ON CONGRUENCES BETWEEN MODULAR FORMS

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Abstract

An elementary lemma on group cohomology is proved. Applied to certain arithmetic subgroups of real lie groups this leads to a general method of establishing congruences between modular forms of different weights. We apply this to establish the existence of certain p-adic families (Hida families) of Siegel modular forms and of modular forms for GL_2 over an imaginary quadratic field. We also use these methods to show that the existence of certain Galois representations one expects to be attached to Siegel modular forms corresponding to holomorphic discrete series would imply their existence for Siegel modular forms corresponding to limit of holomorphic discrete series.

Contents

Introduction

This work is concerned with certain methods of Hida for producing congruences between modular forms. We generalise some of these methods and apply them to the problem of constructing Galois representations attached to modular forms.

Hida in a series of papers (see [Hi1] and the references cited therein) has constructed certain p-adic families of elliptic modular forms. To describe these results fix an odd prime p, an integer N and a Dirichlet character χ defined modulo Np. Fix $\mathbb{Q}^{ac} \subset \mathbb{Q}_p^{ac}$ and $\mathbb{Q}^{ac} \subset \mathbb{C}$. Let $\mathcal{O}$ be the integers of a finite extension of $\mathbb{Q}_p$ in which χ is valued and let $\Lambda = \mathcal{O}[[T]]$. By a Λ-adic form we mean a formal power series $\sum_{n=0}^{\infty} a_n q^n$ with $a_n \in \Lambda$ such that for all pairs (k, α) of an integer $k \geq 2$ and a character $\alpha : (1+p\mathbb{Z})/(1+p^r\mathbb{Z}) \to \mathbb{Q}^{ac\,\times}$, $\sum_{n=0}^{\infty} a_n(\alpha(1+p)(1+p)^k - 1)q^n$ is the Fourier expansion of an elliptic modular form of weight k, level Np^r and character $\chi\omega^{-k}\alpha$, where ω denotes the Teichmuller character. If Λ' is a finite extension of Λ we define the space of Λ'-adic forms to be the space of Λ-adic forms tensored with Λ'. One can define an action of the Hecke operators on Λ-adic forms compatible with specialisation. One can also define a notion of ordinary modular forms, both at the finite and at the Λ-adic level. Roughly speaking the space of ordinary forms over a p-adic ring of integers, or over Λ, is the largest space on which the action of $U_p = [\Gamma_0(Np^?) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Np^?)]$ is invertible, and its complement is that on which U_p is p-adically nilpotent. Hida's main results are that the space of Λ-adic ordinary forms of given level N and character χ is a finite torsion free Λ-module, and moreover any ordinary eigenform of the Hecke algebra of weight k, level Np^r and character $\chi\omega^{-k}\alpha$ (with α as above) can be lifted to an ordinary Λ'-adic eigenform of the Hecke algebra (for some finite Λ'/Λ). Thus there are very systematic families of congruences between ordinary eigenforms.

This work has been generalised by Hida ([Hi2]) and Wiles ([Wi]) to Hilbert modular forms. In this thesis we shall give generalisations to Siegel modular forms (in chapter 3) and to modular forms over imaginary quadratic fields (in chapter 4). Our central technique is a method of comparing modular forms of different weights via group cohomology. This seems to be a very general method and we give an exposition of it in chapter 1. It is a generalisation of techniques of Shimura ([Sh1]) and Hida ([Hi1]). It is based on the fact that if G is an algebraic group, P a parabolic subgroup, M_1 and M_2 $G(\mathbb{Z})$-modules and Γ a congruence subgroup of $G(\mathbb{Z})$ such that $\Gamma \bmod p^r$ is contained in $P(\mathbb{Z}/p^r\mathbb{Z})$ then there may be non-trivial Γ-morphisms between $M_1 \otimes (\mathbb{Z}/p^r\mathbb{Z})$ and $M_2 \otimes (\mathbb{Z}/p^r\mathbb{Z})$ which allow us to compare the Γ-cohomology of the modules M_1 and M_2.

The easiest type of deduction to draw from this method is that the dimension of the space of ordinary modular forms for a given group Γ is bounded independently of the weight as the weight varies over some infinite set. This together with a suitable modular form congruent to one modulo p is enough to produce a lot of congruences between ordinary *eigenforms* of different weights. This is carried out in chapter 2 for Siegel modular forms. In the case of GSp_4 it is applied to the problem of associating Galois representations to modular forms. Specifically to eigenforms of "weight (n_1, n_2)" with $2 \leq n_1 \leq n_2$ one expects to be able to associate certain four dimensional Galois representations. If $3 \leq n_1$ then one hopes to be able to find these representations in the cohomology of certain Shimura varieties. However for the case $n_1 = 2$ no such method is expected to exist. We show how the result for ordinary eigenforms of weight $(2, n_2)$ $(n_2 \geq 2)$ would follow from the result for forms of weight $3 \leq n_1 \leq n_2$.

In chapter 3 we organise these congruences into "Hida families". However we restrict to the case of parallel weight $(k, ..., k)$ and even genus. We follow the method of Wiles ([Wi]) based on finding enough Λ-adic forms by writing down Λ-adic Eisenstein series, multiplying these by suitable modular forms and spectrally decomposing the result. This requires that one already has suitable bounds on the dimension of various spaces of ordinary modular forms involved. This again follows from our results in chapter 1. In the case of Siegel modular forms the calculations required for this method become very messy.

Finally in chapter 4 we consider the case of imaginary quadratic fields. Here we are not able to multiply together modular forms, so we work exclusively with the corresponding cohomology groups. Our results are not as sharp as we would like due to torsion in the homology groups. As a byproduct of our method we can in fact exhibit torsion in the homology of certain sheaves (of "non-parallel weight", so that the torsion free part of the cuspidal part of the first homology vanishes) on quotients of hyperbolic 3-space by certain discrete groups.

It is a pleasure to acknowledge the influence of the work of Hida [Hi1] and that of Wiles [Wi] on this thesis. I have also enjoyed and benefited from many discussions with Fred Diamond and Michael Larsen. Finally I would like to express my great gratitude to my advisor Andrew Wiles for his constant help and encouragement.

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Chapter 1

A Group Cohomological Lemma

1.1 The Lemma

We shall make repeated use of a certain argument to compare the group cohomology of different modules, and hence certain spaces of modular forms of different "weight". We shall describe this method abstractly as it seems to be applicable rather generally. It is an extension of some ideas of Shimura ([Sh2]) and of Hida (see [Hi1]).

If Δ is a semi-group, Γ a subgroup and M a Δ-module then the cohomology groups $H^\bullet(\Gamma, M)$ may be considered as the image of M under the right derived functors of the fixed point functor $N \mapsto N^\Gamma$ from Δ-modules to abelian groups. If Γ_1 and Γ_2 are two subgroups of Δ and if $g \in \Delta$ is such that $[\Gamma_2 : \Gamma_2 \cap g\Gamma_1 g^{-1}]$ is finite then there is a natural transformation:

$$[\Gamma_2 g \Gamma_1] : H^\bullet(\Gamma_1, \_) \longrightarrow H^\bullet(\Gamma_2, \_)$$

determined as being the unique such transformation compatible with the boundary homomorphisms and which coincide in degree zero with:

$$\begin{array}{rcl} M^{\Gamma_1} & \longrightarrow & M^{\Gamma_2} \\ m & \longmapsto & \sum \gamma_i g m \end{array}$$

where:

$$\Gamma_2 = \coprod \gamma_i (\Gamma_2 \cap g \Gamma_1 g^{-1})$$

or equivalently:

$$\Gamma_2 g \Gamma_1 = \coprod \gamma_i g \Gamma_1$$

These are usually called Hecke operators, and the special case $\Gamma_1 \subset \Gamma_2$ and $g = 1$ is called corestriction and denoted *cor*.

If Γ_1, Γ_2 are two groups and M_1, M_2 are modules over Γ_1 and Γ_2 respectively and, if moreover, $\phi : \Gamma_2 \to \Gamma_1$ and $\theta : M_1 \to M_2$ satisfy:

$$\theta((\phi\gamma)m) = \gamma(\theta m)$$

for all $\gamma \in \Gamma_2$ and $m \in M_1$; then there is an induced map:

$$(\phi^*, \theta_*) : H^\bullet(\Gamma_1, M_1) \longrightarrow H^\bullet(\Gamma_2, M_2)$$

It may be defined as the unique map functorially extending:

$$\theta : M_1^{\Gamma_1} \longrightarrow M_2^{\Gamma_2}$$

Returning to the situation in the last paragraph we may factor $[\Gamma_1 g \Gamma_2]$ as:

$$H^\bullet(\Gamma_1, M) \stackrel{(c^*_{g^{-1}}, g_*)}{\longrightarrow} H^\bullet(\Gamma_2 \cap g\Gamma_1 g^{-1}, M) \stackrel{cor}{\longrightarrow} H^\bullet(\Gamma_2, M)$$

where $c_{g^{-1}}(\gamma) = g^{-1}\gamma g$.

We shall now introduce a slight generalisation of Hecke operators. Let Γ_1, Γ_2 again be subgroups of Δ and $g \in \Delta$ be such that $[\Gamma_2 : \Gamma_2 \cap g\Gamma_1 g^{-1}] < \infty$. Let M_1, M_2 be modules for $\langle g, \Gamma_1 \rangle$ and Γ_2 respectively. Let $\theta : gM_1 \to M_2$ be a map of $\Gamma_2 \cap g\Gamma_1 g^{-1}$-modules. Then we can define a map:

$$[\Gamma_2 g \Gamma_1]_\theta : H^\bullet(\Gamma_1, M_1) \longrightarrow H^\bullet(\Gamma_2, M_2)$$

to be the composite:

$$H^\bullet(\Gamma_1, M_1) \stackrel{(c^*_{g^{-1}}, (\theta \circ g)_*)}{\longrightarrow} H^\bullet(\Gamma_2, \cap g\Gamma_1 g^{-1}, M_2) \stackrel{cor}{\longrightarrow} H^\bullet(\Gamma_2, M_2)$$

If we set $M_1 = M_2$ and $\theta = Id$ then we recover the normal Hecke operators.

Lemma 1.1 *Let Δ be a semi-group; $\Gamma_1 \supset \Gamma_2$ subgroups of Δ; $g \in \Delta$ with $[\Gamma_1 : \Gamma_1 \cap g\Gamma_2 g^{-1}] < \infty$; M_1 (resp. M_2) a module for $\langle \Gamma_1, g\rangle$ (resp. $\langle \Gamma_2, g\rangle$); and $j : M_1 \to M_2$ a $\langle \Gamma_2, g\rangle$ morphism such that $j : gM_1 \overset{\sim}{\to} gM_2$. Then there exists $I : H^\bullet(\Gamma_2, M_2) \to H^\bullet(\Gamma_1, M_1)$ such that:*

1. *If there exist elements $\gamma_i \in \Gamma_1$ such that $\Gamma_1 g\Gamma_1 = \coprod \gamma_i g\Gamma_1$ and $\Gamma_1 g\Gamma_2 = \coprod \gamma_i g\Gamma_2$ then $I \circ j_* = [\Gamma_1 g\Gamma_1]$.*

2. *If there exist elements $\delta_i \in \Gamma_1$ such that $\Gamma_1 g\Gamma_2 = (\Gamma_2 g\Gamma_2)\amalg(\coprod \Gamma_2\delta_i g\Gamma_2)$ and $j\delta_i gM_1 = 0$ then $j_* \circ I = [\Gamma_2 g\Gamma_2]$.*

The conditions in 1) and 2) are automatically satisfied if $\Gamma_1 = \Gamma_2$.

Proof: Set $I = [\Gamma_1 g\Gamma_2]_{j|_{gM_1}^{-1}}$. The first part is easy, it follows from the commutativity of the following diagram:

$$\begin{array}{ccccccc}
H^\bullet(\Gamma_1, M_1) & \overset{(c^*_{g^{-1}}, g_*)}{\longrightarrow} & H^\bullet(\Gamma_1 \cap g\Gamma_1 g^{-1}, gM_1) & \longrightarrow & H^\bullet(\Gamma_1 \cap g\Gamma_1 g^{-1}, M_1) & \overset{cor}{\longrightarrow} & H^\bullet(\Gamma_1, M_1) \\
j_* \downarrow & & j_* \downarrow & \searrow res & & \searrow res & \uparrow cor \\
H^\bullet(\Gamma_2, M_2) & \overset{(c^*_{g^{-1}}, g_*)}{\longrightarrow} & H^\bullet(\Gamma_1 \cap g\Gamma_2 g^{-1}, gM_2) & \overset{j_*}{\cong} & H^\bullet(\Gamma_1 \cap g\Gamma_2 g^{-1}, gM_1) & \longrightarrow & H^\bullet(\Gamma_1 \cap g\Gamma_1 g^{-1}, M_1)
\end{array}$$

(The only slight problem is the right hand triangle, but working in the category of $\langle \Gamma_1, g\rangle$-modules we need only check this in degree zero.)

For the second part we must check that the following diagram is commutative:

$$\begin{array}{ccccccc}
H^\bullet(\Gamma_1, M_1) & \overset{j_*}{\cong} & H^\bullet(\Gamma_1 \cap g\Gamma_2 g^{-1}, gM_1) & \longrightarrow & H^\bullet(\Gamma_1 \cap g\Gamma_2 g^{-1}, M_1) & \overset{cor}{\longrightarrow} & H^\bullet(\Gamma_1, M_1) \\
(c^*_{g^{-1}}, g_*) \uparrow & & & j_* \searrow & \downarrow j_* & & \downarrow j_* \\
H^\bullet(\Gamma_2, M_2) & & \overset{(c^*_{g^{-1}}, g_*)}{\longrightarrow} & & H^\bullet(\Gamma_2 \cap g\Gamma_2 g^{-1}, M_2) & \overset{cor}{\longrightarrow} & H^\bullet(\Gamma_2, M_2)
\end{array}$$

The left hand side is easy, the only problem is to check that the two composite maps $H^\bullet(\Gamma_3, N) \to H^\bullet(\Gamma_2, M_1)$ in:

$$\begin{array}{ccccc}
H^\bullet(\Gamma_3, N) & \longrightarrow & H^\bullet(\Gamma_3, M_1) & \overset{cor}{\longrightarrow} & H^\bullet(\Gamma_1, M_1) \\
& & \downarrow j_* & & \downarrow j_* \\
& & H^\bullet(\Gamma_3 \cap \Gamma_2, M_2) & \overset{cor}{\longrightarrow} & H^\bullet(\Gamma_2, M_2)
\end{array}$$

are equal, where $\Gamma_3 = \Gamma_1 \cap g\Gamma_2 g^{-1}$ and $N = gM_1$. (If $\Gamma_1 = \Gamma_2$ this also is easy.) In fact we shall prove this under the following assumptions, which are clearly valid in our case:

$\Gamma_3 \subset \Gamma_1$ and $\Gamma_2 \subset \Gamma_1$ are subgroups of Δ, M_1 is a Γ_1 module, N is a Γ_3-submodule of M_1, M_2 is a Γ_2-module and $j : M_1 \to M_2$ is a morphism over Γ_2 such that $\Gamma_1 = (\Gamma_2\Gamma_3) \amalg (\coprod \Gamma_2\delta_i\Gamma_3)$ with $j\delta_i N = 0$.

To prove this let $\phi \in Z^n(\Gamma_3, N)$ and let $\gamma_1, ..., \gamma_n \in \Gamma_2$. Let $\Gamma_2\Gamma_3 = \coprod h_{k,0}\Gamma_3$ and $\Gamma_2\delta_i\Gamma_3 = \coprod h_{i,k,0}\Gamma_3$ with each $h_{k,0}$ and $h_{i,k,0}$ in Γ_2, so that $jh_{i,k,0}N = 0$ and $\Gamma_2 = \coprod h_{k,0}(\Gamma_2 \cap \Gamma_3)$. Then the image of ϕ in $H^n(\Gamma_2, M_1)$ by the lower route is represented by:

$$(\gamma_1, ..., \gamma_n) \longmapsto \sum h_{k,0} j \circ \phi(h_{k,0}^{-1}\gamma_1 h_{k,1}, ..., h_{k,n-1}^{-1}\gamma_n h_{k,n})$$

where $h_{k,l}$ is defined by $h_{k,l-1}^{-1}\gamma_l h_{k,l} \in \Gamma_3$ and $h_{k,l} = h_{k',0}$ for some k'. Moreover if we define $h_{i,k,l}$ in the similarly, then the image of ϕ by the upper route is represented by:

$$\begin{aligned}(\gamma_1, ..., \gamma_n) \quad \longmapsto \quad & j \sum h_{k,0}\phi(h_{k,0}^{-1}\gamma_1 h_{k,1}, ..., h_{k,n-1}^{-1}\gamma_n h_{k,n}) \\ & \quad + j \sum h_{i,k,0}\phi(h_{i,k,0}^{-1}\gamma_1 h_{i,k,1}, ...) \\ & = \sum h_{k,0} j \circ \phi(h_{k,0}^{-1}\gamma_1 h_{k,1}, ..., h_{k,n-1}^{-1}\gamma_n h_{k,n})\end{aligned}$$

1.2 Some Applications

The basic idea in the application of this lemma is that if G is a reductive group, P a parabolic subgroup, L a Levi component of P, A the split component of its centre, $\Gamma_P(N)$ the inverse image of $P(\mathbb{Z}/N\mathbb{Z})$ under $G(\mathbb{Z}) \to G(\mathbb{Z}/N\mathbb{Z})$, M a G module with weights $\Phi \subset X^*(A)$, $\phi_0 \in \Phi$ a lowest weight with respect to the partial order corresponding to P, and $\nu \in X_*(A)$ is such that $\nu.\phi \geq 0$ for all $\phi \in \Phi$ with equality if and only if $\phi = \phi_0$; then one can define a map $j : M(\mathbb{Z}/N\mathbb{Z}) \to M_{\phi_0}(\mathbb{Z}/N\mathbb{Z})$ of $\Gamma_P(N)$-modules such that $j : \nu(N)M(\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \nu(N)M_{\phi_0}(\mathbb{Z}/N\mathbb{Z})$ (here M_{ϕ_0} denotes the ϕ_0 eigenmodule). By our lemma we then see that $H^\bullet(\Gamma_P(N), M(\mathbb{Z}/N\mathbb{Z}))$ depends up to the action of $[\Gamma_P(N)\nu(N)\Gamma_P(N)]$ only on the action of L on $M_{\phi_0}(\mathbb{Z}/N\mathbb{Z})$. No doubt this can be formalised in this generality, but we shall simply treat several examples.

Example 1.1

We shall prove:

Theorem 1.1 *Fix a prime p and an extension of the p-adic valuation on $\mathbb{Q}$ to $\mathbb{Q}^{ac}$ (i.e. $\mathbb{Q}^{ac} \subset \mathbb{Q}_p^{ac}$) and an integer N. Fix also a constant C. Then the sum of the dimensions of the eigenspaces in $S_k(\Gamma_1(N))$ for the Hecke operator $T_p = [\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(N)]$ for which the corresponding eigenvalue has p-adic valuation less than C is bounded independently of k.*

Note that we shall here employ the standard notation for elliptic modular forms. We shall also use standard facts about them without comment. See for example [Sh2].

Proof: We first reduce to the case $p|N$. So suppose $p \nmid N$ and without loss of generality $k > C^2 + 1$. If $f \in S_k(\Gamma_0(N), \chi)$ is an eigenvalue for T_p with eigenvalue a_p where $\mathrm{val}_p(a_p) < C$ then the equation $X^2 - a_p X + \chi(p)p^{k-1}$ has a root α with $\mathrm{val}_p(\alpha) = \mathrm{val}_p(a_p)$ and a root β with $\mathrm{val}_p(\beta) > C$, and $f(z) - \beta f(pz) \in S_k(\Gamma_1(Np))$ is an eigenvector for $[\Gamma_1(Np) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1(Np)]$ with eigenvalue α. Moreover if $f_1, ..., f_r \in S_k(\Gamma_1(N))$ are linearly independent and if $\beta_1, ..., \beta_r \in \mathbb{C}$ then the functions $f_i(z) - \beta_i f_i(pz)$ are linearly independent in $S_k(\Gamma_1(Np))$. The desired reduction now follows at once.

Thus assume $p|N$. By a theorem of Eichler and Shimura it will do to establish the theorem with $H^1(\Gamma_1(N), S^{k-2}((\mathbb{Q}_p^{ac})^2))$ in place of $S_k(\Gamma_1(N))$, and $T_p = [\Gamma_1(N) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(N)]$. (Here S^n denotes the n^{th} symmetric power.) In fact it will do to consider $H^1(\Gamma_1(Np^r), S^{k-2}((\mathbb{Q}_p^{ac})^2))$ for any $r \geq 0$ (because $p|N$ implies that T_p commutes with restriction from $\Gamma_1(N)$ to $\Gamma_1(Np^r)$). Let $B_k(\mathbb{Q}_p^{ac})$ denote the sum of the eigenspaces of T_p in this cohomology group, which have p-adic valuation less than C. Also let $B_k = B_k(\mathbb{Q}_p^{ac}) \cap H^1(\Gamma_1(Np^r), S^{k-2}(\mathbb{Z}_p^2))^{TF}$ (where TF indicates the torsion free quotient). Then $B_k(\mathbb{Q}_p^{ac}) = B_k \otimes_{\mathbb{Z}_p} \mathbb{Q}_p^{ac}$. Take $M = H^1(\Gamma_1(Np^r), S^n((\mathbb{Z}/p^r\mathbb{Z})^2))$ for some fixed choice of $n > C$ and for $r = n(n+1)$. Then for $k \geq n+2$ we have a natural projection map:

$$j : S^{k-2}((\mathbb{Z}/p^r\mathbb{Z})^2) \longrightarrow S^n((\mathbb{Z}/p^r\mathbb{Z})^2)$$

as $\Gamma_1(Np^r)$-modules. If moreover $g = \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}$ we have:

$$j : gS^{k-2}((\mathbb{Z}/p^r\mathbb{Z})^2) \xrightarrow{\sim} gS^n((\mathbb{Z}/p^r\mathbb{Z})^2)$$

and so by our lemma 1.1 we know that the kernel of:

$$B_k/p^rB_k \hookrightarrow H^1(\Gamma_1(Np^r), S^{k-2}((\mathbb{Z}/p^r\mathbb{Z})^2)) \xrightarrow{j_*} M$$

is killed by T_p^n. Thus $r(\mathrm{rk}\, B_k) - \mathrm{val}_p\,(\det T_p^n) \leq \mathrm{val}_p\,(\#M)$ so that $r(\mathrm{rk}\, B_k) \leq \mathrm{val}_p\,(\#M) + nC(\mathrm{rk}\, B_k)$ and $\mathrm{rk}\, B_k \leq \mathrm{val}_p\,(\#M)$ as desired.

