generalized Dolbeault isomorphism)**.** *Let* $(\Sigma, \partial\Sigma)$ *be a Riemann surface,* $\bar{\partial}_{T\partial\Sigma}$ *the standard* $\bar{\partial}$ *operator on* $(T\Sigma, T\partial\Sigma)$*, restricted to elements with boundary values in* $T\partial\Sigma$*. Then*

$$H^q(\mathcal{O}(E, F)) \simeq H^{0,q}_{\bar{\partial}}(E, F).$$

Proof. Consider the following sequence of sheaves:

$$0 \to \mathcal{O} \longrightarrow \mathcal{A}^0_{\mathbb{R}} \xrightarrow{\bar{\partial}_{\mathbb{R}}} \mathcal{A}^{0,1} \to 0 \tag{16}$$

We would like to verify exactness on stalks. Given a point $p \in \Sigma$, take a coordinate neighbourhood around it, and apply the sequence there. The first map is just an inclusion. Exactness at $\mathcal{A}^0_{\mathbb{R}}$ is obvious – the kernel of $\bar{\partial}_{\mathbb{R}}$ consists exactly of holomorphic functions. The last map is onto, by Lemma 4.3 if $p \in int(\Sigma)$ and Lemma 6.2 if $p \in \partial\Sigma$. Hence the sequence is exact.

Since $\mathcal{O}(E, F)$ is locally free (Theorem 5.1), we can tensor it with (16) without disrupting exactness. Hence

$$0 \to \mathcal{O} \otimes_{\mathcal{O}} \mathcal{O}(E, F) \longrightarrow \mathcal{A}^0_{\mathbb{R}} \otimes_{\mathcal{O}} \mathcal{O}(E, F) \xrightarrow{\bar{\partial}_{\mathbb{R}} \otimes 1} \mathcal{A}^{0,1} \otimes_{\mathcal{O}} \mathcal{O}(E, F) \to 0.$$

Since the operator D_F is locally just $\bar{\partial}_{\mathbb{R}}$, we obtain the short exact sequence

$$0 \to \mathcal{O}(E, F) \longrightarrow \mathcal{A}^0_F(E) \xrightarrow{D_F} \mathcal{A}^{0,1}(E) \to 0.$$

Therefore $(\mathcal{A}^{0,*}, \bar{\partial})$ yields a resolution of $\mathcal{O}$. It is also acyclic, being fine (for these sheaves are modules over $C^\infty(\Sigma, \mathbb{R})$, which has partition of unity). Therefore, it is possible to use it in order to compute the sheaf cohomology $H^q(\mathcal{O}(E,F))$. In other words,

$$H^0(\mathcal{O}(E,F)) \simeq \operatorname{Ker} D_F, \quad H^1(\mathcal{O}(E,F)) \simeq \operatorname{Coker} D_F,$$

as desired.

□

7. Normal bundles

Let (X,ω) be a symplectic manifold with smooth ω-tame integrable complex structure $J \in \mathcal{J}_0$, L a regular Lagrangian (see Section 1.1 for definitions).

Let $\bar{u} : (D, \partial D) \to (X, L)$ be a J-holomorphic disk and ϕ a branched covering of the disk. Define then

$$u = \phi \circ \bar{u}, \quad \deg \phi = d > 1.$$

Consider the short exact sequence (s.e.s):

$$0 \longrightarrow \mathcal{O}(TD, T\partial D) \xrightarrow{d\bar{u}} \mathcal{O}(\bar{u}^*TX, \bar{u}^*TL) \xrightarrow{p} \mathcal{N}_{\bar{u}} \longrightarrow 0,$$

where $\mathcal{N}_{\bar{u}}$ is the cokernel sheaf, and the holomorphic structure on $(\bar{u}^*TX, \bar{u}^*TL)$ is given by $D_{\bar{u}}$ – which is a complex CR operator, since J is integrable (see Remark 1.7).

By Lemma 5.3, there exists a short exact sequence

$$0 \longrightarrow Tor \longrightarrow \mathcal{N}_{\bar{u}} \longrightarrow \mathcal{O}(N_{\bar{u}}) \longrightarrow 0,$$

where $N_{\bar{u}}$ is a rank 2 vector bundle, and Tor is a torsion sheaf arising from the zeroes of $d\bar{u}$, as explained in the proof. More precisely, let s be a section generating TD. If we denote by $G_{\bar{u}}$ the line subbundle of $\bar{u}^*TX$ generated by $d\bar{u}(s)$ (as in Proposition 4.10), then $N_{\bar{u}}$ is precisely the quotient bundle:

$$N_{\bar{u}} = \bar{u}^*TX / G_{\bar{u}}.$$

We omit boundary conditions, although they are implicitly assumed.