Before giving further examples we recall the notion of "Hida idempotent". If M is a $\mathbb{Z}_p$ module with $End_{\mathbb{Z}_p}(M)$ a finite $\mathbb{Z}_p$-module (for example if M or $Hom(M, \mathbb{Q}_p/\mathbb{Z}_p)$ is a finite $\mathbb{Z}_p$-module) and if $U : M \to M$ is an endomorphism then there exists a unique idempotent $e_U \in \mathbb{Z}_p[U] \subset End_{\mathbb{Z}_p}(M)$ such that:

1. U is invertible on $e_U M$
2. U is topologically nilpotent on $(1 - e_U)M$
3. $e_U = \lim_{r\to\infty} U^{r!}$
4. if U commutes with another operator T so does e_U

If M' is a second such module with an operator U' and $T : M \to M'$ is such that $TU = U'T$ then $e_{U'}T = Te_U$. These results are all easy consequences of the discussion in [MW2] (section 4). If A is a $\mathbb{Z}_p$-algebra we can think of $e_U \in End_A(M \otimes A)$. Moreover if M and U are defined over $\mathbb{Z}$, say $M = M_0 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ then $e_U \in End_R(M_0 \otimes_{\mathbb{Z}} R)$ where $R = \mathbb{Z}_p \cap \mathbb{Q}^{ac}$. In particular if we fix $\mathbb{Q}^{ac} \subset \mathbb{C}$ and $\mathbb{Q}^{ac} \subset \mathbb{Q}_p^{ac}$ we may think of $e_U \in End_{\mathbb{C}}(M_0 \otimes \mathbb{C})$. Exactly similar results hold with $\mathbb{Z}_p$ replaced by the completion of the integers of any number field at any finite prime.

Note that we may deduce from the above example the following result of Hida:

Corollary 1.1 *Fix a prime p, an integer N and embeddings $\mathbb{Q}^{ac} \subset \mathbb{C}$ and $\mathbb{Q}^{ac} \subset \mathbb{Q}_p^{ac}$. Let e be the idempotent associated to the Hecke operator T_p (defined as above) on $S_k(\Gamma_1(N))$. Then $\dim eS_k(\Gamma_1(N))$ is bounded independently of k.*

We now consider our second:

Example 1.2

We recall some facts about Sp_{2g}. Fix a maximal torus T in $Sp_{2g}(\mathbb{C})$ consisting of diagonal matrices in the standard representation into GL_{2g}. Fix $X^*(T) \cong \mathbb{Z}^g$ by setting $\vec{n} = (n_1, ..., n_g) : (diag(\mu_1, .., \mu_g, \mu_1^{-1}, .., \mu_g^{-1})) = \mu_1^{n_1} ... \mu_g^{n_g}$. Also fix a Borel B consisting of matrices of the form:

$$\begin{pmatrix} * & 0 & \dots & 0 & * & . & . & * \\ * & * & & 0 & . & & & . \\ \vdots & & \ddots & \vdots & . & & & . \\ * & * & \dots & * & * & . & . & * \\ 0 & . & . & 0 & * & \dots & * & * \\ . & & & . & \vdots & \ddots & & \vdots \\ . & & & . & 0 & & * & * \\ 0 & . & . & 0 & 0 & \dots & 0 & * \end{pmatrix}$$

Then the roots Φ of T on $\mathfrak{sp}_{2g}$, the Lie algebra of Sp_{2g}, are:

- α_{ij} the vector consisting of zeroes except for 1 in the i^{th} place and -1 in the j^{th} place $(i \neq j)$
- β_{ij} the vector consisting of zeroes except for 1 in the i^{th} and j^{th} places $(i > j)$
- β_{ii} the vector consisting of zeroes except for 2 in the i^{th} place
- $\gamma_{ij} = -\beta_{ij}$ $(i \geq j)$

With respect to B the positive roots Φ^+ are the α_{ij} with $i > j$ and the β_{ij}. Note that:

$$\check{\alpha}_{ij} = \begin{pmatrix} \epsilon_{ii} - \epsilon_{jj} & 0 \\ 0 & \epsilon_{jj} - \epsilon_{ii} \end{pmatrix}$$

$$\check{\beta}_{ij} = \begin{pmatrix} \epsilon_{ii} + \epsilon_{jj} & 0 \\ 0 & -\epsilon_{ii} - \epsilon_{jj} \end{pmatrix}$$

$$\check{\beta}_{ii} = \begin{pmatrix} \epsilon_{ii} & 0 \\ 0 & -\epsilon_{ii} \end{pmatrix}$$

$$\check{\gamma}_{ij} = -\check{\beta}_{ij}$$

where ϵ_{ij} is the $g \times g$ matrix with one in the i^{th} row and j^{th} column and zeroes elsewhere. Also $\mathfrak{sp}_{2g}(\mathbb{C})$ has a Chevalley basis consisting of the following elements:

$$X_{ij} = \begin{pmatrix} \epsilon_{ij} & 0 \\ 0 & -\epsilon_{ij} \end{pmatrix} \qquad (i \neq j)$$

$$Y_{ij} = \begin{pmatrix} 0 & \epsilon_{ij} + \epsilon_{ji} \\ 0 & 0 \end{pmatrix} \qquad (i > j)$$

$$Y_{ii} = \begin{pmatrix} 0 & \epsilon_{ii} \\ 0 & 0 \end{pmatrix}$$

$$Z_{ij} =^t Y_{ij} \qquad (i \geq j)$$

together with $\check{\alpha}_{(i+1)i}$ for $i = 1, ..., g-1$ and $\check{\beta}_{11}$.

We shall let $\mathcal{U}_{\mathbb{Z}}$ denote the $\mathbb{Z}$-subalgebra of the universal enveloping algebra of $\mathfrak{sp}_{2g}(\mathbb{C})$ generated by elements of the form $\frac{X_{ij}^n}{n!}$, $\frac{Y_{ij}^n}{n!}$ and $\frac{Z_{ij}^n}{n!}$. If V is an $\mathfrak{sp}_{2g}(\mathbb{C})$ module then by an admissible lattice $L \subset V$ we shall mean a $\mathbb{Z}$ lattice preserved by $\mathcal{U}_{\mathbb{Z}}$. Then it is known that any finite dimensional irreducible $\mathfrak{sp}_{2g}(\mathbb{C})$ module contains an admissible lattice, for example $\mathcal{U}_{\mathbb{Z}}v$ for any lowest weight vector v, and moreover that any admissible lattice is equal to the sum of its intersections with the weight spaces of T in V.

Let R denote $\mathbb{Z}$ or $\mathbb{Z}/N\mathbb{Z}$. Let L be an admissible lattice in V_L where $\rho_L : Sp_{2g}(\mathbb{C}) \to GL_{V_L}$. We can define $G_L(R)$ to be the subgroup of $GL_{L\otimes R}$ generated by elements of the form $\exp X_{ij}$, $\exp Y_{ij}$ and $\exp Z_{ij}$. We shall let $P_L(R)$ denote the subgroup of $G_L(R)$ generated by the $\exp X_{ij}$ and the $\exp Y_{ij}$, and $S_L(R)$ the one generated by the $\exp X_{ij}$. It is known that if L_1 and L_2 are as above with $ker\rho_{L_1} \subset ker\rho_{L_2}$ then there is a unique map $G_{L_1}(R) \to G_{L_2}(R)$ taking $\exp W \in G_{L_1}(R)$ to $\exp W \in G_{L_2}(R)$ for W equal to any X_{ij}, Y_{ij} or Z_{ij}. (See [Sg].) We see that this map takes $P_{L_1}(R)$ to $P_{L_2}(R)$ and $S_{L_1}(R)$ to $S_{L_2}(R)$. In particular we see that $G_L(R)$, $P_L(R)$ and $S_L(R)$ depend only on $ker\rho_L$ up to canonical isomorphism.

If ρ_L is the standard (faithful) $2g$ dimensional representation of $Sp_{2g}(\mathbb{C})$ then $G_L(R) = Sp_{2g}(R)$, $P_L(R) = P(R)$ the subset of matrices of the form $\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}$ with $A \in SL_{2g}(R)$ and $S_L(R) = SL_g(R)$ the subset of these matrices with $B = 0$. We expect that this is well known, but know no references. Briefly one shows by performing row and column operations that $Sp_{2g}(R)$ is generated by matrices of the following forms:

1. $\begin{pmatrix} 1-\epsilon_{ij} & 0 \\ 0 & 1+\epsilon_{ij} \end{pmatrix} = \exp(-X_{ij})$ for $i \neq j$
2. $\begin{pmatrix} 1_g & \epsilon_{ij}+\epsilon_{ji} \\ 0 & 1_g \end{pmatrix} = \exp(Y_{ij})$ for $i > j$
3. $\begin{pmatrix} 1_g & \epsilon_{ii} \\ 0 & 1_g \end{pmatrix} = \exp(Y_{ii})$
4. $\begin{pmatrix} 1_g & 0 \\ \epsilon_{ij}+\epsilon_{ji} & 1_g \end{pmatrix} = \exp(Z_{ij})$ for $i > j$
5. $\begin{pmatrix} 1_g - \epsilon_{ii} - \epsilon_{jj} - \epsilon_{ji} + \epsilon_{ij} & 0 \\ 0 & 1_g - \epsilon_{ii} - \epsilon_{jj} - \epsilon_{ji} + \epsilon_{ij} \end{pmatrix}$
$= \exp(X_{ij})\exp(-X_{ji})\exp(X_{ij})$ for $i \neq j$
6. $\begin{pmatrix} 1_g - \epsilon_{ii} & \epsilon_{ii} \\ -\epsilon_{ii} & 1_g - \epsilon_{ii} \end{pmatrix} = \exp(Y_{ii})\exp(-Z_{ii})\exp(Y_{ii})$

that $P(R)$ is generated by matrices of types 1), 2), 3), and 5); and that $SL_g(R)$ is generated by those of type 1) and 3).

Let:

$$\Gamma_1(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(R) \,|\, C \equiv 0 \bmod N \quad \det A \equiv 1 \bmod N \right\}$$

Then if L is any admissible lattice we have maps:

$$\begin{array}{ccccc} Sp_{2g}(\mathbb{Z}) & \longrightarrow & G_L(\mathbb{Z}/N\mathbb{Z}) & & \\ \bigcup & & \bigcup & & \\ \Gamma_1(N) & \longrightarrow & P_L(\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & S_L(\mathbb{Z}/N\mathbb{Z}) \end{array}$$

compatible with the map $L \to L \otimes \mathbb{Z}/N\mathbb{Z}$. In particular $\Gamma_1(N) \twoheadrightarrow SL_g(\mathbb{Z}/N\mathbb{Z})$ where $\exp X_{ij} \mapsto \exp X_{ij}$ and $\exp Y_{ij} \mapsto 0$.

Recall that the irreducible representations of $Sp_{2g}(\mathbb{C})$ are parametrised by vectors $\vec{n} \in X^*(T)$ with $0 \leq n_1 ... \leq n_g$. We shall denote the set of such vectors $X^*(T)^+$. Let $V_{\vec{n}}$ denote the $Sp_{2g}(\mathbb{C})$ module parametrised by $\vec{n}$, and give it a $GSp_{2g}(\mathbb{C})$ action by letting $\mu 1_{2g}$ act by $\mu^{|\vec{n}|}$, where $|\vec{n}| = \sum n_i$. Note that if $\vec{a} \in X^*(T)$ is a weight of T on $V_{\vec{n}}$ then $\begin{pmatrix} \mu 1_g & 0 \\ 0 & 1_g \end{pmatrix}$ acts on the corresponding weight space $V_{\vec{n}}^{\vec{a}}$ as $\mu^{\frac{1}{2}\sum(n_i+a_i)}$ and that $\frac{1}{2}\sum(n_i + a_i) \in \mathbb{Z}_{\geq 0}$. In particular if $\mu \in \mathbb{Z}$ then $\begin{pmatrix} \mu 1_g & 0 \\ 0 & 1_g \end{pmatrix}$ preserves any admissible lattice.

Now choose $v_{\vec{n}} \in V_{\vec{n}}$ a lowest weight vector. Set $L_{\vec{n}} = \mathcal{U}_{\mathbb{Z}} v_{\vec{n}}$, an admissible lattice in $V_{\vec{n}}$. Let $V'_{\vec{n}} = \bigoplus V_{\vec{n}}^{\vec{a}}$, where the sum is taken over weights $\vec{a}$ with $\sum(n_i + a_i) = 0$. Also let $\mathcal{U}'_{\mathbb{Z}}$ be the subalgebra of $\mathcal{U}_{\mathbb{Z}}$ generated by the elements $\frac{X_{ij}^n}{n!}$ and $L'_{\vec{n}} = \mathcal{U}'_{\mathbb{Z}} v_{\vec{n}}$. Then $L'_{\vec{n}} \subset V'_{\vec{n}}$. In fact it is known that $L_{\vec{n}}$ is spanned over $\mathbb{Z}$ by vectors of the form $\prod_{i>j} \frac{X_{ij}^{a_{ij}}}{a_{ij}!} \prod_{i \geq j} \frac{Y_{ij}^{b_{ij}}}{b_{ij}!} v_{\vec{n}}$ and so we see that $L'_{\vec{n}} = L_{\vec{n}} \cap V'_{\vec{n}}$ and this is a direct summand of $L_{\vec{n}}$ (for an element of the above form lies in $V'_{\vec{n}}$ if and only if $b_{ij} = 0$ for all i, j).

Fix a positive integer N. $L_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z})$ is a $\Gamma_1(N)$-module and this action factors through $P_{L_{\vec{n}}}$. We can also make $L'_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z})$ a $\Gamma_1(N)$ module through the map $\Gamma_1(N) \to S_{L_{\vec{n}}}(\mathbb{Z}/N\mathbb{Z})$. I claim that with these actions the projection map:

$$j : L_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z}) \longrightarrow L'_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z})$$

is a map of $\Gamma_1(N)$-modules. But it will do to show that:

- $j(X_{ij}v) = X_{ij}j(v)$
- $j(Y_{ij}v) = 0$

and these are both clear. Moreover if $g = \begin{pmatrix} N1_g & 0 \\ 0 & 1_g \end{pmatrix}$ then:

$$j : g(L_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z})) \xrightarrow{\sim} g(L'_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z}))$$

Thus if $\Gamma \subset \Gamma_1(N)$ is of finite index, and if U denotes the Hecke operator $[\Gamma g \Gamma]$ we see that our proposition implies that there exists:

$$I : H^\bullet(\Gamma, L'_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z})) \longrightarrow H^\bullet(\Gamma, L_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z}))$$

such that $I \circ j_* = U$ and $j_* \circ I = U$. It is well known that these cohomology groups are finitely generated abelian groups and so if $N = p$ a prime we can associate a Hida idempotent e to U. Then we have that:

$$j_* : eH^\bullet(\Gamma, L_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z})) \xrightarrow{\sim} eH^\bullet(\Gamma, L'_{\vec{n}} \otimes (\mathbb{Z}/N\mathbb{Z}))$$

Now consider $SL_g \subset GSp_{2g}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$. Then we see from the fact that $V'_{\vec{n}} = L'_{\vec{n}} \otimes \mathbb{C}$ that $V'_{\vec{n}}$ is an irreducible $\mathfrak{sl}_g(\mathbb{C})$-module of heighest weight depending only on the $(g-1)$-tuple $(n_1 - n_2, ..., n_1 - n_g)$. Thus if $\vec{m} \in \mathbb{Z}^g$ with $0 \leq m_1 \leq ... \leq m_g$ and $m_i - m_1 = n_i - n_1$ for $i = 2, ..., g$ then we have an isomorphism of $\mathfrak{sl}_g(\mathbb{C})$-modules $V'_{\vec{n}} \xrightarrow{\sim} V'_{\vec{m}}$ such that $v_{\vec{n}} \mapsto v_{\vec{m}}$, and so $L'_{\vec{n}} \xrightarrow{\sim} L'_{\vec{m}}$ preserving the action of the X_{ij}. Thus $eH^\bullet(\Gamma, L_{\vec{n}} \otimes \mathbb{F}_p)$ depends up to canonical isomorphism only on the $(g-1)$-tuple $(n_2 - n_1, ..., n_g - n_1)$. We deduce:

Theorem 1.2 *Let p be a prime, $\Gamma \subset \Gamma_1(p)$ a subgroup of finite index. Fix $\mathbb{Q}^{ac} \subset \mathbb{Q}_p^{ac}$ and $\mathbb{Q}^{ac} \subset \mathbb{C}$. Then we can associate a Hida idempotent e to the action of $[\Gamma \begin{pmatrix} p1_g & 0 \\ 0 & 1_g \end{pmatrix} \Gamma]$ on $H^\bullet(\Gamma, V_{\vec{n}})$ for $\vec{n} \in X^*(T)^+$, and* $\dim eH^\bullet(\Gamma, V_{\vec{n}+m\vec{t}})$ *is bounded independently of $m \geq 0$, where $\vec{t} = (1, ..., 1) \in \mathbb{Z}^g$.*

Proof: Set $\vec{m} = \vec{n} + m\vec{t}$. Then it will do to show that $\dim eH^\bullet(\Gamma, L_{\vec{m}} \otimes \mathbb{Q}_p)$ is so bounded. But we have seen that $\dim eH^\bullet(\Gamma, L_{\vec{m}} \otimes \mathbb{F}_p)$ is so bounded and we have that:

- $eH^\bullet(\Gamma, L_{\vec{m}} \otimes \mathbb{Z}_p) \otimes \mathbb{F}_p \hookrightarrow eH^\bullet(\Gamma, L_{\vec{m}} \otimes \mathbb{F}_p)$
- $\dim eH^\bullet(\Gamma, L_{\vec{m}} \otimes \mathbb{Z}_p) \otimes \mathbb{Q}_p = \dim eH^\bullet(\Gamma, L_{\vec{m}} \otimes \mathbb{Q}_p)$

so the result follows. (The first embedding comes from the long exact sequence corresponding to $0 \to \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \to \mathbb{F}_p \to 0$.)

Chapter 2

Some Congruences between Siegel Modular Forms

2.1 Introduction

In this chapter we are concerned with showing how, starting with a Siegel modular form f of low weight which is an eigenform of the Hecke operators on a certain congruence subgroup of $\Gamma_0(p)$ (those matrices in $Sp_{2g}(\mathbb{Z})$ congruent to $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mod p) which is ordinary in the sense that it is an eigenvalue of a certain Hecke operator U_p (see section 2.2) with eigenvalue a p-adic unit; we can find a series of eigenforms of heigher weight whose eigenvalues under the Hecke operators tend to those of the first form p-adically. We apply this to show how standard conjectures about the existence of p-adic Galois representations corresponding to such forms of genus two, if true for high weight (where one hopes to find the representations in certain p-adic cohomology groups) would also be true for ordinary forms of low weight.

To explain our results more precisely recall that given g integers $0 \leq n_1 \leq ... \leq n_g$ we may consider Siegel modular forms of genus g and of weight $\vec{n} = (n_1, ..., n_g)$. These are holomorphic functions on Siegel modular space valued in the irreducible representation of $GL_g(\mathbb{C})$ with heighest weight $\vec{n}$ and which have certain transformation properties. If $\frac{1}{2}g(g+1) \leq n_1$ then such forms correspond to automorphic representations of $GSp_{2g}(\mathbb{A})$

which are holomorphic discrete series at infinity. In this case one expects (conjecturally) to be able to associate to certain such modular forms (those which are eigenforms of a Hecke algebra) a system of 2^g dimensional l-adic Galois representations, and to be able to find these representations in the cohomology of certain sheaves on canonical models of certain quotients of Siegel modular space. In the case $n_1 = \frac{1}{2}g(g+1) - 1$ such forms correspond to automorphic representations of $GSp_{2g}(\mathbb{A})$ which are limit of holomorphic discrete series at infinity. One still expects to be able to associate Galois representations to such forms, but one can no longer expects to be able to find them geometrically.

We shall show how to reduce the second case to the first in the case of "ordinary" forms of genus two. The restriction on the genus is probably not important but it simplifies some of the arguments and genus two is the case of most interest for us. An eigenform on a congruence subgroup Γ of $\Gamma_0(p)$ is called ordinary at p if its eigenvalue for the Hecke operator $U_p = [\Gamma \begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} \Gamma]$ is a p-adic unit. An eigenform of genus two for a congruence subgroup Γ of $Sp_4(\mathbb{Z})$ which is dense in $Sp_4(\mathbb{Z}_p)$ is called ordinary at p if a certain quartic polynomial $Q_p(X)$ (see 2.4) associated to f has distinct roots the ratio of no two of which equals p and one of which is a p-adic unit (in the case $n_1 \geq 2$ this is certainly true if $Q_p(X)$ has two distinct roots which are p-adic units).

Our method is as follows. One can write down a theta series θ of weight $(p-1, ..., p-1)$ which is congruent to one modulo p. Multiplying a form f by high p powers of θ produces very congruent forms of heigher weight. This along with some background on Siegel modular forms is discussed in section 2.2. Unfortunately this does not seem to be enough for the applications to Galois representations, the problem being that if one starts with an eigenform of the Hecke operators one does not obtain an *eigenform* highly congruent to it. To overcome this problem in the case of a congruence subgroup contained in $\Gamma_0(p)$ one uses the fact that the number of ordinary eigenforms on Γ of level $\vec{n} + m(1, ..., 1)$ is bounded independently of m. This is proved by embedding the cusp forms in a certain cohomology group and using the results of the first chapter to relate these as m varies. From this it is not difficult to see (by an argument of Wiles using Fitting ideals) that we can lift $f\theta^{p^m}$ to an eigenform of weight $\vec{n} + p^m(p-1)(1,...,1)$ with eigenvalues congruent to those of f modulo $p^{\frac{m+1}{C}}$ for some C independent of m, as we wanted. This argument is discussed in section 2.3. The application to Galois representations is discussed in section 2.5. In section 2.4 we show how to construct from an eigenform Γ on a general congruence subgroup which is dense in $Sp_4(\mathbb{Z}_p)$ an eigenform on $\Gamma \cap \Gamma_0(p)$ which is also an eigenform for U_p. We use this to generalise the results about Galois representations to congruence subgroups not contained in $\Gamma_0(p)$.

2.2 Review of Siegel Modular Forms

Fix an integer $g \geq 1$. Let $GSp_{2g}(R)$ denote the set of $\alpha \in GL_{2g}(R)$ such that:

$$\alpha \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} {}^t\alpha = \nu(\alpha) \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

for some $\nu : GSp_{2g}(R) \to R^\times$. Let $Sp_{2g}(R)$ be those elements of $GSp_{2g}(R)$ in the kernel of ν. Let $G(R) = GSp_{2g}(R)$, $G_\infty = G(\mathbb{R})$, $G_\infty^+ = \nu^{-1}\mathbb{R}_{>0}^\times \subset G_\infty$, $G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G_\infty^+$, and $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the adeles (resp. finite adeles) of $\mathbb{Q}$. Let U_∞ denote the group of:

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL_{2g}(\mathbb{R})$$

such that A^tB is symmetric and $A^tA + B^tB$ is a nonzero scalar, or:

$$U_\infty = \left\{(\alpha, \nu) \in GL_g(\mathbb{C}) \times \mathbb{C}^\times \mid (\alpha, \nu)^* = (\alpha, \nu)\right\}$$

where $(\alpha, \nu)^* = (\nu^c({}^t\alpha^c)^{-1}, \nu^c)$ (c denoting complex conjugation), and the correspondence is given by:

$$\beta = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto (A - iB, \nu(\beta))$$

Thus U_∞ is a real form of $GL_g(\mathbb{C}) \times \mathbb{C}^\times$. It is in fact the $g \times g$ unitary similitudes.

Let $\mathcal{Z} = \mathcal{Z}_g$ denote the set of symmetric complex $g \times g$ matrices $x + \sqrt{-1}y$ with y positive definite. Then G_∞^+ acts on $\mathcal{Z}$ by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : z \to (Az+B)(Cz+D)^{-1}$. If $z_0 = (\sqrt{-1})1_g$

then $\alpha \mapsto \alpha z_0$ gives a bijection $G_\infty^+/U_\infty \stackrel{\sim}{\to} \mathcal{Z}$. We define a map $J : G_\infty^+ \times \mathcal{Z} \to GL_g(\mathbb{C}) \times \mathbb{C}^\times$ by:

$$\left(\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, z\right) \longrightarrow (Cz + D, \nu(\alpha))$$

so that:

$$J(\alpha\beta, z) = J(\alpha, \beta z) J(\beta, z)$$

If ρ is a finite dimensional representation of $GL_g(\mathbb{C}) \times \mathbb{C}^\times$ on a complex vector space V then we set $J_\rho = \rho \circ J : G_\infty^+ \times \mathcal{Z} \to Aut(V)$.

If $U \subset G(\mathbb{A}_f)$ is an open compact subgroup we let $\mathcal{S}_\rho(U)$ denote the space of functions $\phi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to V$ such that:

- $\phi(guu_\infty) = \rho(u_\infty)^{-1}\phi(g)$ for all $g \in G(\mathbb{A})$, $u \in U$ and $u_\infty \in U_\infty$
- if $h \in G(\mathbb{A})$ then the function:

$$\begin{array}{rcl} f_h : \mathcal{Z} & \longrightarrow & V \\ \alpha z_0 & \longmapsto & J_\rho(\alpha, z_0)\phi(h\alpha) \end{array}$$

  where $\alpha \in G_\infty^+$, is holomorphic. (It is easily checked that this function is well defined.)
- $\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(nh)dn = 0$ where N is the unipotent radical of any proper parabolic subgroup and where dn is any invariant measure on $N(\mathbb{Q})\backslash N(\mathbb{A})$

We set $\mathcal{S}_\rho = \bigcup \mathcal{S}_\rho(U)$ as U ranges over open compact subgroups. Then $G(\mathbb{A}_f)$ acts on $\mathcal{S}_\rho$ on the right by $(\phi|g)(h) = \phi(hg^{-1})$, and $\mathcal{S}_\rho(U) = \mathcal{S}_\rho^U$. We define similarly $\mathcal{M}_\rho(U)$ by omitting the last condition (assuming $g > 1$, which is the only case we shall be concerned with as the case $g = 1$ is well known).

Similarly if $\Gamma \subset G_\infty^+$ is a discrete subgroup we set $S_\rho(\Gamma)$ to be the set of holomorphic functions $f : \mathcal{Z} \to V$ such that:

- $f|\gamma = f$ for all $\gamma \in \Gamma$
- $\lim_{\lambda \to +\infty} (f|\gamma) \begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix} = 0$ for all $\gamma \in G(\mathbb{Q})^+$ and all $z \in \mathcal{Z}_{g-1}$

where for $\gamma \in G_\infty^+$ we define:

$$(f|\gamma)(z) = J_\rho(\gamma, z)^{-1} f(\gamma z)$$

Similarly we may define $M_\rho(\Gamma)$ by dropping the last condition (if $g > 1$).

Assume $U \subset G(\mathbb{A}_f)$ is an open compact subgroup such that $(\nu U)\mathbb{Q}^\times \mathbb{R}_{>0}^\times = \mathbb{A}^\times$. Then if we set $\Gamma_U = U \cap G(\mathbb{Q})^+$ we have an isomorphism:

$$\mathcal{S}_\rho(U) \cong S_\rho(\Gamma_U)$$

given by:

$$\phi \longmapsto (f_\phi : \alpha z_0 \mapsto J_\rho(\alpha, z_0)\phi(\alpha))$$

and inversely by:

$$f \longmapsto (\phi_f : \gamma u \alpha \mapsto J_\rho(\alpha, z_0)^{-1} f(\alpha z_0))$$

where $\alpha \in G_\infty^+$, $u \in U$ and $\gamma \in G(\mathbb{Q})$. The second map is well defined as, by the strong approximation theorem and our assumption on U, we have that $G(\mathbb{A}) = G(\mathbb{Q})UG_\infty^+$. If $h \in G(\mathbb{A}_f)$ and $h = u\gamma$ with $u \in U$ and $\gamma \in G(\mathbb{Q})^+$ then:

$$\phi_f|h = \phi h|\gamma \quad \text{and} \quad f_{\phi|h} = f_\phi|\gamma$$

Now let U and U' be open compact subgroups of $G(\mathbb{A}_f)$ and let $g \in G(\mathbb{A}_f)$, then we define a Hecke operator:

$$\begin{aligned} [UgU'] : \mathcal{S}_\rho(U) \quad &\longrightarrow \quad \mathcal{S}_\rho(U') \\ \phi \quad &\longmapsto \quad \sum \phi|g_i \end{aligned}$$

where $UgU' = \coprod Ug_i$. If U and U' also satisfy the condition of the last paragraph we can think of $[UgU'] : S_\rho(\Gamma_U) \to S_\rho(\Gamma_{U'})$. It is given by $f \mapsto \sum f|\gamma_i$ where $g_i = u_i\gamma_i$ with $\gamma_i \in G(\mathbb{Q})^+$ and $u_i \in U$. Equivalently we may write $g = u\gamma$ with $\gamma \in G(\mathbb{Q})^+$ and $u \in U$, and then the γ_i's may be defined by $\Gamma_U \gamma \Gamma_{U'} = \coprod \Gamma_U \gamma_i$.

It is well known that if V is irreducible there is an inner product on V, say $\langle\, , \rangle$, such that:

$$\langle \rho(\alpha)v_1, \rho(\alpha)v_2 \rangle = \nu(\alpha)^\mu \langle v_1, v_2 \rangle$$

for all $\alpha \in U_\infty$ and $v_1, v_2 \in V$, and where μ depends only on ρ (in fact $\rho(x1_g, x^2) = x^\mu$). Now define an inner product on $\mathcal{S}_\rho(U)$ by:

$$\langle \phi_1, \phi_2 \rangle = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})/UU_\infty} \langle \phi_1(g), \phi_2(g) \rangle ||\nu(g)||^{-\mu} dg$$

where dg is an invariant measure on $G(\mathbb{Q})\backslash G(\mathbb{A})/UU_\infty$ and $||.|| : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{R}^\times_{>0}$ by $x \mapsto \prod |x_v|_v$. This is easily checked to be well defined. Moreover the adjoint of g is $||\nu(g)||^{-\mu} g^{-1}$ (if the two measures are normalised correctly) and the adjoint of $[UgU']$ is $||\nu(G)||^{-\mu}[U'g^{-1}U]$.

We now introduce some specific representations ρ. We recall first some multilinear algebra. Let $0 \leq n_1 \leq ... \leq n_g$ be integers, and set $\vec{n} = (n_1, ..., n_g)$ and $|\vec{n}| = n_1 + ... + n_g$. Also let $\vec{t} = (1, ..., 1)$. $S_{|\vec{n}|}$, the symmetric group on $|\vec{n}|$ letters acts on $M^{\otimes|\vec{n}|}$. There is an element c in $\mathbb{Z}[S_{|\vec{n}|}]$ (unique up to ± 1) satisfying:

- $c^2 = \mu c$ for some scalar μ
- $c = \sum_{\sigma \in S_{|\vec{n}|}} \delta_\sigma \sigma$ where $\delta_\sigma = 0$, 1 or -1
- $c\sigma = c$ if σ preserves the sets $\{1, ..., n_1\}$, $\{n_1 + 1, ..., n_1 + n_2\}$, etc.
- $\sigma c = (-)^\sigma c$ (where $(-)^\sigma$ denotes the sign of sigma) if σ preserves sets of the form $\{n_{i_{j-1}} + j, n_{i_{j-1}+1} + j, ..., |\vec{n}| - n_g + j\}$ where i_j is the least index such that $n_{i_j} \geq j$ (where $n_0 = 0$).

We can think of $c \in End(M^{\otimes|\vec{n}|})$. Let $\bigotimes^{\vec{n}} M$ denote $cM^{\otimes|\vec{n}|}$. Then $\bigotimes^{\vec{n}}$ commutes with localisation and if $\alpha : M \to N$ is linear we get a linear map $\otimes^{\vec{n}}(\alpha) : \bigotimes^{\vec{n}}(M) \to \bigotimes^{\vec{n}}(N)$. If R is any ring this gives a natural action of $GL_g(R)$ on $\bigotimes^{\vec{n}}(R^g)$, and if R is a field of characteristic zero this representation is known to be irreducible (see [We]).

Let $W_{\vec{n}}$ denote the $GL_g(R) \times R^\times$ module $\bigotimes^{\vec{n}}(R^g)$ with the above action of $GL_g(R)$ and on which $R^\times$ acts via $\lambda \mapsto \lambda^{\frac{1}{2}g(g+1)-|\vec{n}|}$. Let $\rho_{\vec{n}}$ denote the corresponding representation, let $W_{\vec{n}}$ denote $W_{\vec{n}}(\mathbb{C})$ and drop the ρ when $\rho_{\vec{n}}$ is used as a sub- or super-script.

If R is a ring denote the ring of formal power series:

$$\sum_{h \in symm^*_g(\mathbb{Z})^{\geq 0}} a_h q^h$$

in an indeterminate q by $R[[q]]_g$. Here $symm_g^*(\mathbb{Z})$ denotes the semigroup of $g \times g$ symmetric integral matrices with even diagonal entries, and a superscript ≥ 0 (resp. > 0) indicates those which are positive semi-definite (resp. positive definite). If M is an R-module we define $M[[q]]_g = R[[q]]_g \otimes M$.

If $f \in M_\rho(\Gamma)$ there is an integer N depending only on Γ such that $f(z + Nh) = f(z)$ for $h \in symm_g(\mathbb{Z})$. Thus we have a Fourier expansion:

$$f(z) = \sum_{h \in symm_g^*(\mathbb{Z})^{\geq 0}} a_h(f) \exp(\pi\sqrt{-1}N^{-1}\mathrm{tr}(hz))$$

and so we get an embedding $M_\rho(\Gamma) \hookrightarrow W_{\vec{n}}[[q^{1/N}]]_g$. $f \in S_\rho(\Gamma)$ if and only if $a_h(f|\gamma) = 0$ for all $\gamma \in G(\mathbb{Q})^+$ and for all $\det h = 0$. If $R \subset \mathbb{C}$ we define $M_{\vec{n}}(\Gamma, R) = M_{\rho_{\vec{n}}}(\Gamma) \cap W_{\vec{n}}(R)[[q^{1/N}]]_g$. We define $S_{\vec{n}}(\Gamma, R)$ similarly, and for U an open compact subgroup of $G(\mathbb{A}_f)$ with $(\nu U)\mathbb{Q}^\times\mathbb{R}_{>0}^\times = \mathbb{A}^\times$ we define $\mathcal{S}_{\vec{n}}(U, R)$ to be $S_{\vec{n}}(\Gamma_U, R)$, and similarly for $\mathcal{M}_{\vec{n}}(U, R)$.

Lemma 2.1 *Let $\Gamma \subset G(\mathbb{Q})^+$ be a discrete congruence subgroup, then there is a finite abelian extension $K/\mathbb{Q}$ such that $S_{\vec{n}}(\Gamma) = S_{\vec{n}}(\Gamma, \mathcal{O}_K) \otimes \mathbb{C}$, and similarly $M_{\vec{n}}(\Gamma) = M_{\vec{n}}(\Gamma, \mathcal{O}_K) \otimes \mathbb{C}$.*

Proof: It will clearly do to show the following:

1. $M_{\vec{n}}(\Gamma)$ is finite dimensional

2. $M_{\vec{n}}(\Gamma) = M_{\vec{n}}(\Gamma, \mathbb{Q}^{ab}) \otimes \mathbb{C}$

3. $S_{\vec{n}}(\Gamma) = S_{\vec{n}}(\Gamma, \mathbb{Q}^{ab}) \otimes \mathbb{C}$

4. if $f \in M_{\vec{n}}(\Gamma, \mathbb{Q}^{ab})$ then there exists $0 \neq C \in \mathbb{Q}^{ab}$ with $Cf \in M_{\vec{n}}(\Gamma, \mathcal{O}_{\mathbb{Q}^{ab}})$

1) is well known. If $M_{\vec{n}}$ denotes the union over all congruence subgroups Γ of $M_{\vec{n}}(\Gamma)$ then Shimura has proved that $M_{\vec{n}} = M_{\vec{n}}(\mathbb{Q}) \otimes \mathbb{C}$ and that $G(\mathbb{Q})^+$ preserves $M_{\vec{n}}(\mathbb{Q}^{ab})$ (see [Sh3]). 2) and 3) follow from this. Finally it will do to establish 4) in the special case when Γ is equal to the set of matrices in $Sp_{2g}(\mathbb{Z})$ which are congruent to 1_{2g} modulo N for some $N > 3$.

It is known (see for example [Fa2]) that there is a separated scheme $\mathcal{M}$ and a principally polarised abelian scheme $\mathcal{A}/\mathcal{M}$ of relative dimension g together with an isomorphism α :

$(\boldsymbol{\mu}_N^g \times (\mathbb{Z}/N\mathbb{Z})^g) \times_{\mathbb{Z}} \mathcal{M} \xrightarrow{\sim} \mathcal{A}[N]$ taking the standard pairing $(\boldsymbol{\mu}_N^g \times (\mathbb{Z}/N\mathbb{Z})^g)^2 \to \boldsymbol{\mu}_N$ to the Weil pairing $\mathcal{A}[N]^2 \to \boldsymbol{\mu}_N$ (i.e. a level N structure), with the property that if A/S is a principally polarised abelian scheme with level N structure, then there is a unique map $S \to \mathcal{M}$ such that $A \cong \mathcal{A} \times_{\mathcal{M}} S$ and the polarisation and level N structure on A come by pulling back that on $\mathcal{A}$. If A/S is an abelian scheme, let $\omega_{A/S}$ denote the direct image of the sheaf of relative differentials $\Omega_{A/S}$. Then $\omega_{A/S}$ is quasicoherent on S, and if $T \to S$ then $\omega_{A\times T/T}$ is the inverse image of $\omega_{A/S}$. (These are easy from the definition of Ω and the fact that $A \to S$ is quasicompact.) In particular $\omega_{A/S}$ is the inverse image of $\omega_{\mathcal{A}/\mathcal{M}}$ under the canonical map $S \to \mathcal{M}$, and we get a map $\omega_{\mathcal{A}/\mathcal{M}}(\mathcal{M}) \to \omega_{A/S}(S)$.

Now consider $\mathbb{Z}[[q^{1/N}]]$. There is an $h \in symm_g^*(\mathbb{Z})$ such that if $f = q^h$ and $R = \mathbb{Z}[[q^{1/N}]][f^{-1}]$, then it is known (the Mumford construction, but see [Fa2]) that there is an abelian scheme $A/\mathrm{spec}\ R$ with a canonical isomorphism $\omega_{A/R} \cong R^g$. Thus we obtain a map $\omega_{\mathcal{A}/\mathcal{M}}(\mathcal{M}) \to R^g$. Moreover if $\vec{n}$ is as above we get a map $(\bigotimes^{\vec{n}} \omega_{\mathcal{A}/\mathcal{M}})(\mathcal{M}) \to \bigotimes^{\vec{n}}(R^g)$. If we tensor over $\mathbb{Z}$ with $\mathbb{C}$ we get a commutative diagram:

$$\begin{array}{ccc} (\bigotimes^{\vec{n}} \omega_{\mathcal{A}/\mathcal{M}})(\mathcal{M}) & \longrightarrow & \bigotimes^{\vec{n}}(R^g) \\ \downarrow & & \downarrow \\ (\bigotimes^{\vec{n}} \omega_{\mathcal{A}_{\mathbb{C}}/\mathcal{M}_{\mathbb{C}}})(\mathcal{M}_{\mathbb{C}}) & \longrightarrow & \bigotimes^{\vec{n}}(R \otimes \mathbb{C})^g \end{array}$$

It is further known that $(\bigotimes^{\vec{n}} \omega_{\mathcal{A}_{\mathbb{C}}/\mathcal{M}_{\mathbb{C}}})(\mathcal{M}_{\mathbb{C}}) = M_{\vec{n}}(\Gamma)$ and that the map $(\bigotimes^{\vec{n}} \omega_{\mathcal{A}_{\mathbb{C}}/\mathcal{M}_{\mathbb{C}}})(\mathcal{M}_{\mathbb{C}}) \to \bigotimes^{\vec{n}}(\mathbb{C}[[q^{1/N}]]_g[f^{-1}]^g)$ is just the normal q-expansion. Finally as $\omega_{\mathcal{A}/\mathcal{M}}$ is quasicoherent and $\mathbb{C}$ is flat over $\mathbb{Z}$, $\bigotimes^{\vec{n}}(\omega_{\mathcal{A}_{\mathbb{C}}/\mathcal{M}_{\mathbb{C}}}(\mathcal{M}_{\mathbb{C}})) = \bigotimes^{\vec{n}}(\omega_{\mathcal{A}/\mathcal{M}}(\mathcal{M})) \otimes \mathbb{C}$. Thus the image of $\bigotimes^{\vec{n}}(\omega_{\mathcal{A}/\mathcal{M}}(\mathcal{M}))$ in $M_{\vec{n}}(\Gamma)$ spans $M_{\vec{n}}(\Gamma)$ and each element has a Fourier expansion with coefficients in $\mathbb{Z}$ as desired.

Corollary 2.1 *With the notation as in the lemma, if $\Gamma = \Gamma_U$ for $U \subset \prod GSp_{2g}(\mathbb{Z}_l)$ an open compact subgroup normalised by $i \prod \mathbb{Z}_l^\times$ where $i : \mathbb{G}_m \to GSp_{2g}$ by $t \mapsto \begin{pmatrix} 1_g & 0 \\ 0 & t1_g \end{pmatrix}$, then we may take $K = \mathbb{Q}$.*

Proof: It will do to show that $S_{\vec{n}}(\Gamma, \mathbb{Q}^{ab})$ is stable under the action of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, and similarly for $M_{\vec{n}}(\Gamma, \mathbb{Q}^{ab})$. We shall only treat the first case, which is marginally harder.

Shimura (see [Sh3]) defines an action of $G(\mathbb{A}_f)$ on $M_{\vec{n}}(\mathbb{Q}^{ab})$ such that in particular every function is stabilised by an open subgroup, $G(\mathbb{Q})^+$ has its normal action, and if $t \in \prod \mathbb{Z}_l^\times$ then $f^{it} = f^{(t^{-1},\mathbb{Q}^{ab}/\mathbb{Q})}$, where $(\ ,\mathbb{Q}^{ab}/\mathbb{Q})$ is the Artin symbol. Let $f \in S_{\vec{n}}(\Gamma, \mathbb{Q}^{ab})$ and $t \in \prod \mathbb{Z}_l^\times$, we must show that $f^{it} \in S_{\vec{n}}(\Gamma, \mathbb{Q}^{ab})$.

Firstly let $\alpha \in G(\mathbb{Q})^+$ and $h \in symm_g^*(\mathbb{Z})^{\geq 0}$ with $\det h = 0$. Then we can find $u \in stab_{G(\mathbb{A}_f)}(f)$, $\beta \in G(\mathbb{Q})^+$ and $s \in \prod \mathbb{Z}_l^\times$ such that $i(t)\alpha = u\beta i(s)$, so we see that:

$$a_h(f^{i(t)}|\alpha) = a_h(f^{u\beta i(s)}) = a_h(f|\beta)^{(s^{-1},\mathbb{Q}^{ab}/\mathbb{Q})} = 0$$

Thus $f^{i(t)} \in S_{\vec{n}}(\mathbb{Q}^{ab})$. Secondly let $\alpha \in \Gamma$, then we can find $\beta \in G(\mathbb{Q})^+$, $s \in \prod \mathbb{Z}_l^\times$ and $u \in W = \{x \in \prod G(\mathbb{Z}_l) | x \equiv 1_{2g} \bmod N\} \subset stab_{G(\mathbb{A}_f)}(f) \cap U$ for some N, such that $i(t)\alpha = u\beta i(s)$. Then we see that $\beta \in Sp_{2g}(\mathbb{Z}).\{1, i(-1)\}$ and $\nu(u)s = \pm t$. Without loss of generality we may assume $s = t$, $\nu(u) = 1$ and $\beta \in Sp_{2g}(\mathbb{Z})$. In fact in this case:

$$\begin{aligned} \beta &\in Sp_{2g}(\mathbb{Z}) \cap W.i(t)Ui(t)^{-1} \\ &\subset Sp_{2g}(\mathbb{Z}) \cap U = \Gamma \end{aligned}$$

and so:

$$f^{i(t)}|\alpha = f^{u\beta i(t)} = (f|\beta)^{i(t)} = f^{i(t)}$$

and we are done.