Lemma 7.1. *$\bar{u}$ is an immersion, and $\mathcal{O}(N_{\bar{u}}) \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.*

Proof. By the direct sum property (see Theorem 3.3),

$$\mu(N_{\bar{u}}) = \mu(\bar{u}^*TX) - \mu(G_{\bar{u}}).$$

Since L is Fukaya, $\mu(\bar{u}^*TX) = 0$ and so $\mu(N_{\bar{u}}) = -\mu(G_{\bar{u}})$. By Proposition 4.10,

$$\begin{aligned} -\mu(N_{\bar{u}}) = \mu(G_{\bar{u}}) &= \sum ord_z(d\bar{u}(s)) \\ &= \sum ord_z(d\bar{u}) + \sum ord_z(s) \\ &= t + \mu(TD) = t + 2. \end{aligned} \tag{17}$$

Applying the sheaf of sections on the s.e.s.

$$0 \longrightarrow G_{\bar{u}} \xrightarrow{i} \bar{u}^*TX \xrightarrow{\pi} N_{\bar{u}} \longrightarrow 0 \tag{18}$$

we get

$$H^1(\mathcal{O}(\bar{u}^*TX)) \longrightarrow H^1(\mathcal{O}(N_{\bar{u}})) \longrightarrow 0.$$

By assumption, L is regular. That is, $D_{\bar{u}}$ is onto. Therefore, $H^1(\mathcal{O}(\bar{u}^*TX)) = 0$. It follows that $H^1(\mathcal{O}(N_{\bar{u}})) = 0$.

By the Birkhoff factorization (Theorem 4.1), $N_{\bar{u}} \simeq L_k \oplus L_l$. By (17), $k + l = -2 - t$. Yet on the other hand,

$$0 = H^1(\mathcal{O}(N_{\bar{u}})) = H^1(\mathcal{O}(k) \oplus \mathcal{O}(l)),$$

so $k, l \geq -1$. Therefore $k = l = -1$, and $t = 0$. This precisely means that $N_{\bar{u}} \simeq L_{-1} \oplus L_{-1}$, and $\bar{u}$ is an immersion.

□

Note that it follows from $\bar{u}$ being an immersion that $\mathcal{N}_{\bar{u}} = \mathcal{O}(N_{\bar{u}})$ and $TD \simeq G_{\bar{u}}$.

Proposition 7.2. $H^0\big(\mathcal{O}(N_u)\big) = 0$.

Proof. Consider the pullback by ϕ of the s.e.s. (18):

$$0 \longrightarrow \phi^*G_{\bar{u}} \xrightarrow{i} \phi^*\bar{u}^*TX \xrightarrow{\pi} \phi^*N_{\bar{u}} \longrightarrow 0.$$

First, note that $du = d\bar{u} \cdot d\phi$, therefore $\phi^*(\bar{u}^*TX) = u^*TX$. Recall that s was a section generating TD. Then $d\phi(s)$ generates ϕ^*TD, which is mapped to $\phi^*G_{\bar{u}}$ by $d\bar{u}$. That is, $\phi^*G_{\bar{u}}$ is generated by $d\bar{u}(d\phi(s))$. But so is G_u, hence the two bundles are identical. Comparing this sequence with

$$0 \longrightarrow G_u \xrightarrow{i} u^*TX \xrightarrow{\pi} N_u \longrightarrow 0,$$

we conclude that

$$N_u = \phi^*N_{\bar{u}}.$$

It follows that $\mathcal{O}(N_u) = \mathcal{O}(-d) \oplus \mathcal{O}(-d)$. Then

$$H^0(\mathcal{O}(N_u)) = H^0(\mathcal{O}(-d) \oplus \mathcal{O}(-d)) = 0,$$

where the last equality is because $-d \leq -2 \leq -1$. □

References

[1] Sheldon Katz. On the finiteness of rational curves on quintic threefolds. *Compositio Math.*, 60(2):151–162, 1986.

[2] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[3] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.

[4] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.

[5] Herbert Clemens. Some results about Abel-Jacobi mappings. In *Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982)*, volume 106 of *Ann. of Math. Stud.*, pages 289–304. Princeton Univ. Press, Princeton, NJ, 1984.

[6] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.

[7] Laurent Lazzarini. Relative frames on J-holomorphic curves. *Journal of Fixed Point Theory and Applications*, pages 1–44, 2010. 10.1007/s11784-010-0004-1.

[8] Yong-Geun Oh. Fredholm theory of holomorphic discs under the perturbation of boundary conditions. *Math. Z.*, 222(3):505–520, 1996.

[9] Urs Frauenfelder. Gromov convergence of pseudoholomorphic disks. *J. Fixed Point Theory Appl.*, 3(2):215–271, 2008.

[10] Yong-Geun Oh and Ke Zhu. Embedding property of J-holomorphic curves in Calabi-Yau manifolds for generic J. *Asian J. Math.*, 13(3):323–340, 2009.

[11] Yong-Geun Oh. Super-rigidity and finiteness of embedded J-holomorphic curves on calabi-yau threefolds. arXiv:0807.3152v3 [math.SG], 2010.

[12] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[13] Hassler Whitney. Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.*, 36(1):63–89, 1934.

[14] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1990. Oxford Science Publications.

[15] V. I. Arnol′d. On a characteristic class entering into conditions of quantization. *Funkcional. Anal. i Priložen.*, 1:1–14, 1967.

[16] Allen Hatcher. Vector bundles and K-theory. `http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html`.

[17] John R. Silvester. Determinants of block matrices. *Mathematical Gazette*, 84:460–467, 2000.

[18] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[19] L. Lazzarini. Existence of a somewhere injective pseudo-holomorphic disc. *Geom. Funct. Anal.*, 10(4):829–862, 2000.

[20] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2008. With a new appendix by Oscar Garcia-Prada.