We shall now introduce some particular Hecke operators of special importance for us. Fix a rational prime p and consider an open compact subgroup $U = U_1 \times U_2 \subset \prod_{l \neq p} GSp_{2g}(\mathbb{Z}_l) \times GSp_{2g}(\mathbb{Z}_p)$ satisfying:

- $U \supset \begin{pmatrix} 1_g & 0 \\ 0 & (\prod \mathbb{Z}_l^\times) 1_g \end{pmatrix}$
- there is an integer $r = r(U) \geq 1$ such that U_2 is the set of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_{2g}(\mathbb{Z})$ with $C \equiv 0 \bmod p^r$ and $(A \bmod p^r)$ and $(D \bmod p^r)$ lying in some finite set of possibilities.

Let $N' = N'(U)$ be the smallest positive integer such that U contains all elements of $\prod GSp_{2g}(\mathbb{Z}_l)$ which are congruent to one modulo N', and write $N'(U) = N(U)p^r$ (or if no confusion can arrise $N' = Np^r$. Then we define:

$$U_p = [U \begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} U] = [U \begin{pmatrix} 1_g & 0 \\ 0 & \pi_p 1_g \end{pmatrix} U]$$

where π_p denotes a uniformizer in $\mathbb{Z}_p$. Then we have:

Lemma 2.2 *Let U be as described above. Then $\mathcal{S}_{\vec{n}}(U) = \mathcal{S}_{\vec{n}}(U, \mathbb{Z}) \otimes \mathbb{C}$ and the Hecke operator U_p preserves $\mathcal{S}_{\vec{n}}(U, \mathbb{Z})$. In fact $(\sum a_h q^{h/N})|U_p = \sum a_{ph} q^{h/N}$. Thus we can define a Hida idempotent e on $\mathcal{S}_{\vec{n}}(U)$ as in section 1.2.*

<u>*Proof:*</u>Let $\mathcal{X}$ be a set of representatives for $symm_g(\mathbb{Z})$ modulo p, such that each $X \in \mathcal{X}$ is congruent to zero modulo N. Then:

$$U \begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} U = \coprod_{X \in \mathcal{X}} U \begin{pmatrix} 1_g & X \\ 0 & p1_g \end{pmatrix}$$

as follows easily from the fact that if $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U$ there is an $X \in \mathcal{X}$ with $B \equiv AX \bmod p$ (A is invertible modulo p) and from the equality:

$$\begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & p^{-1}(B - AX) \\ pC & D - CX \end{pmatrix} \begin{pmatrix} 1_g & X \\ 0 & p1_g \end{pmatrix}$$

Now note that for $h \in symm_g^*(\mathbb{Z})$:

$$\sum_{X \in \mathcal{X}} \exp(\pi\sqrt{-1}N^{-1}p^{-1}\mathrm{tr}(hX)) = \begin{cases} p^{\frac{1}{2}g(g+1)} & \text{if } h \in p\ symm_g^*(\mathbb{Z}) \\ 0 & \text{otherwise} \end{cases}$$

(If $h \in p\ symm_g^*(\mathbb{Z})$ this is clear. If not pick $Y \in \mathcal{X}$ with $2Np \not| \mathrm{tr}(hY)$, and then:

$$\begin{aligned} \textstyle\sum_{X \in \mathcal{X}} \exp(\pi\sqrt{-1}N^{-1}p^{-1}\mathrm{tr}(hX)) &= (\textstyle\sum_{X \in \mathcal{X}} \exp(\pi\sqrt{-1}N^{-1}p^{-1}\mathrm{tr}(hX))) \\ &\quad \exp(\pi\sqrt{-1}N^{-1}p^{-1}\mathrm{tr}(hY)) \end{aligned}$$

and the result follows.) Now:

$$\begin{aligned} &(\textstyle\sum a_h \exp(\pi\sqrt{-1}N^{-1}\mathrm{tr}(hz)))|U_p \\ &= \rho_{\vec{n}}(p1_g, p)^{-1} \textstyle\sum a_h \exp(\pi\sqrt{-1}N^{-1}p^{-1}\mathrm{tr}(hz)) \sum_X \exp(\pi\sqrt{-1}N^{-1}p^{-1}\mathrm{tr}(hX)) \\ &= p^{-\frac{1}{2}g(g+1)} \textstyle\sum_{p|h} a_h p^{\frac{1}{2}g(g+1)} \exp(\pi\sqrt{-1}N^{-1}p^{-1}\mathrm{tr}(hz)) \\ &= \textstyle\sum a_{ph} \exp(\pi\sqrt{-1}N^{-1}\mathrm{tr}(hz)) \end{aligned}$$

Keep the notation of the lemma. Then $Z = Z(U) = ((\mathbb{Z}/N'\mathbb{Z})^{\times})^g$ acts on $\mathcal{S}_{\vec{n}}(U)$ by $(a_1, ..., a_g) \mapsto \sigma_a = \mathrm{diag}(\tilde{a_1}, ..., \tilde{a_g}, \tilde{a_1}^{-1}, ..., \tilde{a_g}^{-1})$ where $\tilde{a_i} \in \prod \mathbb{Z}_l^*$ with $(\tilde{a_i})_l$ equal to 1 if $l \not| N'$ and a_i if $l|N'$. We can decompose $\mathcal{S}_{\vec{n}}(U) = \bigoplus_{\chi \in \check{Z}} \mathcal{S}_{\vec{n}}(U)^{\chi}$. Let $R = \mathbb{Q}^{ab} \cap \mathcal{O}_{\mathbb{Q}_p^{ac}}$, and $\mathcal{S}_{\vec{n}}(U, \chi, R) = \mathcal{S}_{\vec{n}}(U)^{\chi} \cap \mathcal{S}_{\vec{n}}(U, R)$. It follows from the results of Shimura discussed above that $\mathcal{S}_{\vec{n}}(U)^{\chi} = \mathcal{S}_{\vec{n}}(U, \chi, U) \otimes_R \mathbb{C}$. Let $\mathbb{T} = \mathbb{T}(U)$ be the abstract double coset algebra over $\mathbb{Z}$ generated by the operators $[UxU]$ where $x \in M_{2g}(\mathbb{Z}_l) \cap GSp_{2g}(\mathbb{Q}_l)$ for all primes $l \not| N'$. It is known that $\mathbb{T}$ is commutative (see for example [A2]). Moreover $\mathbb{T}$ acts on $\mathcal{S}_{\vec{n}}(U)$, and each element $T \in \mathbb{T}$ acts as a normal operator (i.e. it commutes with its adjoint). Thus the elements of $\mathbb{T}$ can be simultaneously diagonalised. The action of $\mathbb{T}$ commutes with that of Z, and if $r(U) \geq 1$ these both commute with the action of U_p. I claim that $\mathbb{T}$ preserves $\mathcal{S}_{\vec{n}}(U, \chi, R)$.

To see this let $x \in M_{2g}(\mathbb{Z}_l) \cap GSp_{2g}(\mathbb{Q})$ then we can write $UxU = \coprod Ux_i$ where $x_i \in M_{2g}(\mathbb{Z}_l) \cap GSp_{2g}(\mathbb{Q}_l)$ is of the form:

$$\begin{pmatrix} a_1 & 0 & \dots & 0 & * & * & \dots & * \\ * & a_2 & & 0 & * & * & & * \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ * & * & \dots & a_g & * & * & \dots & * \\ 0 & 0 & \dots & 0 & ba_1^{-1} & * & \dots & * \\ 0 & 0 & \dots & 0 & 0 & ba_2^{-1} & \dots & * \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & ba_g^{-1} \end{pmatrix}$$

(see for instance [A2]). We may further suppose that b and each a_i lies in $l^{\mathbb{Z}} \subset \mathbb{Z}_l$. Then consider an element $x_i' \in M_{2g}(\mathbb{Z}) \cap GSp_{2g}(\mathbb{Q})$ defined by:

- x_i' lies in the same Borel as x_i was required to lie in
- $x_i' \equiv \mathrm{diag}(a_1, ..., a_g, ba_1^{-1}, ..., ba_g^{-1}) \bmod N'$
- $x_i' \equiv x_i \bmod l^?$

Then for ? large enough $Ux_i = U\sigma_{a^{-1}}x_i'$ so that $f|x_i = \chi(a^{-1})f|x_i'$ and x_i' clearly preserves $R[[q^{1/N}]]_g$.

Finally we introduce a particular modular form which we shall need later.

Lemma 2.3 *Fix a rational prime p, and set:*

$$\Gamma_0(p) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \;\middle|\; C \equiv 0 \bmod p \right\}$$

Then there is an element $\theta \in M_{(p-1)\vec{t}}(\Gamma_0(p), \mathbb{Z})$ with $a_0(\theta) = 1$ and $p|a_h(\theta)$ for all $h \neq 0$.

Proof:This argument is due to Hida in the case $g = 1$.

Let Q_n denote the $(n-1) \times (n-1)$ matrix:

$$\begin{pmatrix} 2 & -1 & 0 & . & . & 0 \\ -1 & 2 & -1 & & & 0 \\ 0 & -1 & 2 & & & 0 \\ . & & & . & & . \\ . & & & & . & . \\ 0 & 0 & 0 & . & . & 2 \end{pmatrix}$$

then $Q_n \in symm^*_{n-1}(\mathbb{Z})$ and an easy induction on n shows that $\det Q_n = n$ and hence in particular Q_n is positive definite. Let:

$$\theta_1 = \sum_{X \in M_{(p-1),g}(\mathbb{Z})} \exp(\pi\sqrt{-1}\mathrm{tr}({}^tXQ_pXz))$$

Then theorems 2 and 3 of [AM] imply that $\theta = \theta_1^2 \in M_{(p-1)\vec{t}}(\Gamma_0(p), \mathbb{Z})$. It is clear that $a_0(\theta) = a_0(\theta_1)^2 = 1$. Now let ζ be the $(p-1) \times (p-1)$ matrix:

$$\begin{pmatrix} 0 & 0 & 0 & . & . & 0 & -1 \\ 1 & 0 & 0 & & & 0 & -1 \\ 0 & 1 & 0 & & & 0 & -1 \\ . & & & . & & & . \\ . & & & & . & & . \\ 0 & 0 & 0 & & & 0 & -1 \\ 0 & 0 & 0 & . & . & 1 & -1 \end{pmatrix}$$

It has characteristic polynomial $X^{p-1} + ... + 1$ and its eigenvalues are the nontrivial p^{th} roots of one. Thus $\mathbb{Z}/p\mathbb{Z}$ acts on $M_{(p-1),g}(\mathbb{Z})$ by $m : X \mapsto \zeta^m X$ and the orbits are either $\{0\}$ or have cardinality p. It is easily checked that ${}^t\zeta Q_p \zeta = Q_p$, so that ${}^tXQ_pX$ is constant on such orbits. Then $p|a_h(\theta_1)$ for $h \neq 0$, from which the result follows.

Corollary 2.2 *Let U be an open compact subgroup of $G(\mathbb{A}_f)$ satisfying the conditions described before lemma 2.2. Let $R = \mathbb{Q}^{ab} \cap \mathcal{O}_{\mathbb{Q}_p^{ac}}$ and χ be a character on $Z(U)$. Then there is a map:*

$$i_m : e\mathcal{S}_{\vec{n}}(U, \chi, R) \hookrightarrow e\mathcal{S}_{\vec{n}+(p-1)p^{m-1}\vec{t}}(U, \chi, R)$$

such that for all $h \in symm_g^*(\mathbb{Z})$ $a_h(i_m f) \equiv a_h(f) \bmod p^m$ and infact for all $T \in \mathbb{T}(U)$ we have $a_h((i_m f)|T) \equiv a_h(f|T) \bmod p^m$. (Recall that $\vec{t} = (1, ..., 1)$.)

Proof:Set $i_m(f) = e(\theta^{p^m} f)$. Then $\theta^{p^m} f \equiv f \bmod p^m$, so $U_p^{r!}(\theta^{p^m} f) \equiv U_p^{r!} f \bmod p^m$ and hence $e(\theta^{p^m} f) \equiv ef = f \bmod p^m$.

Also if $T \in \mathbb{T}$ recall that $f|T = \sum \zeta_i f|x_i'$ where ζ_i is a root of unity and $x_i' \in M_{2g}(\mathbb{Z}) \cap GSp_{2g}(\mathbb{Q})$ is as described in the discussion after lemma 2.2. But if $|_{\vec{n}}$ denotes the action as for modular forms of weight $\vec{n}$, then:

$$(\sum a_h q^{h/N})|_{\vec{n}+a\vec{t}} x_i' = \lambda_i^a (\sum a_h q^{h/N})|_{\vec{n}} x_i'$$

for some integer λ_i depending only on x_i'. Thus:

$$i_m(f)|_{\vec{n}+(p-1)p^{m-1}\vec{t}} x_i \equiv i_m(f|_{\vec{n}} x_i) \bmod p^m$$

and so $i_m(f)|T \equiv i_m(f|T) \bmod p^m$ as desired.

2.3 Relation to Cohomology

Our aim in this section is to relate our spaces of automorphic forms to certain cohomology groups and use this to prove an analogue of theorem 1.2 for automorphic forms. We then apply this to find congruences between eigenforms of the Hecke operators in different weights.

We fix some notation:

$$\begin{aligned}
\mathfrak{sp}_{2g} &= \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in M_{2g} \middle| \ {}^tB = B,\ {}^tC = C \right\} \\
\mathfrak{k} &= \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M_{2g} \middle| \ {}^tB = B,\ {}^tA = -A \right\} \\
\mathfrak{p} &= \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in M_{2g} \middle| \ {}^tB = B,\ {}^tA = A \right\} \\
\mathfrak{a} &= \left\{ \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix} \in M_{2g} \middle| \ \Lambda \text{ is diagonal} \right\}
\end{aligned}$$

Then $\mathfrak{sp}_{2g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k}(\mathbb{R})$ is the Lie algebra of $K_\infty = U_\infty \cap Sp_{2g}(\mathbb{R})$, and $\mathfrak{a} \subset \mathfrak{k} \subset \mathfrak{sp}_{2g}$ is a Cartan subalgebra. We fix $\mathfrak{a}(\mathbb{C})' \cong \mathbb{C}^g$ by $\vec{x} : diag(\lambda_1, ..., \lambda_g) \mapsto \sqrt{-1}\sum x_i\lambda_i$. Then if T is the maximal torus of $Sp_{2g}(\mathbb{C})$ considered in example 1.2, and $\mathfrak{t}(\mathbb{C})$ is its Lie algebra, we may conjugate $\mathfrak{t}(\mathbb{C})$ to $\mathfrak{a}(\mathbb{C})$ such that the map:

$$\mathbb{Z}^g \cong X^*(T) \subset \mathfrak{t}(\mathbb{C})' \xrightarrow{\sim} \mathfrak{a}(\mathbb{C})' \cong \mathbb{C}^g$$

is the canonical inclusion. Thus the roots of $\mathfrak{a}$ on $\mathfrak{sp}_{2g}$ are just the vectors α_{ij} $(i \neq j)$, β_{ij} $(i \leq j)$ and γ_{ij} $(i \leq j)$ as in example 1.2. The roots Φ_c of $\mathfrak{a}$ on $\mathfrak{k}$ are the α_{ij}, and those (Φ_n) on $\mathfrak{p}$ are the β_{ij} and the γ_{ij}. Choose the same order we chose in example 1.2. Let W denote the Weyl group of $\mathfrak{sp}_{2g}$ and W_n the subset of elements w such that $\Phi_c^+ \subset w\Phi^+$ where $\Phi_c^+ = \Phi_c \cap \Phi^+$. Let $\mathcal{U}$ denote the universal enveloping algebra of $\mathfrak{sp}_{2g}(\mathbb{C})$, and $Z(\mathcal{U})$ its centre. Then the homomorphisms $Z(\mathcal{U}) \to \mathbb{C}$ are parametrised by $\mathfrak{a}(\mathbb{C})'/W$ (the Harish-Chandra parametrisation). We shall denote this correspondence $\lambda \leftrightarrow \chi_\lambda$. We now restate a special case of theorem 10 of [Fa1] in a slightly different notation:

Let:

- $\Gamma \subset G_\infty^+$ be a torsion free discrete subgroup
- $w \in W_n$ of length $l(w)$
- Δ be the basis corresponding to Φ^+, and $\delta_G = (1, ..., g)$ half the sum of the positive roots

- $\lambda \in \mathfrak{a}(\mathbb{C})'$ be a dominant (integral) weight with respect to Δ
- $W(\mu)$ denote the irreducible K_∞ module with highest weight μ
- $\mathcal{C}_\mu$ be the space of cuspidal C^∞ functions $f : \Gamma\backslash Sp_{2g}(\mathbb{R}) \to W(\mu)$ such that $f(hk) = k^{-1}f(h)$ for all $h \in Sp_{2g}(\mathbb{R})$ and $k \in K_\infty$
- $\mathcal{C}_\mu^{\chi_\nu}$ be the subspace of $\mathcal{C}_\mu$ that transform by χ_ν under the action of $Z(\mathcal{U})$
- $V(\lambda)$ be the irreducible $Sp_{2g}(\mathbb{C})$ module of heighest weight λ
- $\mathbf{V}(\lambda)$ the sheaf on $\Gamma\backslash\mathcal{Z}$ defined by setting $\mathbf{V}(\lambda)(U)$ to be the set of C^∞ functions $f : \tilde{U} \to V(\lambda)$ such that $f(\gamma z) = \gamma f(z)$ for all $z \in \mathcal{Z}$ and $\gamma \in \Gamma$, and where $\tilde{U}$ is the pre-image of U under the map $\mathcal{Z} \to \Gamma\backslash\mathcal{Z}$
- $H_P^\bullet$ denote the image of the cohomology of compact support in the cohomology, or equivalently the kernel of the map from the cohomology to the cohomology of the boundary of the Borel-Serre compactification

then:

$$\mathcal{C}_{w(\lambda+\delta_G-w^{-1}\delta_G)}^{\chi_{\lambda+\delta_G}} \hookrightarrow H_P^{l(w)}(\Gamma\backslash\mathcal{Z}, \mathbf{V}(\lambda)) \hookrightarrow H^{l(w)}(\Gamma, V(\lambda))$$

Moreover the Hecke operator $[\Gamma g\Gamma]$ corresponds to the Heck operator $[\Gamma g^{-1}\Gamma]$, for $g \in Sp_{2g}(\mathbb{R})$ for which these operators make sense.

Now take $w : (x_1, ..., x_n) \mapsto (-x_n, ..., -x_1)$. Note that $\delta_G - w^{-1}\delta_G = (g+1)\vec{t}$. Let $\delta_K = \frac{1}{2}(1-g, 3-g, ..., g-1)$ be half the sum of the elements of Φ_c^+. Then the representation $\rho_{\vec{n}}$ of $K_\infty \subset U_\infty$ defined in the last section has heighest weight $w\vec{n}$, which is dominant with respect to $w\Phi^+$. Thus $Z(\mathcal{U})$ acts on $S_{\vec{n}}(\Gamma)$ via $\chi_{w\vec{n}+2w\delta_K-w\delta_G} = \chi_{w\vec{n}+\delta_G}$ so that for Γ torsion free and for $\vec{n}$ with $g+1 \leq n_1 \leq ... \leq n_g$:

$$\begin{aligned} S_{\vec{n}}(\Gamma) \hookrightarrow \mathcal{C}_{w\vec{n}}^{\chi_{\vec{n}+w^{-1}\delta_G}} &\hookrightarrow H_P^{\frac{1}{2}g(g+1)}(\Gamma\backslash\mathcal{Z}, \mathbf{V}(\vec{n}-(g+1)\vec{t})) \\ &\hookrightarrow H^{\frac{1}{2}g(g+1)}(\Gamma, V(\vec{n}-(g+1)\vec{t})) \end{aligned}$$

and so for any discrete Γ:

$$S_{\vec{n}}(\Gamma) \hookrightarrow H^{\frac{1}{2}g(g+1)}(\Gamma, V(\vec{n}-(g+1)\vec{t}))$$

(We may choose a normal subgroup Γ' of finite index which is torsion free and then take Γ/Γ' invariants, using the inflation restriction sequence on the right.) Moreover if $g \in G(\mathbb{Q})^+$ then the action of $[\Gamma g \Gamma]$ on $S_{\vec{n}}(\Gamma)$ corresponds to that of $[\Gamma {}^{\iota}g \Gamma]$ on $H^{\frac{1}{2}g(g+1)}(\Gamma, V(\vec{n} - (g+1)\vec{t}))$, where ${}^{\iota}g = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} {}^t g \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}^{-1}$. To see this we need only check that for $\mu \in \mathbb{R}^{\times}_{>0}$ $[\Gamma \mu 1_{2g} \Gamma]$ acts on both by $\mu^{|\vec{n}| - g(g+1)}$, which is easy. Now we have:

Proposition 2.1 *Let $U \subset G(\mathbb{A}_f)$ be an open compact subgroup satisfying the conditions stated before lemma 2.2 with $r(U) \geq 1$, let $\vec{n}$ be such that $0 \leq n_1 ... \leq n_g$ and let e denote the Hida idempotent. Then there is a constant C such that:*

$$\dim e\mathcal{S}_{\vec{n}+m\vec{t}}(U) < C$$

for all $m \geq 0$, where $\vec{t} = (1, ..., 1)$.

_Proof:_Without loss of generality we may restrict to $m \geq g+1$. Then by theorem 1.2 we can choose C such that $\dim eH^{\frac{1}{2}g(g+1)}(\Gamma_U, V(\vec{n} + m'\vec{t})) < C$ for all $m' \geq 0$. Now fix m. Also choose a finite $\mathbb{Z}$ module $M \subset \mathbb{C}$ such that:

$$S_{\vec{n}+m\vec{t}}(\Gamma_U, \mathbb{Z}) \hookrightarrow H^{\frac{1}{2}g(g+1)}(\Gamma_U, L_{\vec{n}+(m-g-1)\vec{t}})^{TF} \otimes M$$

Then:

$$\begin{aligned} eS_{\vec{n}+m\vec{t}}(\Gamma, \mathbb{Z}) &\hookrightarrow e(H^{\frac{1}{2}g(g+1)}(\Gamma_U, L_{\vec{n}+(m-g-1)\vec{t}})^{TF} \otimes M) \\ &= (eH^{\frac{1}{2}g(g+1)}(\Gamma_U, L_{\vec{n}+(m-g-1)\vec{t}})^{TF}) \otimes M \\ &\subset eH^{\frac{1}{2}g(g+1)}(\Gamma_U, V_{\vec{n}+(m-g-1)\vec{t}}) \end{aligned}$$

and so $\dim S_{\vec{n}+m\vec{t}}(\Gamma_U, \mathbb{Z}) < C$ and we are done.

We can now deduce our first main result:

Theorem 2.1 *Let $U \subset G(\mathbb{A}_f)$ be an open compact subgroup satisfying the conditions stated before lemma 2.2 with $r(U) \geq 1$, let $\vec{n}$ be such that $0 \leq n_1 \leq ... \leq n_g$, let $R = \mathbb{Q}^{ab} \cap \mathcal{O}_{\mathbb{Q}^{ac}}$, and let e denote the Hida idempotent. Let $f \in e\mathcal{S}_{\vec{n}}(U)^{\chi}$ be an eigenform for $\mathbb{T}$ with eigenvalues given by $\lambda : \mathbb{T} \to R$. Then we can find $f_m \in e\mathcal{S}_{\vec{n}+a_m\vec{t}}(U)^{\chi}$ such that:*

- *f_m is an eigenvalue for $\mathbb{T}$ with eigenvalues $\lambda_m : \mathbb{T} \to R$*

- *$a_m \to \infty$ as $m \to \infty$*

- $\sup_{\mathbb{T}} |\lambda(T) - \lambda_m(T)|_p \to 0$ *as $m \to \infty$*

Proof: We may assume $f \in e\mathcal{S}_{\vec{n}}(U, \chi, R)$ and that if $\mu \in \mathbb{Q}^{ab}$ and $\mu f \in \mathcal{S}_{\vec{n}}(U, \chi, R)$ then $\mu \in R$. Then by corollary 2.2 we can find $f'_m \in e\mathcal{S}_{\vec{n}+a_m\vec{t}}(U, \chi, R)$ with $f_m|T \equiv f|T \bmod p^{r'_m}$ for all $T \in \mathbb{T}$, where a_m and r'_m tend to infinity with m. Let $\mathbb{T}_m$ denote the image of $\mathbb{T}$ in $End(e\mathcal{S}_{\vec{n}+a_m\vec{t}}(U, \chi, R))$. Then we get $\lambda'_m : \mathbb{T}_m \twoheadrightarrow R/p^{r'_m}R$ with $\lambda'_m \equiv \lambda \bmod p^{r_m}$. Call its kernel I_m. Let C be the bound from the last proposition (proposition 2.1). Then we can find less than C functions $h_{m,i} \in e\mathcal{S}_{\vec{n}+a_m\vec{t}}(U, \chi, R)$ which span $e\mathcal{S}_{\vec{n}+a_m\vec{t}}(U, \chi, R)$ and such that each $h_{m,i}$ is an eigenform for $\mathbb{T}_m$ with eigenvalues given by $\lambda_{m,i}$ say. Then $V_m = \oplus R h_{m,i}$ is a faithful $\mathbb{T}_m$ module. Thus, if Fitt denotes the Fitting ideal (see, for example, the appendix of [MW1]), $\mathrm{Fitt}_{\mathbb{T}_m}(V_m) = 0$ and so:

$$\begin{aligned} 0 &= \mathrm{Fitt}_{\mathbb{T}_m/I_m}(V_m/I_mV_m) \\ &= \textstyle\prod_i \mathrm{Fitt}_{R/p^{r'_m}R}(R/\lambda_{m,i}(I_m)) \\ &= \textstyle\prod_i \lambda_{m_i}(I_m) \subset R/p^{r'_m}R \end{aligned}$$

Thus for some i, $\mathrm{val}_p(\lambda_{m,i}I_m) \geq r'_m/C$. Let $f_m = h_{m,i}$ and we are done.

2.4 Some Lemmas on Hecke Operators

The discussion in the other sections is principally concerned with “ordinary” forms, i.e. modular forms in the image of the Hida idempotent e acting on a space of modular forms for an open compact subgroup contained in $U_0(p)$, the set of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\prod G(\mathbb{Z}_l)$ with $C \equiv 0 \bmod p$. It is often more interesting to consider a modular form f on an open compact subgroup containing $G(\mathbb{Z}_p)$. In this section we give a criterion for $ef \in \mathcal{S}(U \cap U_0(p))$ not to vanish. In fact we shall only treat the case of genus two. The calculations are already very messy, and this is the case of principal interest for us.

In the case of genus one the answer to this question is very easy. If $f \in S_k(\Gamma_1(N))$, with N prime to p, is an eigenform for T_p and S_p, say with eigenvalues a_p and d_p, then $ef \in S_k(\Gamma_1(Np))$ is non-zero if and only if one root of the equation $Q(X) = X^2 - a_p X + p d_p$ is a p-adic unit. To prove this (at least in the case that this polynomial has distinct roots) one writes down two forms f_1 and f_2 in $S_k(\Gamma_1(Np))$ which are both eigenforms for U_p with eigenvalues the roots of $Q(X)$, and such that f is a linear combination of the two (and not a multiple of either one separately).

For the rest of this chapter we assume that $g = 2$. Also let N be an integer and U the open compact subgroup of $\prod GSp_4(\mathbb{Z}_l)$ consisting of matrices congruent to $\begin{pmatrix} 1_2 & 0 \\ 0 & \lambda 1_2 \end{pmatrix}$ modulo N. Let p be a prime not dividing N. Then we define Hecke operators:

- $T(p) = T_p = \left[U \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi_p & 0 \\ 0 & 0 & 0 & \pi_p \end{pmatrix} U \right]$
- $T_{p^2} = \left[U \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi_p^2 & 0 \\ 0 & 0 & 0 & \pi_p^2 \end{pmatrix} U \right]$
- $R_p = \left[U \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi_p & 0 & 0 \\ 0 & 0 & \pi_p^2 & 0 \\ 0 & 0 & 0 & \pi_p \end{pmatrix} U \right]$
- $S_p = \left[U \begin{pmatrix} \pi_p & 0 & 0 & 0 \\ 0 & \pi_p & 0 & 0 \\ 0 & 0 & \pi_p & 0 \\ 0 & 0 & 0 & \pi_p \end{pmatrix} U \right]$
- $T(p^2) = T_{p^2} + R_p + S_p$

where π_p denotes a uniformiser in $\mathbb{Z}_p$. Also let $Q_p(X)$ be the formal polynomial whose coefficients are Hecke operators given by:

$$X^4 - T_p X^3 + (T_p^2 - T(p^2) - p^2 S_p)X^2 - p^3 T_p S_p X + p^6 S_p^2$$

Recall the following formulae (see [A2]):

- $T_p^2 - T(p^2) - p^2 S_p = pR_p + p(p^2+1)S_p = (p+1)^{-1}(pT_p^2 - pT_{p^2} - (p^4-1)S_p)$
- $pR_p = T_p^2 - T(p^2) - p(p^2+p+1)S_p$
- $pT_{p^2} = (p+1)T(p^2) + p^2(p+1)S_p - T_p^2$

Also recall that $\mathcal{S}_{\vec{n}}$ is a direct sum, $\bigoplus \pi$, of irreducible admissible representations $\pi = \bigotimes \pi_l$ of $GSp_4(\mathbb{A}_f)$. If $p \not| N$ and if $\pi^U \neq (0)$ then π_p is spherical and so is the unique spherical irreducible subquotient of some unramified principal series representation. (See [C] for this and the facts quoted below about such representations.)

We first discuss the action of Hecke operators on unramified principal series representations, and then we apply these results to spaces of cusp forms. Fix the Borel:

$$B = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

and the maximal torus T consisting of diagonal matrices. Describe unramified characters on $T(\mathbb{Q}_p)$ by triples (χ_1, χ_2, ψ) of unramified characters on $\mathbb{Q}_p^\times$, where:

$$(\chi_1, \chi_2, \psi) : \mathrm{diag}(\lambda, \mu, \nu\lambda^{-1}, \nu\mu^{-1}) \longmapsto \chi_1(\lambda)\chi_2(\mu)\psi(\nu)$$

In particular let δ denote the character taking this matrix to $|\lambda^2\mu^4\nu^{-3}|_p$. By the unramified principal series corresponding to (χ_1, χ_2, ψ) we mean the representation on the space of locally constant functions $\theta : GSp_4(\mathbb{Q}_p) \to \mathbb{C}$ satisfying $\theta(bh) = ((\chi_1, \chi_2, \psi)\delta^{\frac{1}{2}})(b)\theta(h)$ for all $b \in B(\mathbb{Q}_p)$ and $h \in GSp_4(\mathbb{Q}_p)$; where the action is given by $(\theta g)(h) = \theta(hg^{-1})$. We shall denote this representation by $\pi(\chi_1, \chi_2, \psi)$ and its irreducible spherical subquotient by $\sigma(\chi_1, \chi_2, \psi)$. Note this action is twisted from that in [C]. Then it is easy to see that $\pi(\chi_1, \chi_2, \psi)^{GSp_4(\mathbb{Z}_p)} = \mathbb{C}\Theta_{\chi_1,\chi_2,\psi}$ where:

$$\Theta(bk) = \Theta_{\chi_1,\chi_2,\psi}(bk) = ((\chi_1, \chi_2, \psi)\delta^{\frac{1}{2}})(b)$$

for $b \in B(\mathbb{Q}_p)$ and $k \in GSp_4(\mathbb{Z}_p)$. That this is a good definition follows from the Iwasawa decomposition. Then we can compute that:

- $\Theta|T_p = p^{\frac{3}{2}}\psi(p^{-1})(1 + \chi_1(p^{-1}) + \chi_2(p^{-1}) + \chi_1\chi_2(p^{-1}))\Theta$
- $\Theta|S_p = \psi(p^{-2})\chi_1\chi_2(p^{-1})\Theta$
- $\Theta|R_p = p^2\psi(p^{-2})(\chi_1(p^{-1}) + \chi_2(p^{-1}) + \chi_1\chi_2(p^{-1}) + \chi_1^2\chi_2(p^{-1}) + \chi_1\chi_2^2(p^{-1}))\Theta$
  $- \psi(p^-2)\chi_1\chi_2(p^{-1})\Theta$

These follow from the coset decompositions (see [A2]):

- $GSp_4(\mathbb{Z}_p)\mathrm{diag}(1, 1, p, p)GSp_4(\mathbb{Z}_p) = \coprod GSp_4(\mathbb{Z}_p)\alpha$ as α runs over the matrices:
  - $\begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & y & z \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$ for $x, y, z = 0, ..., p-1$
  - $\begin{pmatrix} p & 0 & 0 & 0 \\ -i & 1 & 0 & z \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & p \end{pmatrix}$ for $i, z = 0, ..., p-1$
  - $\begin{pmatrix} 1 & 0 & x & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ for $x = 0, ..., p-1$

– $\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- $GSp_4(\mathbb{Z}_p)\mathrm{diag}(p,p,p,p)GSp_4(\mathbb{Z}_p) = GSp_4(\mathbb{Z}_p)\mathrm{diag}(p,p,p,p)$

- $GSp_4(\mathbb{Z}_p)\mathrm{diag}(1,p,p^2,p)GSp_4(\mathbb{Z}_p) = \coprod GSp_4(\mathbb{Z}_p)\beta$ as β runs over the matrices:

  – $\begin{pmatrix} 1 & 0 & x & y \\ 0 & p & py & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$ for $x = 0, ..., p^2 - 1$ and $y = 0, ..., p - 1$

  – $\begin{pmatrix} p & 0 & 0 & py \\ i & 1 & y & z \\ 0 & 0 & p & pi \\ 0 & 0 & 0 & p^2 \end{pmatrix}$ for $i, y = 0, ..., p - 1$ and $z = 0, ..., p^2 - 1$

  – $\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p^2 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

  – $\begin{pmatrix} p^2 & 0 & 0 & 0 \\ -pi & p & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & p \end{pmatrix}$ for $i = 0, ..., p - 1$

  – $\begin{pmatrix} p1_2 & B \\ 0 & p1_2 \end{pmatrix}$ where B runs over non-zero symmetric 2×2 integral matrices modulo p which satisfy $\det B \equiv 0 \bmod p$

Then we easily conclude that:

- $\Theta|(pR_p + p(p^2+1)S_p) = p^3\psi(p^{-2})(\chi_1(p^{-1}) + \chi_2(p^{-1}) + 2\chi_1\chi_2(p^{-1}) + \chi_1^2\chi_2(p^{-1}) + \chi_1\chi_2^2(p^{-1}))\Theta$
- $\Theta|Q_p(X) = (X - p^{\frac{3}{2}}\psi(p^{-1}))(X - p^{\frac{3}{2}}\psi\chi_1(p^{-1}))(X - p^{\frac{3}{2}}\psi\chi_2(p^{-1}))(X - p^{\frac{3}{2}}\psi\chi_1\chi_2(p^{-1}))\Theta$

Now let Γ denote the subgroup of elements of $GSp_4(\mathbb{Z}_p)$ which are congruent to a matrix of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ modulo p. Then $GSp_4(\mathbb{Q}_p) = \coprod_1^4 B(\mathbb{Q}_p)w_i\Gamma$, where:

$$w_1 = 1_4 \qquad w_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad w_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Thus $\pi(\chi_1, \chi_2, \psi)^\Gamma$ has a basis consisting of functions f_1, f_2, f_3, f_4 where f_i is supported on $B(\mathbb{Q}_p)w_i\Gamma$ and where $f_i(w_i) = 1$. We shall represent the function $\sum \mu_i f_i$ by the row vector $(\mu_1, ..., \mu_4)$, so in particular Θ is represented by $(1,1,1,1)$. We shall calculate the matrix representing the action of the Hecke operator $U_p = [\Gamma \mathrm{diag}(1,1,p,p)\Gamma]$ with respect to this basis. It is easy to see that it is represented by $(f_j|U_p(w_i))$, and we claim that this is:

$$p^{\frac{1}{2}}\psi(p^{-1}) \begin{pmatrix} p & p-1 & p-1 & p-1 \\ 0 & p\chi_1(p^{-1}) & (p-1)\chi_1(p^{-1}) & (p-1)\chi_1(p^{-1}) \\ 0 & 0 & p\chi_2(p^{-1}) & (p-1)\chi_2(p^{-1}) \\ 0 & 0 & 0 & p\chi_1\chi_2(p^{-1}) \end{pmatrix}$$

To see this first note that:

$$(f_j|U_p)(w_i) = \sum_X \chi_1\chi_2\psi^2(p^{-1}) f_j\left(w_i \begin{pmatrix} p1_2 & X \\ 0 & 1_2 \end{pmatrix}\right)$$

where X runs over 2×2 integral matrices modulo p. To calculate these values we write:

$$w_i \begin{pmatrix} p1_2 & X \\ 0 & 1_2 \end{pmatrix} = b(i,X) w_{k(i,X)} \gamma(i,X)$$

where $b \in B(\mathbb{Q}_p)$, $\gamma \in \Gamma$ and $k = 1, 2, 3$ or 4. Then:

$$f_j \left(w_i \begin{pmatrix} p1_2 & X \\ 0 & 1_2 \end{pmatrix} \right) = \begin{cases} ((\chi_1, \chi_2, \psi)\delta^{\frac{1}{2}})(b(i,X)) & \text{if } j = k(i,X) \\ 0 & \text{otherwise} \end{cases}$$

The b's, γ's and k's are given by the following formulae:

- $w_1 \begin{pmatrix} p1_2 & X \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} p1_2 & X \\ 0 & 1_2 \end{pmatrix} w_1$
- If $x \not\equiv 0 \bmod p$, say $ax \equiv 1 \bmod p$ then:

$$w_2 \begin{pmatrix} p & 0 & x & y \\ 0 & p & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 0 & -a & -ay \\ 0 & p & -ay & z - ay^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} w_1 \begin{pmatrix} -a & 0 & \frac{1}{p}(1-ax) & 0 \\ ay & 1 & \frac{y}{p}(1-ax) & 0 \\ -p & 0 & -x & y \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $w_2 \begin{pmatrix} p & 0 & 0 & y \\ 0 & p & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ y & p & 0 & z \\ 0 & 0 & p & -y \\ 0 & 0 & 0 & 1 \end{pmatrix}$
- If $z \not\equiv 0 \bmod p$, say $az \equiv 1 \bmod p$, then:

$$w_3 \begin{pmatrix} p & 0 & x & y \\ 0 & p & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 0 & x - ay^2 & -ay \\ 0 & p & -ay & -a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} w_1 \begin{pmatrix} 1 & -ay & 0 & \frac{y}{p}(1-az) \\ 0 & -a & 0 & \frac{1}{p}(1-az) \\ 0 & 0 & 1 & 0 \\ 0 & -p & -y & -z \end{pmatrix}$$

- If $y \not\equiv 0 \bmod p$, say $ay \equiv 1 \bmod p$, then:

$$w_3 \begin{pmatrix} p & 0 & x & y \\ 0 & p & y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & p & 0 & a^2x \\ 0 & 0 & p & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} w_2 \begin{pmatrix} 0 & a & \frac{1}{p}(ay-1) & 0 \\ -a & a^2x & \frac{ax}{p}(ay-1) & \frac{1}{p}(1-ay) \\ p & 0 & x & y \\ 0 & -p & -y & 0 \end{pmatrix}$$

- $w_3 \begin{pmatrix} p & 0 & x & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix} w_3$

- If X is invertible modulo p, say $YX \equiv 1_2 \bmod p$ then:

$$w_4 \begin{pmatrix} p1_2 & X \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} p1_2 & -Y \\ 0 & 1_2 \end{pmatrix} w_1 \begin{pmatrix} -Y & \frac{1}{p}(1_2 - YX) \\ -p & -X \end{pmatrix}$$

- If $xz = y^2$ and $z \not\equiv 0 \bmod p$, say $az \equiv 1 \bmod p$, then:

$$w_4 \begin{pmatrix} p & 0 & x & y \\ 0 & p & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -az & p & 0 & a \\ 0 & 0 & p & az \\ 0 & 0 & 0 & 1 \end{pmatrix} w_2 \begin{pmatrix} 1 & ya & \frac{1}{p}(ay^2 - x) & \frac{y}{p}(1-az) \\ 0 & a & 0 & \frac{1}{p}(1-az) \\ 0 & 0 & 1 & 0 \\ 0 & -p & y & z \end{pmatrix}$$

- If $x \not\equiv 0 \bmod p$, say $ax \equiv 1 \bmod p$, then:

$$w_4 \begin{pmatrix} p & 0 & x & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} w_3 \begin{pmatrix} a & 0 & \frac{1}{p}(1-ax) & 0 \\ 0 & 1 & 0 & 0 \\ -p & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $w_4 \begin{pmatrix} p1_2 & 0 \\ 0 & 1_2 \end{pmatrix} = \begin{pmatrix} 1_2 & 0 \\ 0 & p1_2 \end{pmatrix} w_4$

From these results an easy calculation gives the matrix for the action of U_p. Note that in particular the eigenvalues of U_p on $\pi(\chi_1, \chi_2, \psi)^\Gamma$ are $p^{\frac{3}{2}}\psi(p^{-1})$, $p^{\frac{3}{2}}\chi_1\psi(p^{-1})$, $p^{\frac{3}{2}}\chi_2\psi(p^{-1})$ and $p^{\frac{3}{2}}\chi_1\chi_2\psi(p^{-1})$.

Finally we are in a position to deduce the main result of this section:

Lemma 2.4 *Let $f \in \mathcal{S}_{\vec{n}}(U)$, where U is an open compact subgroup of $\prod GSp_4(\mathbb{Z}_l)$ consisting of all matrices congruent to $\begin{pmatrix} 1_2 & 0 \\ 0 & \mu 1_2 \end{pmatrix}$ modulo some integer N; be an eigenvector of the Hecke operators S_l, T_l and $T(l^2)$ for all primes $l \not| N$, say $f|T = \lambda(T)f$. Fix a prime $p \not| N$. Assume that $\lambda(Q_p(X))$ has distinct roots the ratio of no two of which is p. Then if α is a root of $\lambda(Q_p(X))$ we can find a non-zero form $f' \in \mathcal{S}_{\vec{n}}(U \cap U_0(p))$ which is an eigenform of the Hecke operators S_l, T_l and $T(l^2)$ for $l \not| Np$ with eigenvalues given by λ, and such that $f'|U_p = \alpha f'$. In particular if α is a p-adic unit then $ef' = f'$.*

Proof: We may write $\mathcal{S}_{\vec{n}} = \bigoplus \pi$ with $\pi = \bigotimes \pi_l$ being irreducible admissible representations of $G(\mathbb{A}_f)$. We may assume without loss of generality that $f \in \pi^U$ for some π. Then π_p is the spherical subquotient of an unramified principle series representation $\pi(\chi_1, \chi_2, \psi)$, where $\{p^{\frac{3}{2}}\psi(p^{-1}), p^{\frac{3}{2}}\psi\chi_1(p^{-1}), p^{\frac{3}{2}}\psi\chi_2(p^{-1}), p^{\frac{3}{2}}\psi\chi_1\chi_2(p^{-1}),\}$ is the set of roots of $\lambda(Q_p(X))$. By our assumption that $\lambda(Q_p(X))$ has distinct roots, (χ_1, χ_2, ψ) is regular (i.e. its conjugates under the Weyl group are distinct, i.e. $\chi_1 \neq \chi_2$; $\chi_1, \chi_2, \chi_1\chi_2 \neq 1$) and satisfies the condition of theorem 3.10 of [C], namely:

$$\left.\begin{matrix} (\chi_1, \chi_2, \psi) \\ (\chi_1^{-1}, \chi_2^{-1}, \psi\chi_1\chi_2) \end{matrix}\right\} \left\{\begin{matrix} \mathrm{diag}(p^{-1}, p, p, p^{-1}) \\ \mathrm{diag}(p, 1, p^{-1}, 1) \\ \mathrm{diag}(1, p, 1, p^{-1}) \\ \mathrm{diag}(p, p, p^{-1}, p^{-1}) \end{matrix}\right\} \neq p$$

i.e. $\chi_2\chi_1^{-1}(p) \neq p^{\pm 1}$, $\chi_1(p) \neq p^{\pm 1}$, $\chi_2(p) \neq p^{\pm 1}$ and $\chi_2\chi_1(p) \neq p^{\pm 1}$. Thus $\pi(\chi_1, \chi_2, \psi)$ is irreducible and so the result now follows from the above discussion.

Remark: If $\lambda(Q_p(X))$ has two distinct roots which are p-adic units and if $2 \leq n_1 \leq n_2$ then it certainly satisfies the condition of the lemma.

Before finishing this section we give one further computation. If $\mathcal{H}$ is the Hecke algebra of double cosets in $GSp_4(\mathbb{Z}_p)\backslash GSp_4(\mathbb{Q}_p)/GSp_4(\mathbb{Z}_p)$ and if $\mathcal{H}'$ is the Hecke algebra of double cosets in $GL_2(\mathbb{Z}_p)\backslash GL_2(\mathbb{Q}_p)/GL_2(\mathbb{Z}_p)$ then there is an injection: $\mathcal{H} \hookrightarrow \mathcal{H}'[x^{\pm 1}]$ (see for example [Fa2]). An easy calculation shows that this map is given explicitly by:

$$[GSp_4(\mathbb{Z}_p)gGSp_4(\mathbb{Z}_p)] \longmapsto x^{\mathrm{val}_\nu(g)} \sum GL_2(\mathbb{Z}_p)a_i$$

where $GSp_4(\mathbb{Z}_p)gGSp_4(\mathbb{Z}_p) = \coprod GSp_4(\mathbb{Z}_p)\begin{pmatrix} a_i & b_i \\ 0 & \nu(g)^t a_i^{-1} \end{pmatrix}$. Thus if we let t_p denote $[GL_2(\mathbb{Z}_p)\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}GL_2(\mathbb{Z}_p)]$ and s_p denote $[GL_2(\mathbb{Z}_p)p1_2GL_2(\mathbb{Z}_p)]$ we have that:

- $T_p \longmapsto (p^3 + pt_p + s_p)x$
- $S_p \longmapsto s_p x^2$
- $R_p \longmapsto (p^3 t_p + s_p t_p + (p^2 - 1)s_p)x^2$
- $Q_p(X) \longmapsto (X - s_p x)(X - p^3 x)(X^2 - pt_p xX + p^3 s_p x^2)$

These follow from the decompositions given above. In particular we have that $Q_p(s_p x) = 0$.

2.5 Main Results

We continue to assume that $g = 2$. We fix an integer N and we fix U to be the subgroup of $\prod GSp_4(\mathbb{Z}_l)$ consisting of matrices congruent to $\begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}$ modulo N. We fix also a prime p not dividing N. We shall let U' denote any subgroup of the form:

$$\left\{ x \in U \,\middle|\, x \equiv \begin{pmatrix} \alpha_i & * \\ 0 & \lambda^t \alpha_i^{-1} \end{pmatrix} \bmod p^r \ \text{ for some } \ i \right\}$$

where $r \geq 1$ and $\{\alpha_i\} \subset GL_2(\mathbb{Z}/p^r\mathbb{Z})$ is some set. Also let $\mathbb{T}_M$ denote the double coset algebra over $\mathbb{Z}$ generated by the Hecke operators T_l, S_l and $T(l^2)$ for all primes $l \nmid M$. Then we have:

Proposition 2.2 *Let U' be as above and $3 \leq n_1 \leq n_2$. Then there is a constant C such that if $f \in e\mathcal{S}_{\vec{n}+m\vec{t}}(U')$ (recall that $\vec{t} = (1,1)$) is an eigenform of the ring of Hecke operators $\mathbb{T}_{Np}$, say $f|T = \lambda(T)f$; then there is an integer $a \geq 1$, a finite extension $E/\mathbb{Q}_p$ such that $a[E : \mathbb{Q}_p] < C$, and a continuous representation:*

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GL_a(\mathcal{O}_E)$$

such that ρ is unramified outside Np and such that if $l \not| Np$ is a prime then $\lambda(Q_l)(\mathrm{Frob}_l) = 0$.

Proof:We need only construct a representation valued in $GL_a(E)$, because as ρ is continuous and $\mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q})$ is compact ρ will stabilise some compact $\mathcal{O}_E$ module spanning E^a, which then must be free.

In section 2.3 we define a representation $V(\vec{n} + (m-3)\vec{t})$ of $GSp_4/\mathbb{Z}$. This gives us a locally constant etale p-adic sheaf $\mathbf{V}_p(\vec{n}+(m-3)\vec{t})$ on a certain smooth model $\mathcal{M}_M/\mathbb{Z}[\frac{1}{M}]$ of $\mathcal{Z}/\Gamma_M$, where $M = Np^{r+1}$ and Γ_M denotes the set of matrices in $Sp_4(\mathbb{Z})$ which are congruent to one modulo M. (See [Fa2].) Let W denote $H^3_{et}(\mathcal{M}_M \times \mathrm{spec}\, \mathbb{Q}, \mathbf{V}_p(\vec{n} + (m-3)\vec{t}))$. Then $\mathbb{T}_{Np}$, U_p and $\mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q})$ all act on W and these actions commute with one another. The action of $\mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q})$ is unramified outside Np and if $l \not| Np$ then $Q_l(\mathrm{Frob}_l) = 0$. For these assertions we refer the reader to [Fa2]. The operator U_p is not treated there, but can be treated in an exactly analogous manner. Note also that theorem one of the section "Hecke Operators and Frobenius" in [Fa2] implies that $Q_l(\mathrm{Frob}_l) = 0$ because we have seen at the end of section 2.4 that $Q_l(xs_l) = 0$.

Now consider $(eW) \otimes_{\mathbb{T}_{Np}} E$, where E is the finite extension of $\mathbb{Q}_p$ generated by the image of λ and has a $\mathbb{T}_{Np}$ action via λ. We must show that $1 \leq \dim_{\mathbb{Q}_p}(eW) \otimes_{\mathbb{T}_{Np}} E \leq C$, for C some constant independent of m and f. However $(eW) \otimes_{\mathbb{T}_{Np}} E \cong eH^3(\Gamma_M, V(\vec{n}+(m-3)\vec{t})) \otimes_{\mathbb{T}_{Np}} E$ and so $\dim_{\mathbb{Q}_p}(eW) \otimes_{\mathbb{T}_{Np}} E \leq \dim_{\mathbb{Q}_p} eW \leq C$ for some constant C as in theorem 1.2. Moreover if we fix $E \hookrightarrow \mathbb{C}$ compatible with the p-adic valuation on $\mathbb{Q}^{ac} \subset \mathbb{C}$ and if we let $\mathbb{C}(\lambda)$ denote the one dimensional complex $\mathbb{T}_{Np}$ module with the action via λ, then we see that:

$$\begin{aligned} ((eW) \otimes_{\mathbb{T}_{Np}} E) \otimes \mathbb{C} &\cong eH^3(\Gamma_M, V(\vec{n} + (m-3)\vec{t})) \otimes_{\mathbb{T}_{Np}} \mathbb{C}(\lambda) \\ &\supset e\mathcal{S}_{\vec{n}}(U') \otimes_{\mathbb{T}_{Np}} \mathbb{C}(\lambda) \neq (0) \end{aligned}$$

and we are done.

The following conjectural refinements of this proposition are well known:

Conjecture 2.1 *Let U' be as above and $3 \leq n_1 \leq n_2$. Then there is a constant C such that if $f \in e\mathcal{S}_{\vec{n}+m\vec{t}}(U')$ is an eigenform of the ring of Hecke operators $\mathbb{T}_{Np}$, say $f|T = \lambda(T)f$; then there is a finite extension $E/\mathbb{Q}_p$ such that $[E : \mathbb{Q}_p] < C$, and a continuous representation:*

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GSp_4(\mathcal{O}_E)$$

such that ρ is unramified outside Np and such that if $l \not| Np$ is a prime then $\lambda(Q_l)(\mathrm{Frob}_l) = 0$.

Conjecture 2.2 *Let U' be as above and $3 \leq n_1 \leq n_2$. Then there is a constant C such that if $f \in e\mathcal{S}_{\vec{n}+m\vec{t}}(U')$ is an eigenform of the ring of Hecke operators $\mathbb{T}_{Np}$, say $f|T = \lambda(T)f$; then there is a finite extension $E/\mathbb{Q}_p$ such that $[E : \mathbb{Q}_p] < C$, and a continuous representation:*

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GSp_4(\mathcal{O}_E)$$

such that ρ is unramified outside Np and such that if $l \not| Np$ is a prime then Frob_l has characteristic polynomial $\lambda(Q_l)$.

Now we can state the main result of this chapter:

Theorem 2.2 *Let U and U' be as above and let $2 \leq n_1 \leq n_2$. Let $\chi : ((\mathbb{Z}/Np^r)^\times)^2 \to \mathbb{C}^\times$ be a character. Assume one of the following is true:*

1. *$f \in e\mathcal{S}_{\vec{n}}(U')^\chi$ is an eigenform of $\mathbb{T}_{Np}$, say $f|T = \lambda(T)f$*
2. *$f \in \mathcal{S}_{\vec{n}}(U)^\chi$ is an eigenform of $\mathbb{T}_N$, say $f|T = \lambda(T)f$, and $\lambda(Q_p(X))$ has distinct roots no quotient of two of which equals p and one of which is a p-adic unit.*
3. *$f \in \mathcal{S}_{\vec{n}}(U)^\chi$ is an eigenform of $\mathbb{T}_N$, say $f|T = \lambda(T)f$, and $\lambda(Q_p(X))$ has two distinct roots which are p-adic units*

Then there is a finite extension $E/\mathbb{Q}_p$, an integer $a \geq 1$ and a continuous representation:

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GL_a(\mathcal{O}_E)$$

which is unramified outside Np and such that if $l \not| Np$ is a prime then $\lambda(Q_l)(\mathrm{Frob}_l) = 0$.

Proof:By lemma 2.4 of section 2.4 the second two cases reduce at once to the first case. So consider the first case. We may find forms $f_m \in e\mathcal{S}_{\vec{n}+b_m\vec{t}}(U')^\chi$ which are eigenvalues of $\mathbb{T}_{Np}$, say $f_m|T = \lambda_m(T)f$, and such that as m goes to infinity $b_m \to \infty$ and $\sup_{\mathbb{T}_{Np}} |\lambda_m(T) - \lambda(T)|_p \to 0$. (Lemma 2.1 of section 2.3.) Then for any positive integer s we may find a positive integer a_s, a finite extension $E_s/\mathbb{Q}_p$ with $a_s[E_s : \mathbb{Q}_p] < C$ (where C is as in the last proposition, but for $(n_1 + 1, n_2 + 1)$) and a continuous representation $\rho_s : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \to GL_{a_s}(\mathcal{O}_{E_s}/p^s)$ which is unramified outside Np and such that $\lambda(Q_l)(\mathrm{Frob}_l) = 0$. However there are only finitely many extensions $E : \mathbb{Q}_p$ of degree less than C, so we may assume that $E_s = E$ and $a_s = a$ are independent of s.

We shall recursively define infinite subsets $I_t \subset I_{t-1} \subset \mathbb{T}$ such that if $s_1, s_2 \in I_t$ then $\rho_{s_1} \equiv \rho_{s_2} \bmod p^t$. This is possible as $(\rho_s \bmod p^t)$ factors through $\mathrm{Gal}(K/\mathbb{Q})$ for some Galois extension $K/\mathbb{Q}$ unramified outside Np and of degree bounded independently of s. It is known that there are only finitely many such extensions and so there is a finite Galois extension $L/\mathbb{Q}$ through which all the $(\rho_s \bmod p^t)$ factor. Now there are only finitely many maps $\mathrm{Gal}(L/\mathbb{Q}) \to GL_a(\mathcal{O}_E/p^t)$ so for infinitely many $s \in I_{t-1}$ the maps $(\rho_s \bmod p^t)$ must be equal as desired. Now we define $\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \to GL_a(\mathcal{O}_E)$ by the requirement that $\rho(\sigma) \equiv \rho_s(\sigma) \bmod p^t$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q})$ and all t and all $s \in I_t$. This is easily checked to be a good definition of a morphism with the desired properties.

If we assume the conjectures mentioned above the same method gives the following conjectural strengthening of this theorem:

Theorem 2.3 *Assume conjecture 2.1 (resp. 2.2). Let U and U' be as above and let $2 \leq n_1 \leq n_2$. Let $\chi : ((\mathbb{Z}/Np^r)^\times)^2 \to \mathbb{C}^\times$ be a character. Assume one of the following is true:*

1. *$f \in e\mathcal{S}_{\vec{n}}(U')^\chi$ is an eigenform of $\mathbb{T}_{Np}$, say $f|T = \lambda(T)f$*
2. *$f \in \mathcal{S}_{\vec{n}}(U)^\chi$ is an eigenform of $\mathbb{T}_N$, say $f|T = \lambda(T)f$, and $\lambda(Q_p(X))$ has distinct roots no quotient of two of which equals p and one of which is a p-adic unit.*

3. $f \in \mathcal{S}_{\vec{n}}(U)^{\chi}$ *is an eigenform of* $\mathbb{T}_N$*, say* $f|T = \lambda(T)f$*, and* $\lambda(Q_p(X))$ *has two distinct roots which are p-adic units*

Then there is a finite extension $E/\mathbb{Q}_p$ *and a continuous representation:*

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GSp_4(\mathcal{O}_E)$$

which is unramified outside Np *and such that if* $l \nmid Np$ *is a prime then* $\lambda(Q_l)(\mathrm{Frob}_l) = 0$ *(resp.* Frob_l *has characteristic polynomial* $\lambda(Q_l(X))$*).*

Chapter 3

p-adic Families of Siegel Modular Forms

3.1 Introduction

In this chapter we develop a theory of Hida families for Siegel modular forms of even genus.

To explain our results fix an odd prime p, an even positive integer g and embeddings $\mathbb{Q}^{ac} \subset \mathbb{C}$, $\mathbb{Q}^{ac} \subset \mathbb{Q}_p^{ac}$. The assumption that g be even is probably not essential. It is made because our computations with Eisenstein series and Rankin's method are dependent on the parity of g, and because $g = 2$ is the case of greatest interest for us. Let $M_k(N, \chi)$ denote the space of Siegel modular forms of genus g, weight k, level N and character χ. This space has a $\mathbb{Z}[\zeta]$ structure for some root of unity ζ and this allows us to define $M_k(N, \chi, A)$ for any algebra A containing enough roots of unity so that we can consider χ as valued in A. Any element of $M_k(N, \chi, A)$ has a formal Fourier expansion $\sum a_h \exp(\pi i \, tr(hz))$ with $a_h \in A$ and where h runs over the set S of integral, positive semi-definite $g \times g$ symmetric matrices with even diagonal entries. If $p|N$ we shall let U_p denote the Hecke operator $[\Gamma_0(N) \begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} \Gamma_0(N)]$ where $\Gamma_0(N)$ is defined in section 3.2. If $\mathcal{O}$ denotes the integers of a suitably large extension of $\mathbb{Q}_p$ we define $M_k^\circ(N, \chi, \mathcal{O})$ to be the largest submodule of $M_k(N, \chi, \mathcal{O})$ on which U_p is an automorphism. It is in fact a direct summand, which we shall call the ordinary part. We can carry this notion over to other settings, for example we can define $M_k^\circ(N, \chi, \mathbb{C})$ using the embeddings $\mathbb{Q}^{ac} \subset \mathbb{C}$ and $\mathbb{Q}^{ac} \subset \mathbb{Q}_p^{ac}$. Note that if $f \in M_k(N, \chi)$ is an eigenform of U_p, say $f|U_p = a_p f$, then $f \in M_k^\circ(N, \chi)$ if and only if a_p is a p-adic unit. We shall also define in section 3.2 a certain commutative ring of Hecke operators $\mathbb{T}_N$ which acts semi-simply on $M_k(N, \chi)$ and commutes with the action of U_p.

Now let N be prime to p. Let Λ be the power series ring $\mathcal{O}[[T]]$ with $\mathcal{O}$ as above. By a Λ-*adic form* of level N and character χ (defined modulo Np), we shall mean a formal expansion:

$$\sum_{h \in S} a_h q^h$$

with coefficients in Λ and such that for all but finitely many pairs (k, α) with $k \geq g+1$ and:

$$\alpha : (\mathbb{Z}/Np^? \mathbb{Z})^\times \to (1 + p\mathbb{Z})/(1 + p^? \mathbb{Z}) \to \mathbb{Q}^{ac\,\times}$$

we have that:

$$\sum_{h \in S} a_h(\alpha(1+p)(1+p)^k - 1) \exp(\pi i \, tr(hz))$$

is the Fourier expansion of an element of $M_k^\circ(Np^?, \chi\omega^{-k}\alpha, \mathbb{Q}_p^{ac})$. We shall denote the space of such forms $\mathcal{M}^\circ(N, \chi)$. We can define an action of $\mathbb{T}_{Np}$ on $\mathcal{M}^\circ(N, \chi)$ compatible with specialisation.

Our first main theorem states that $\mathcal{M}^\circ(N, \chi)$ is a finite free Λ module. The main point of the proof is that $\dim M_k^\circ(Np, \chi)$ is bounded independently of k. The second main theorem states that if $f \in M_k^\circ(Np^?, \chi\omega^{-k}\alpha)$ is an eigenvector of $\mathbb{T}_{Np}$, say $f|T(n) = \lambda(n) f$, then we can find an integer M (divisible by N) and a form $F \in \mathcal{M}^\circ(M, \chi) \otimes \mathcal{R}$ ($\mathcal{R}$ the integers of some extension of the field of fractions of Λ) which is an eigenvector for $\mathbb{T}_{Mp}$, say $F|T(n) = \boldsymbol{\lambda}(n) F$, such that $\boldsymbol{\lambda}(n) \equiv \lambda(n)$ modulo some prime of $\mathcal{R}$ lying above $(1 + T - \alpha(1+p)(1+p)^k$. To prove this we follow a method of Wiles (see [Wi]). One first writes down some Λ-adic Eisenstein series, one then multiplies them by a certain form of low weight (we use a theta series) and uses Rankin's method to show that if f is of high enough weight it will occur in the spectral decomposition of this product. For this we must use our first theorem. To extend the result to all weights k one shows that if f is an ordinary eigenform of weight k we can find ordinary eigenforms arbitrarily congruent to f with weights of the form $k+(p-1)p^?$.

In the last section we show how to use this theory to rederive some results of the last chapter about Galois representations. We also show how the standard conjectures on the existence of Galois representations one expects to associate to Siegel modular forms would imply similar results about Λ-adic representations.

3.2 Review of Siegel Modular Forms II

We set up a theory of Siegel modular forms in a more classical setting than the last chapter. This is better suited to the purposes of this chapter.

We shall fix throughout an odd prime p and embeddings $\mathbb{Q}^{ac} \hookrightarrow \mathbb{C}$ and $\mathbb{Q}^{ac} \hookrightarrow \mathbb{Q}_p^{ac}$. We shall also fix a positive integer g. For our main results we must assume that g is even, and we shall in fact always assume this except in sections one and four where we shall need to consider all g to make a certain induction argument work.

We shall let $\mathcal{Z}_g$ denote the Siegel space of genus g, that is:

$$\mathcal{Z}_g = \{x + iy \,|\, x, y \in symm_g(\mathbb{R}),\, y > 0\}$$

Here we write $a > 0$ if a is a positive definite symmetric real matrix. Also if A is a ring $symm_g(A)$ denotes the module of $g \times g$ symmetric matrices over A. Moreover $symm_g^*(A)$ will denote the sub-module whose diagonal entries are in $2A$ and we shall use the superscript ≥ 0 to denote the sub-semigroup of positive semi-definite elements when this makes sense.

A couple more notes on notation. We shall use ϵ_{ij} to denote the $g \times g$ matrix that has one at the intersection of the i^{th} row and j^{th} column and zeroes elsewhere, and 1_g to denote the $g \times g$ identity matrix. We shall let $\nu : GSp_{2g} \to \mathbb{G}_m$ denote the character such that $\left(\nu\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) 1_g = A^tD - B^tC$. Also if R is an integral domain F_R will denote its field of fractions.

The group $GSp_{2g}^{+}(\mathbb{R})$ acts on $\mathcal{Z}_g$ by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : z \longmapsto (Az+B)(Cz+D)^{-1}$$

If $f : \mathcal{Z}_g \to \mathbb{C}$ is a function, k is a non-negative integer and $\gamma \in GSp_{2g}^{+}(\mathbb{R})$ we define a transform of f by γ by the formulae:

- $f|_k\gamma(z) = \nu(\gamma)^{\frac{gk}{2}} j(\gamma, z)^{-k} f(\gamma z)$
- $j\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, z\right) = \det(Cz+D)$

If Γ is a subgroup of $Sp_{2g}(\mathbb{Z})$ of finite index we define a space of modular forms, denoted $M_k(\Gamma)$, to be the set of holomorphic functions $f : \mathcal{Z}_g \to \mathbb{C}$ such that $f|_k\gamma = f$ for all $\gamma \in \Gamma$. If $g = 1$ we must supplement these conditions with a growth condition, but this is well known. If we wish to indicate the genus we shall write $M_k^{(g)}(\Gamma)$. In this work we shall be concerned with two special congruence subgroups of $Sp_{2g}(\mathbb{Z})$ which we shall denote:

- $\Gamma_0(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \,|\, C \equiv 0 \bmod N\}$
- $\Gamma_1(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \,|\, C \equiv 0 \bmod N,\ \det D \equiv 1 \bmod N\}$

We shall decompose:

$$M_k(\Gamma_1(N) = \bigoplus_{\chi} M_k(N, \chi)$$

as χ runs over characters $\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathbb{Q}^{ac\times}$, and where $M_k(N, \chi)$ denotes the space of holomorphic functions $f : \mathcal{Z}_g \to \mathbb{C}$ such that $f|_k\gamma = \chi(\gamma)f$ for all $\gamma \in \Gamma_0(N)$. Here $\chi : \Gamma_0(N) \to \mathbb{Q}^{ac\times}$ by $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \chi(\det D)$. Again we need a growth condition if $g = 1$.

Any element of $M_k(\Gamma_1(N))$ has a Fourier expansion:

$$\sum a_h(f) \exp(\pi i \, tr(hz))$$

as h runs over $symm_g^*(\mathbb{Z})^{\geq 0}$. If R is a ring we shall denote by $R[[q]]_g$ the ring of formal power series $\sum a_h q^h$ as h runs over the semigroup $symm_g^*(\mathbb{Z})^{\geq 0}$, and where $a_h \in R$. With this notation we see that:

$$\begin{array}{rcl} M_k(\Gamma_1(N)) & \hookrightarrow & \mathbb{C}[[q]]_g \\ f & \longmapsto & \sum a_h(f) q^h \end{array}$$

If $R \subset \mathbb{C}$ is a sub-$\mathbb{Z}$-module we define $M_k(\Gamma_1(N), R)$ to be those elements of $M_k(\Gamma_1(N))$ whose Fourier expansion lies in $R[[q]]_g \subset \mathbb{C}[[q]]_g$. It is a result of Shimura that $M_k(\Gamma_1(N), \mathbb{C})$ $= M_k(\Gamma_1(N), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ and so for any $\mathbb{Z}$-module R we may consistently define $M_k(\Gamma_1(N), R) =$ $M_k(\Gamma_1(N), \mathbb{Z}) \otimes_{\mathbb{Z}} R$. Similarly if $\mathcal{O}_\chi$ denotes the extension of $\mathbb{Z}$ generated by the image of χ then $M_k(N, \chi, \mathcal{O}_\chi) \otimes_{\mathcal{O}_\chi} \mathbb{C} = M_k(N, \chi)$ so if R is any $\mathcal{O}_\chi$ module we can define $M_k(N, \chi, R) = M_k(N, \chi, \mathcal{O}_\chi) \otimes_{\mathcal{O}_\chi} R$. Note also that this implies that $Aut(\mathbb{C}/\mathbb{Q})$ acts on $M_k(\Gamma_1(N))$ by:

$$f^\sigma(z) = \sum a_h(f)^\sigma \exp(\pi i\, tr(hz))$$

where $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$. In fact $M_k(N, \chi)^\sigma = M_k(N, \sigma \circ \chi)$.

If f is a modular form of genus g for some congruence subgroup Γ of $Sp_{2g}(\mathbb{Z})$ we define a modular form $\Phi(f)$ of genus $g-1$ by:

$$\Phi(f)(z) = \lim_{\lambda \to +\infty} f \begin{pmatrix} z & 0 \\ 0 & i\lambda \end{pmatrix}$$

Φ is called the Siegel operator. Its action on Fourier expansions is given by:

$$\Phi(f) = \sum a_{h'}(f) \exp(\pi i\, tr(hz))$$

where $h' = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$. We can embed $Sp_{2(g-1)}$ into Sp_{2g} by:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} A & & B & \\ & 1 & & 0 \\ \hline C & & D & \\ & 0 & & 1 \end{array}\right)$$

If Γ is a congruence subgroup of Sp_{2g} set $\Phi(\Gamma) = \Gamma \cap Sp_{2(g-1)}$. Then:

$$\Phi : M_k^{(g)}(\Gamma) \longrightarrow M_k^{(g-1)}(\Phi(\Gamma))$$

and:

$$\Phi : M_k^{(g)}(N, \chi) \longrightarrow M_k^{(g-1)}(N, \chi)$$

We call a modular form of weight k cuspidal if for all $\gamma \in Sp_{2g}(\mathbb{Z})$ we have that $\Phi(f|_k\gamma) = 0$. We denote the subspace of $M_k(\Gamma)$ consisting of cusp forms by $S_k(\Gamma)$. Similarly we use the notation $S_k(N, \chi)$. The rationality results described above remain true for cusp forms. If $Sp_{2g}(\mathbb{Z}) = \amalg_I \Gamma\delta_i$ then we have a left exact sequence:

$$0 \to S_k^{(g)}(\Gamma) \to M_k^{(g)}(\Gamma) \to \bigoplus_I M_k^{(g-1)}(\Phi(\delta_i^{-1}\Gamma\delta_i))$$

and similarly for $M_k(N, \chi)$.

If $f \in S_k(\Gamma)$ and $g \in M_k(\Gamma)$ we can define an inner product $\langle f, g\rangle_\Gamma$ by:

$$\int_{\Gamma\backslash\mathcal{Z}_g} f(z)\overline{g(z)}(\det y)^k \, dz$$

where $z = x + iy$ and $dz = (\det y)^{-g-1} \prod_{1\leq\alpha\leq\beta\leq g} dx_{\alpha\beta}\, dy_{\alpha\beta}$. Then if $\gamma \in GSp_{2g}^+(\mathbb{R})$ we have that $\langle f|_k\gamma, g|_k\gamma\rangle_{\gamma^{-1}\Gamma\gamma} = \langle f, g\rangle_\Gamma$. Similarly if $f \in S_k(N, \chi)$ and $g \in M_k(N, \chi)$ we set:

$$\langle f, g\rangle_N = \int_{\Gamma_0(N)\backslash\mathcal{Z}_g} f(z)\overline{g(z)}(\det y)^k \, dz$$

If U denotes an operator on $S_k(N, \chi)$ we shall let U^* denote its adjoint with respect to this inner product. It will be convenient to introduce a slight variant variant of this inner product. Set:

$$W_N = \begin{pmatrix} 0 & -1_g \\ N1_g & 0 \end{pmatrix}$$

then for $f \in S_k(N, \chi)$ and $g \in M_k(N, \chi)$ we define:

$$(f, g)_N = \langle f|W_N, g^c\rangle_N$$

where c denotes complex conjugation. This is a $\mathbb{C}$ bilinear form which restricted to $S_k(N, \chi)^2$ is non-degenerate. If U is an operator on $S_k(N, \chi)$ we shall let $U^\dagger$ denote its transpose with respect to this pairing.

We shall now recall some facts about the theory of Hecke operators (see [A2] for details). If $g \in GSp_{2g}^{+}(\mathbb{Q})$ and $g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mod N we define the Hecke operator:

$$[\Gamma_0(N)g\Gamma_0(N)] : M_k(N,\chi) \longrightarrow M_k(N,\chi)$$

by:

$$f \longmapsto \nu(g)^{\frac{gk}{2}-\frac{g(g+1)}{2}} \sum_I \chi'(g_i) f|_k g_i$$

where:

- $\Gamma_0(N)g\Gamma_0(N) = \amalg \Gamma_0(N)g_i$
- $\chi' \begin{pmatrix} A & B \\ NC & D \end{pmatrix} = \chi(\det A)$ this being given the value 0 if $(\det A, N) \neq 1$.

Lemma 3.1 *If* $f_1 \in S_k(N,\chi),\ f_2 \in M_k(N,\chi),\ g = \begin{pmatrix} A & B \\ NC & D \end{pmatrix} \in GSp_{2g}^{+}(\mathbb{Q})$ *and* $A, B, C, D \in M_{g\times g}(\mathbb{Z})$ *then:*

$$\langle (f_1|[\Gamma_0(N)g\Gamma_0(N)])|W_N, f_2\rangle_N = \langle f_1|W_N, f_2|[\Gamma_0(N)g^*\Gamma_0(N)]\rangle$$

where:

$$g^* = \begin{pmatrix} N^{-1}1_g & 0 \\ 0 & 1_g \end{pmatrix} {}^t g \begin{pmatrix} N1_g & 0 \\ 0 & 1_g \end{pmatrix} = \begin{pmatrix} {}^tA & {}^tC \\ N{}^tB & {}^tD \end{pmatrix}$$

In particular if g *is as above and if further:*

$$f^c|[\Gamma_0(N)g\Gamma_0(N)] = (f|[\Gamma_0(N)g\Gamma_0(N)])^c$$

then $[\Gamma_0(N)g\Gamma_0(N)]^\dagger = [\Gamma_0(N)g\Gamma_0(N)]$.

Proof: Let $\Gamma_0(N)g\Gamma_0(N) = \amalg \Gamma_0(N)g\gamma_i$ with $\gamma_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \Gamma_0(N)$. Then:

$$\begin{aligned}
&\langle (f_1|[\Gamma_0(N)g\Gamma_0(N)])|W_N, f_2\rangle_N \\
= \; & \textstyle\sum_i \nu(g\gamma_i)^? \chi(AA_i)[\Gamma_0(N) : \Gamma_0(N) \cap \gamma_i^{-1} g^{-1} \Gamma_0(N) g\gamma_i]^{-1} \\
& \langle f_1|W_N(W_N^{-1} g\gamma_i W_N), f_2\rangle_{\Gamma_0(N)\cap\gamma_i^{-1}g^{-1}\Gamma_0(N)g\gamma_i} \\
= \; & \tfrac{\nu(g)^? \chi(A)}{[\Gamma_0(N):\Gamma_0(N)\cap g^{-1}\Gamma_0(N)g]} \textstyle\sum_i \langle f_1|W_N(W_N^{-1}(g^*)^{-1}, \chi(D_i) f_2|\gamma_i^*\rangle_{\Gamma_0(N)\cap g^{-1}\Gamma_0(N)g} \\
= \; & \nu(g)^? \chi(A) \langle f_1|W_N(g^*)^{-1}, f_2\rangle_{\Gamma_0(N)\cap g^{-1}\Gamma_0(N)g}
\end{aligned}$$

where $? = \frac{gk}{2} - \frac{g(g+1)}{2}$. Similarly:

$$\langle f_1|W_N, f_2|[\Gamma_0(N)g^*\Gamma_0(N)]\rangle_N = \nu(g^*)^? \chi(A) \langle f_1|W_N, f_2|g^*\rangle_{g^{*-1}\Gamma_0(N)g^*\cap\Gamma_0(N)}$$

and so the lemma follows.

Let $\Delta_0(N)$ be the semigroup of elements γ in $GSp_{2g}^+(\mathbb{Q}) \cap M_{2g}(\mathbb{Z})$ with $(N, \det\gamma) = 1$ and $\gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mod N. Let $\mathbb{T}_N^*$ denote the Hecke algebra which is spanned over $\mathbb{Z}$ by the double $\Gamma_0(N)$ cosets contained in $\Delta_0(N)$. Then $\mathbb{T}_N^*$ acts on $M_k(N,\chi)$ and this action preserves $S_k(N,\chi)$. Moreover if $N|M$ then $\mathbb{T}_M^* \hookrightarrow \mathbb{T}_N^*$ and if N and M have the same prime factors this is an isomorphism. This map is also compatible with the inclusion $M_k(N,\chi) \hookrightarrow M_k(M,\chi)$. Any element T of $\mathbb{T}_N^*$ can be written as $\sum n_i \Gamma_0(N) g_i$ with $g_i = \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix}$ and then:

$$a_h(f|T) = \sum n_i \nu(g_i)^{-\frac{g(g+1)}{2}} \chi(\det A_i)(\det A_i)^k \exp(\pi i \, tr(hA_i^{-1}B_i)) a_{D_i h A_i^{-1}}(f)$$

where the sum is taken over those i such that $D_i h A_i^{-1} \in symm_g^*(\mathbb{Z})$. In particular $\mathbb{T}_N^*$ preserves $M_k(N,\chi,\mathbb{Q}^{ac})$. It is also known that $S_k(N,\chi)$ has a basis of eigenforms for $\mathbb{T}_N^*$, which are orthogonal with respect to the Petersson inner product. We see that:

- If $M \subset M_k(N,\chi)$ is preserved by $\mathbb{T}_N^*$ so is $M^\perp \subset S_k(N,\chi)$
- If U is an operator on $S_k(N,\chi)$ which commutes with the action of $\mathbb{T}_N^*$ then U^* also commutes with $\mathbb{T}_N^*$.

For n a positive integer prime to N let $T(n) \in \mathbb{T}_N^*$ denote the sum of all $\Gamma_0(N)$ double cosets in $\Delta_0(N)$ on which ν takes the value n. We let $\mathbb{T}_N$ denote the subring of $\mathbb{T}_N^*$ generated by these operators. We have that:

$$\begin{aligned} a_h(f|T(n)) &= n^{-\frac{g(g+1)}{2}} \chi(n^g) n^{gk} \textstyle\sum_{d_1|d_2|\dots d_g|a} d_1^g d_2^{g-1} \dots d_g \\ & \textstyle\sum_D \chi(\det D)^{-1} (\det D)^{-k} a_{n^{-1}Dh^tD} \end{aligned}$$

where the second sum is taken over a set of representatives for:

$$GL_g(\mathbb{Z}) \backslash GL_g(\mathbb{Z}) \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_g \end{pmatrix} GL_g(\mathbb{Z})$$

In particular $\mathbb{T}_N$ preserves $M_k(N, \chi, \mathcal{O}_{\chi,\wp})$ for $\wp$ any prime of $\mathcal{O}_\chi$ dividing N. Also if $T \in \mathbb{T}_N$ then $T^\dagger = T$. This follows from lemma 3.1 and the fact that $\Delta_0(N)^* = \Delta_0(N)$.

For all these results we refer the reader to [A2]. We also note that any eigenform of $\mathbb{T}_N^*$ is also an eigenform of each algebra $L_{0,l}^g(N)$ ($l \not| N$) considered in [A1] (this follows from [A2] theorem 4.1.8).

If $p|N$ we shall also consider the operators $U_{p^r} = [\Gamma_0(N) \begin{pmatrix} 1_g & 0 \\ 0 & p^r 1_g \end{pmatrix} \Gamma_0(N)]$. We list some properties of U_{p^r}:

Lemma 3.2 *Let $p|N$ and let χ be defined modulo N. Then $U_p \in End(M_k(N, \chi))$ satisfies:*

1. *if $N|M$ then the action of U_p is compatible with $M_k(N, \chi) \hookrightarrow M_k(M, \chi)$*
2. *$a_h(f|U_{p^r}) = a_{p^r h}(f)$*
3. *$U_{p^r} = U_p^r$*
4. *$U_p M_k(Np, \chi) \subset M_k(N, \chi)$*
5. *U_p commutes with the action of $\mathbb{T}_N$*
6. *$U_p^\dagger = U_p$*
7. *if $T \in \mathbb{T}_N^*$ then some power of U_p commutes with T.*

Proof: 1. and 2. follow from the fact that:

$$\Gamma_0(N)\begin{pmatrix} 1_g & 0 \\ 0 & p^r 1_g \end{pmatrix}\Gamma_0(N) = \amalg \Gamma_0(N)\begin{pmatrix} 1_g & B \\ 0 & p^r 1_g \end{pmatrix}$$

where B runs over any set of representatives for the mod p^r congruence classes of $symm_g(\mathbb{Z})$. 3) follows from 2). 4) follows from the fact that:

$$\Gamma_0(Np)\begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix}\Gamma_0(N) = \amalg \Gamma_0(Np)\begin{pmatrix} 1_g & B \\ 0 & p1_g \end{pmatrix}$$

where B is as above. 5) follows from 2) and the formula for $a_h(f|T(n))$, and 6) follows from 2) and lemma 3.1. We do not prove these coset decompositions here. The proof is essentially the same as that given in lemma 3.4 below.

7) follows from these decompositions and the facts that if $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Delta_0(N)$, $\mu = \nu(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix})$, $X \equiv 0 \bmod \mu$ is a symmetric integral matrix, and $p^r \equiv 1 \bmod \mu$ then:

$$\begin{pmatrix} 1_g & X \\ 0 & p^r 1_g \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} 1_g & \mu^{-1t}D((1-p^r)C + XD) \\ 0 & p^r 1_g \end{pmatrix}$$

and that $X \mapsto \mu^{-1t}D((1-p^r)B + XD)$ is a permutation of the modp^r congruence classes of symmetric integral matrices.

Note that in particular U_p preserves $M_k(N, \chi, \mathcal{O}_\chi)$.

We now want to define the Hida idempotent associated to U_p which will be essential in what follows. First recall the following lemma:

Lemma 3.3 *Let $\mathcal{O}$ denote the integers of a finite extension of $\mathbb{Q}_p$ and M a finite $\mathcal{O}$-module. Let U be an operator on M, then there exists a unique idempotent e_U in $End_{\mathcal{O}}(M)$ which commutes with U and such that U is an automorphism on eM and is topologically nilpotent on $(1-e)M$. Moreover $e = \lim_{r\to\infty} U^{r!}$. If M' and U' satisfy corresponding conditions and if $\alpha : M \to M'$ is such that $\alpha U = U'\alpha$ then $\alpha e = e\alpha$.*

If $\mathcal{O}_0$ is a number field, M_0 a finite $\mathcal{O}_0$-module, $\wp$ a prime of $\mathcal{O}_0$ above p and $U \in End_{\mathcal{O}_0}(M_0)$ then there is a ring R contained in a number field with $\mathcal{O}_0 \subset R \subset \mathcal{O}_{0,\wp}$ and an idempotent $e_U \in End_R(M_0 \otimes R)$ such that e_U considered as an element of $End_{\mathcal{O}_{0,\wp}}(M_0 \otimes \mathcal{O}_{0,\wp})$ coincides with the idempotent associated to U above.

In particular in these circumstances we can think of $e \in End_{\mathbb{Q}^{ac}}(M_0 \otimes \mathbb{Q}^{ac})$, as we have fixed $\mathbb{Q}^{ac} \subset \mathbb{Q}_p^{ac}$.

We shall be interested in the case $M_0 = M_k(N, \chi, \mathcal{O}_\chi)$ with $p|N$ and $U = U_p$. We shall denote by e the corresponding idempotent, which we shall call the Hida idempotent. We can think of e acting on $M_k(N, \chi, \mathcal{O}_{\chi,\wp})$ or on $M_k(N, \chi)$. In either case we have the following properties:

- If $N|M$ the action of e is compatible with $M_k(N, \chi) \hookrightarrow M_k(M, \chi)$
- $eM_k(Np^r, \chi) = eM_k(N, \chi)$
- e commutes with the action of $\mathbb{T}_N^*$
- $e^\dagger = e$

Finally we must study the operator U_p and the Hida idempotent e in a slightly different setting. Let N now be an integer prime to p.Now let Γ denote one of the following:

- $\Gamma_0 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \,|\, B \equiv 0 \bmod N,\, C \equiv 0 \bmod Np, \right.$
  $\left. A \equiv D \equiv 1 \bmod N \right\}$
- $\Gamma_1 = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0 \,|\, \det D \equiv 1 \bmod p \right\}$
- $\Gamma_1(Np^r)$

Any element of $M_k(\Gamma)$ has a Fourier expansion $\sum a_h(f) q^h$ where h runs over elements of $N^{-1} symm_g^*(\mathbb{Z})$. The theory of rationality carries over exactly to this situation. We define operators U_{p^r} to be the Hecke operators associated to the double cosets $\Gamma \begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} \Gamma$.

Then we have:

Lemma 3.4 *With the notation as above:*

1. $U_{p^r} = U_p^r$
2. $a_h(f|U_p) = a_{ph}(f)$
3. U_p *preserves* $M_k(\Gamma, \mathbb{Z})$
4. *the action of* U_p *is compatible with the inclusions:*

$$\begin{array}{ccc} M_k(\Gamma_1) & \supset & M_k(Np, \chi) \\ \cup & & \\ M_k(\Gamma_0) & \supset & M_k(Np, 1) \end{array}$$

and:

$$M_k(\Gamma_1(Np^r)) \supset M_k(Np^r, \chi)$$

<u>*Proof:*</u>These all follow from the facts that:

$$\Gamma \begin{pmatrix} 1_g & 0 \\ 0 & p1_g \end{pmatrix} \Gamma = \amalg \Gamma \begin{pmatrix} 1_g & B \\ 0 & p1_g \end{pmatrix}$$

where B runs over a set of representatives for the mod p^r congruence classes of $symm_g(\mathbb{Z})$ each chosen $\equiv 0$ mod N. This decomposition follows from the following equations:

- $\begin{pmatrix} 1_g & B \\ 0 & p^r 1_g \end{pmatrix} = \begin{pmatrix} 1_g & 0 \\ 0 & p^r 1_g \end{pmatrix} \begin{pmatrix} 1_g & B \\ 0 & 1_g \end{pmatrix}$
- $\begin{pmatrix} 1_g & X \\ 0 & p^r 1_g \end{pmatrix} \begin{pmatrix} 1_g & Y \\ 0 & p^r 1_g \end{pmatrix}^{-1} = \begin{pmatrix} 1_g & p^{-r}(X - Y) \\ 0 & 1_g \end{pmatrix}$

- If $\begin{pmatrix} A & B \\ pC & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$ then ${}^tDB \in symm_g(\mathbb{Z})$ and if further $X \in symm_g(\mathbb{Z})$ with $X \equiv^t DB \bmod p^r$ and $X \equiv 0 \bmod N$ then:

$$\begin{pmatrix} 1_g & 0 \\ 0 & p^r 1_g \end{pmatrix} \begin{pmatrix} A & B \\ pC & D \end{pmatrix} = \begin{pmatrix} A & p^{-r}(B - AX) \\ p^{r+1}C & D - pCX \end{pmatrix} \begin{pmatrix} 1_g & X \\ 0 & p^r \end{pmatrix}$$

  where $B - AX \equiv A({}^tDB - X) \equiv 0 \bmod p^r$.

In particular we can introduce the Hida idempotent in this context, and it will be compatible with the other contexts considered earlier. We finish this section with a simplification of the condition that an ordinary modular form is cuspidal:

Lemma 3.5 *Let N be prime to p. Let Γ (or $\Gamma^{(g)}$) denote either Γ_0 or Γ_1 as defined above. Let $\Gamma^{(g-1)}$ denote the corresponding group in genus $g-1$. Then there is an integer m such that for all $k \geq \frac{g(g+1)}{2}$ we have a left exact sequence:*

$$0 \to eS_k(\Gamma^{(g)}) \to eM_k(\Gamma^{(g)}) \to eM_k(\Gamma^{(g-1)})^m$$

Proof:Let $\Gamma \backslash Sp_{2g}(\mathbb{Z}) = \amalg_I \Gamma\delta_i$, so that we have an exact sequence:

$$0 \to S_k(\Gamma) \to M_k(\Gamma) \xrightarrow{\phi} \bigoplus_I M_k(\Phi(\delta_i^{-1}\Gamma\delta_i))$$

Let J be the subset of I consisting of those i for which $\Gamma\delta_i \subset \Gamma_0(p)$. Let $\Omega \subset \mathbb{Q}^{ac}$ be the finite $\mathbb{Z}[\zeta_p]$-module generated by the coefficients $f|\delta_i$ where $f \in M_k(\Gamma, \mathbb{Z})$ and $i \in I$. Then we make the following claims:

1. $f \in M_k(\Gamma, \mathbb{Z})$ and $\phi(f) \in \bigoplus_{I \backslash J} M_k(\Phi(\delta_i^{-1}\Gamma\delta_i), p^r\Omega)$ implies that $\phi(f|U_p) \in \bigoplus_I M_k(\Phi(\delta_i^{-1}\Gamma\delta_i), p^{r+1}\Omega)$

2. $f \in M_k(\Gamma)$ and $\alpha \in \Gamma_0(p)$ implies that $\alpha^{-1}\Gamma\alpha = \Gamma$ and that $f|\alpha|U_p = f|U_p|\alpha$

3. $f \in M_k(\Gamma, \mathbb{Z})$ and $\phi(f) \in \bigoplus_{I \backslash J} M_k(\Phi(\delta_i^{-1}\Gamma\delta_i), p^r\Omega)$ implies that $\phi(f|U_p) \in \bigoplus_{I \backslash J} M_k(\Phi(\delta_i^{-1}\Gamma\delta_i), p^(r+1)\Omega)$

4. The following sequence is left exact:

$$0 \to eS_k(\Gamma) \to eM_k(\Gamma) \to \bigoplus_J M_k(\Gamma^{(g-1)})$$

5. $\Phi : M_k(\Gamma^{(g)}) \to M_k(\Gamma^{(g-1)})$ is compatible with the action of U_p.

Note that from 2), 4) and 5) the lemma follows at once. Also note that 5) follows easily from the description of U_p in terms of its action on Fourier expansions; and that 3) and 4) follow easily from 1) and 2).

First we prove 1). Note that for $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$:

$$f|U_p|\alpha = p^{\frac{gk}{2} - \frac{g(g+1)}{2}} \sum_X f| \begin{pmatrix} A + XC & B + XD \\ pC & pD \end{pmatrix}$$

and that:

$$\begin{pmatrix} A + XC & B + XD \\ pC & pD \end{pmatrix} = \begin{pmatrix} (A + XC)E^{-1} & B' \\ pCE^{-1} & D' \end{pmatrix} \begin{pmatrix} E & Y \\ 0 & p\,^tE^{-1} \end{pmatrix}$$

with:

- $\begin{pmatrix} (A + XC)E^{-1} & B' \\ pCE^{-1} & D' \end{pmatrix} \in Sp_{2g}(\mathbb{Z})$
- $\begin{pmatrix} E & Y \\ 0 & p\,^tE^{-1} \end{pmatrix} \in M_{2g \times 2g}(\mathbb{Z})$
- E upper triangular

(That we can find such a decomposition can be easily deduced from proposition 1.3.7 of [A2].) We now consider two cases:

1. $p \nmid \det E$, *i.e.* $E \in GL_g(\mathbb{Z})$

   In this case:

$$\begin{pmatrix} A + XC & B + XD \\ pC & pD \end{pmatrix} = \begin{pmatrix} A + XC & B'\,^tE^{-1} \\ pC & D'\,^tE^{-1} \end{pmatrix} \begin{pmatrix} 1_g & E^{-1}Y \\ 0 & p1_g \end{pmatrix}$$

with $\begin{pmatrix} A+XC & B'^tE^{-1} \\ pC & D'^tE^{-1} \end{pmatrix} \in \Gamma_0(p)$. Thus:

$$\Phi\left(f\Big|\begin{pmatrix} A+XC & B+XD \\ pC & pD \end{pmatrix}\right) = 0$$

2. $p|\det E$

   In this case:

$$p^{\frac{gk}{2}-\frac{g(g+1)}{2}} f\Big|\begin{pmatrix} A+XC & B+XD \\ pC & pD \end{pmatrix} =$$
$$p^{k-\frac{g(g+1)}{2}}(p^{-1}\det E)^k \sum a_h \exp(\pi i\, tr(p^{-1}hY^tE))\exp(\pi i\, tr(p^{-1}\,{}^tEhEz))$$

   where:

$$f\Big|\begin{pmatrix} (A+XC)E^{-1} & B' \\ pCE^{-1} & D' \end{pmatrix} = \sum a_h \exp(\pi i\, tr(hz))$$

In either case $\phi(f|U_p) \in \bigoplus_I M_k(\Gamma, p^{r+1}\Omega)$ as E upper triangular implies that ${}^tE^{-1}\begin{pmatrix} h & \\ & 0 \end{pmatrix}E^{-1} = \begin{pmatrix} h' & \\ & 0 \end{pmatrix}$.

Finally for claim 2), let $X \equiv 0 \bmod N$ and $\begin{pmatrix} A & B \\ pC & D \end{pmatrix} \in \Gamma_0(p)$. Then:

$$\begin{pmatrix} A & B \\ pC & D \end{pmatrix}\begin{pmatrix} 1_g & X \\ 0 & p1_g \end{pmatrix}\begin{pmatrix} A & B \\ pC & D \end{pmatrix}^{-1}\begin{pmatrix} 1_g & AX^tA - A^tB \\ 0 & p1_g \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} A^tD - B^tCp^2 - pAX^tC & -A^tB + B^tAp + AX^tA \\ p(C^tD - D^tCp - CX^tCp) & p(-C^tB + D^tA + CX^tA) \end{pmatrix}$$
$$\begin{pmatrix} 1_g & p^{-1}(A^tB - AX^tA) \\ 0 & p^{-1}1_g \end{pmatrix}$$

- $\in Sp_{2g}(\mathbb{Z})$ as $A^tD(A^tB - AX^tA) - A^tB + AX^tA \equiv 0 \bmod p$
- $\equiv \begin{pmatrix} A' & * \\ 0 & D' \end{pmatrix} \bmod p$ and if $\det A \equiv \det D \equiv 1 \bmod p$ then $\det A' \equiv \det D' \equiv 1 \bmod p$.

- $\equiv \begin{pmatrix} 1_g & 0 \\ 0 & 1_g \end{pmatrix} \bmod N$

The result now follows as $X \mapsto AX^tA - A^tB$ is a permutation of $symm_g(\mathbb{Z}/p\mathbb{Z})$.

3.3 Some Lemmas on Eisenstein and Theta Series

In this section we collect some results that we need concerning Eisenstein series, theta series and Rankin's method. These results are based on work of Andrianov and of Shimura. First we introduce some Eisenstein series:

- $E(z,s;k,\phi,b) = (\det 2y)^s \sum \phi(\det D)\det(Cz+D)^{-k}|\det(Cz+D)|^{-2s}$ where $z = x+iy$, and where the sum runs over $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ representing the cosets $\Gamma_0\backslash\Gamma_0(b)$.
- $E^*(z,s;k,\phi,b) = E(\frac{-1}{z},s;k,\phi,b)z^{-k}$
- $\widetilde{E}_{k,\phi,b}(z) = \sum_{\substack{h\,\in\, symm_g^*(\mathbb{Z}) \\ h>0}} a_h \exp(\pi i\, tr(hz))$ with:

$$a_h = L(1+\tfrac{g}{2}-k, \widetilde{\phi^{-1}\xi_h}) \left(\prod_{\substack{l\,\in\, P(h) \\ l \nmid b}} M_{l,h}(\tfrac{\phi(l)}{l^k}) \right) (\det h)^{k-\frac{g+1}{2}} f_{\widetilde{\phi\xi_h}}^{\frac{g}{2}-k}$$
$$\tau(\widetilde{\phi\xi_h}) \prod_{\substack{l \nmid f_{\phi\xi_h} \\ l \mid b}} (1-\widetilde{\phi\xi_h}(l)l^{\frac{g}{2}-k})$$

  where

  - $\widetilde{\psi}$ is the primitive character corresponding to ψ
  - if ψ is a primitive character f_ψ denotes its conductor and
  $$\tau(\psi) = \sum_{x\in(\mathbb{Z}/f_\psi\mathbb{Z})^\times} \psi(x)\exp(\frac{2\pi i x}{f_\psi})$$
  denotes the corresponding Gauss sum
  - $\xi_h = \zeta_h\theta^{\frac{g}{2}}$ where ζ_h is the character corresponding to $\mathbb{Q}(\sqrt{\det h})/\mathbb{Q}$ and θ is the character corresponding to $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$

– $P(h)$ is a finite set of primes depending only on $\mathbb{Q}^\times h$, as defined in §10 of [Fe].

– for $l \in P(h)$ $M_{l,h}(X) \in \mathbb{Z}[X]$ is a polynomial depending only on l and $\mathbb{Q}^\times h$ as defined in §10 of [Fe]

Lemma 3.6 *Assume $k \geq \frac{g}{2}+1$ and that ϕ is a character modulo b, with $\phi(-1)=(-1)^k$. Assume also that either $k > \frac{g}{2}+2$ or $\phi^2 \neq 1$. Then:*

$$C\tilde{E}_{k,\phi,b}(z) = E^*(z,0;k,\phi,b) \mid [b] \in M_k(\Gamma_0(b),\phi)$$

with $C \neq 0$ a constant.

Proof:$E^*(x+iy,s;k,\phi,b) = \sum_{bh\in symm_g^*(\mathbb{Z})} a_h(y,s)\exp(\pi i\, tr(hx))$ where $a_h(y,s)$ has been calculated by Shimura and Feit. In particular $a_h(y,s)$ is finite (resp. zero) at $s=0$ if $A_h(s)$ is finite (resp. zero) at $s=0$, where, if h has r_+ positive eigenvalues, r_- negative eigenvalues and rank $r = r_+ + r_-$:

$$\begin{aligned} A_h(s) &= \frac{\Gamma_{g-r}(k-\frac{g+1}{2}+2s)}{\Gamma_{g-r_-}(k+s)\Gamma_{g-r_+}(s)}\left(L(k+2s,\phi)\prod_{i=1}^{\frac{g}{2}} L(2k-2i+4s,\phi^2)\right)^{-1} \\ &\cdot \begin{cases} L(k-g+\frac{r}{2}+2s,\phi\xi_h)\prod_{i=1}^{\frac{g-r}{2}} L(2k-2g+r-1+2i+4s,\phi^2) \\ \prod_{i=0}^{\frac{g-r-1}{2}} L(2k-2g+r+2i+4s,\phi^2) \end{cases}\end{aligned}$$

according as r is even or odd. Here $\Gamma_t(s) = \prod_{i=1}^{t-1}\Gamma(s-\frac{i}{2})$. (See §10 of [Fe].) The only terms that contribute a zero or pole at $s=0$ are:

- $\Gamma_{g-r_+}(s)^{-1}$ which contributes a zero of order the integer part of $\frac{g-r_++1}{2}$
- the L-series in the numerator which can contribute at most a simple pole.

Thus $A_h(s)$ has a pole at $s=0$ only if $g=r_+$ and the L-series in the numerator have a pole, i.e. $k = 1+\frac{g}{2}$, $\phi\xi_h = 1$, which is a case we have excluded. Moreover $A_h(0)=0$ unless $g=r_+$ or one of the following hold:

- $g = r_+ + 1$, $k = \frac{g}{2}+1$, $\phi^2 = 1$
- $g = r_+ + 2$, $k = \frac{g}{2}+2$, $\phi\xi_h = 1$

- $g = r_+ + 2$, $k = \frac{g}{2} + 1$, $\phi^2 = 1$

However our assumptions exclude these last three possibilities. Thus we have:

$$E^*(x+iy, 0; k, \phi, b) = \sum_{\substack{h \,\in\, symm_g^*(\mathbb{Z}) \\ h > 0}} a_{b^{-1}h}(y, 0) \exp(\pi i \, tr(hx))$$

where using [Fe] §10 and the functional equation for L-series associated to Dirichlet characters:

$$a_h(y, o) = 2^? \pi^? i^? (\det b)^? \gamma L^{-1} a_h e^{-\pi \, tr(hy)}$$

with each ? a rational number independent of h and with:

- $\gamma = \Gamma_g(k)^{-1} \Gamma\left(\frac{k+\mu-\frac{g}{2}}{2}\right)^{-1} \Gamma\left(\frac{1-k+\mu+\frac{g}{2}}{2}\right)$
- $L = L(k, \phi) \prod_{i=1}^{\frac{g}{2}} L(2k - 2i, \phi^2)$
- $\mu = 0$ or 1 and $\mu \equiv k - \frac{g}{2} \bmod 2$, so that $\phi\xi_h(-1) = (-1)^\mu$.

From this the desired equality follows at once. Moreover we see that $E^*(z, 0; k, \phi, b) \mid [b]$ is holomorphic. Finally it lies in $M_k(\Gamma_0(b), \phi)$ because it is equal to $E(z, 0; k, \phi, b) \mid W_b$ and $E(z, s; k, \phi, b)$ transforms under $\Gamma_0(b)$ by ϕ^{-1}.

We now introduce some theta series. Let ψ be a character modulo r and let $Q \in symm_g^*(\mathbb{Z})$, $Q > 0$. We shall let $s(Q)$, the *step* of Q, denote the smallest positive integer such that $s(Q)Q^{-1} \in symm_g^*(\mathbb{Z})$. Now set:

$$\theta_{Q,\psi}(z) = \sum_{N \in M_{g \times g}(\mathbb{Z})} \psi(\det N) \exp(\pi i \, tr({}^t N Q N z))$$

We first record some transformation properties of $\theta_{Q,\psi}$. The proofs are based on work of Andrianov. In fact part one is due to him in the case ψ primitive, but unfortunately we don't think this method goes over exactly to the case of ψ imprimitive.

Lemma 3.7 *1. $\theta_{Q,\psi}(z) \in M_{\frac{g}{2}}(\Gamma_0(r^2 s(Q)), \psi\chi_Q)$ where χ_Q is a character of order two modulo $s(Q)$ with $\chi_Q(-1) = (-1)^{\frac{g}{2}}$.*

2. If ψ is primitive modulo r then $\theta_{Q,\psi} \mid W_{r^2 s(Q)} = C\theta_{s(Q)Q^{-1},\psi^{-1}}$ for some constant C.

Proof: Following Andrianov we introduce for $T \in M_{g\times g}(\mathbb{Z})$ with $QT \equiv 0 \bmod s(Q)$ a theta series:

$$\theta(z, Q : T) = \sum_{N\in M_{g\times g}(\mathbb{Z})} \exp(\pi i\, tr({}^t(N + s(Q)^{-1}T)Q(N + s(Q)^{-1}T)z))$$

Then:

$$\theta_{Q,\psi}(z) = \sum_{M\in M_{g\times g}(\mathbb{Z}) \bmod r} \psi(\det M)\theta(z, r^2 Q : rs(Q)M)$$

Now from §1.3.3 of [A2] we know that:

1. If $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(s(Q))$ then:

$$\theta(z, Q : T) \mid_{\frac{g}{2}} \gamma = \chi_Q(\det D) \exp(\pi i\, tr(s(Q)^{-2} A^t B^t TQT))\theta(z, Q : TA)$$

2. $\theta(z, Q : T) \mid W_{s(Q)} = C_1 \sum_{\substack{QU \,\equiv\, 0 \bmod s(Q) \\ U \bmod s(Q)}} \exp(2\pi i\ tr(s(Q)^{-2}\, {}^tTQU))\theta(z, s(Q)Q : U)$
   with C_1 a non-zero constant independent of T.

Thus if $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(r^2 s(Q))$ then:

$$\begin{aligned} \theta_{Q,\psi} \mid_{\frac{g}{2}} \gamma(z) &= \sum \psi(\det M)\chi_Q(\det M) \exp(\pi i\, tr(A^t B^t MQM)) \\ &\quad\ \theta(z, r^2 Q : rs(Q)MA) \\ &= \chi_Q \psi(\det D) \sum \psi(\det(MA))\theta(z, r^2 Q : rs(Q)MA) \\ &= \chi_Q \psi(\det D)\theta_{Q,\psi}(z) \end{aligned}$$

(Here we have used the fact that $(A^t B)({}^t MQM)$ has trace an even integer.)

Now assume ψ is primitive. For $A \in M_{g\times g}(\mathbb{Z})$ set:

$$\tau_A(\psi) = \sum_{M\in M_{g\times g}(\mathbb{Z}/r\mathbb{Z})} \psi(\det M) \exp(\frac{2\pi i}{r}\ tr(MA))$$

We shall assume for the moment the following two facts:

- $A \in GL_g(\mathbb{Z}/r\mathbb{Z})$ implies that $\tau_A(\psi) = \psi(\det A)^{-1}\tau_1(\psi)$
- $A \notin GL_g(\mathbb{Z}/r\mathbb{Z})$ implies that $\tau_A(\psi) = 0$.

Then:

$$\begin{aligned} \theta_{Q,\psi} \mid W_{r^2 s(Q)} &= C_1 \sum \psi(\det M) \sum_{\substack{QT \equiv 0 \bmod s(Q) \\ T \bmod r^2 s(Q)}} \exp(\tfrac{2\pi i}{r}\ tr({}^t M \tfrac{QT}{s(Q)})) \\ &\quad \theta(z, r^4 s(Q) Q : T) \\ &= C_1 \sum_T \tau_{\frac{QT}{s(Q)}}(\psi)\theta(z, r^4 s(Q)Q : T) \\ &= C_1 \tau_1(\psi) \sum_T \psi^{-1}(\det \tfrac{QT}{s(Q)})\theta(z, r^4 s(Q)Q : T) \end{aligned}$$

but:

$$\theta(z, r^2 Q : T) = \sum_N \exp(\pi i\, tr(z^t(r^2 QN + \frac{QT}{s(Q)})s(Q)Q^{-1}(r^2 QN + \frac{QT}{s(Q)})))$$

so that:

$$\begin{aligned} \theta_{Q,\psi} \mid W_{r^2 s(Q)}(z) &= C_1 \tau_1(\psi) \sum \psi^{-1}(\det X) \\ &\quad \sum_N \exp(\pi i\, tr(z^t(r^2 QN + X)s(Q)Q^{-1}(r^2 QN + X))) \end{aligned}$$

where $X = X(T) = \frac{QT}{s(Q)}$ and where T runs over elements of $M_{g\times g}(\mathbb{Z})$ modulo $r^2 s(Q)$ with $QT \equiv 0 \bmod s(Q)$.

Now let I be the right ideal in $M_{g\times g}(\mathbb{Z})$ generated by $r^2 Q$. Then the sets $\{T \bmod r^2 s(Q) \mid QT \equiv 0 \bmod s(Q)\}$ and $\{X \bmod I\}$ are in bijection by the maps:

$$\begin{aligned} T &\longmapsto s(Q)^{-1}QT \\ s(Q)Q^{-1}X &\longleftarrow X \end{aligned}$$

Thus:

$$\begin{aligned} \theta_{Q,\psi} \mid W_{r^2 s(Q)}(z) &= C_1 \tau_1(\psi) \sum_{X \bmod I} \psi^{-1}(\det X) \\ &\quad \sum_{N \equiv X \bmod I} \exp(\pi i\, tr(z^t N s(Q) Q^{-1} N)) \\ &= C_1 \tau_1(\psi)\theta_{s(Q)Q^{-1},\psi^{-1}} \end{aligned}$$

Now we must return to the two claims about the sums $\tau_A(\psi)$. The first is easy on making a change of variable. For the second we shall show the existence of $B \in GL_g(\mathbb{Z}/r\mathbb{Z})$ such that $BA = A$ and $\psi(\det B) \neq 1$. From this one sees easily that $\tau_A(\psi) = \psi(\det B)\tau_A(\psi)$ and so

$\tau_A(\psi) = 0$. To find such a B let M be the submodule of $(\mathbb{Z}/r\mathbb{Z})^g$ generated by the columns of A. What we require is $\beta \in Aut((\mathbb{Z}/r\mathbb{Z})^g)$ with $\beta \mid_M = Id$ and with $\psi(\det \beta) \neq 1$. However we can pick a basis $e_1, ..., e_g$ of $(\mathbb{Z}/r\mathbb{Z})^g$ such that $M = (\lambda_1 e_1, ..., \lambda_g e_g)$ with $(\lambda_1, r) \neq 1$. Then let β be represented by :

$$\begin{pmatrix} \alpha & 0 & . & . & 0 \\ 0 & 1 & & & 0 \\ . & & . & & . \\ . & & & . & . \\ 0 & 0 & . & . & 1 \end{pmatrix}$$

with $\alpha \equiv 1 \bmod \frac{r}{(\lambda_1, r)}$ and $\psi(\alpha) \neq 1$. This is the desired β and completes the proof of the lemma.

We now introduce a formal q-expansion which looks like an Eisenstein series:

$$E_{\phi,b,s}(k,\alpha) = \sum_{\substack{(\det h, p^\infty) = p^s \\ h \in symm_g^*(\mathbb{Z}) \\ h > 0}} a_h q^h$$

where:

- b is prime to p
- $\alpha : (\mathbb{Z}/p^t\mathbb{Z})^\times \xrightarrow{\langle\rangle} (1 + p\mathbb{Z})/(1 + p^t\mathbb{Z}) \longrightarrow \mathbb{Q}^{ac\,\times}$
- $\phi(\mathbb{Z}/bp\mathbb{Z})^\times \longrightarrow \mathbb{Q}^{ac\,\times}$ and $\phi = \phi^{(p)}\phi_{(p)}$ corresponding to $(\mathbb{Z}/bp\mathbb{Z})^\times \cong (\mathbb{Z}/b\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times$

and where:

$$\begin{aligned} a_h \quad = \quad & (\det h)^{-\frac{g+1}{2}} f^{\frac{g}{2}}_{\widetilde{\phi^{(p)}\xi_h^{(p)}}} \phi_{(p)} \xi(f_{\widetilde{\phi^{(p)}\xi_h^{(p)}}}) \tau(\widetilde{\phi^{(p)}\xi_h^{(p)}}) L_{(p)}(1 + \tfrac{g}{2} - k, \phi^{-1}\widetilde{\alpha\omega^{-k}}\xi_h) \\ & \alpha^{-1}(f_{\widetilde{\phi^{(p)}\xi_h^{(p)}}}) \langle f_{\widetilde{\phi^{(p)}\xi_h^{(p)}}} \rangle^{-k} \alpha(p^{-s}\det h) \langle p^{-s} \det h \rangle^k \\ & \prod_{\substack{l \in P(h) \\ l \nmid bp}} M_{l,h}(\phi(l)\alpha(l^{-1})\langle l \rangle^{-k}) \prod_{\substack{l \nmid f_{\widetilde{\phi^{(p)}\xi_h^{(p)}}} \\ l \mid b}} (1 - l^{\frac{g}{2}} \widetilde{\phi\xi_h}(l) \alpha(l^{-1}) \langle l \rangle^{-k}) \end{aligned}$$

Here we have used the notation $\xi_h = \xi\xi_h^p$ where ξ_h^p is defined modulo a number prime to p and where $\xi = \omega^{s\frac{p-1}{2}}$. Also $L_{(p)}$ denotes the L-series with the Euler factor at p removed.

Now we prove that these are not too far off from being modular forms.

Lemma 3.8 *Let b be prime to p, $\phi : (\mathbb{Z}/bp\mathbb{Z})^\times \to \mathbb{Q}^{ac\times}$, $\phi = \phi^{(p)}\phi_{(p)}$ as above with $\phi^{(p)2} \neq 1$, $\phi(-1) = 1$. Let $\alpha : (\mathbb{Z}/p^{t_1}\mathbb{Z})^\times \overset{\langle\rangle}{\to} (1+p\mathbb{Z})/(1+p^{t_1}\mathbb{Z}) \to \mathbb{Q}^{ac\times}$ also be as above. Let $Q \in symm_g^*(\mathbb{Z})$ with $(\det Q, p^\infty) = p^{t_2}$. Set $t = \max(t_1, t_2+1)$. Also let $\psi : (\mathbb{Z}/r\mathbb{Z})^\times \to \mathbb{Q}^{ac\times}$ be such that $s(Q)r^2 \mid bp^t$ and $p \mid r$. Finally assume that $k \geq \frac{g}{2}+1$. Then:*

$$\begin{aligned} CU_p^t(\theta_{Q,\psi}E_{\phi,b,t_2}(k,\alpha)) &= U_p^t(\theta_{Q,\psi\alpha^2\omega^{-2k}}\tilde{E}_{k,\phi\alpha^{-1}\omega^k,bp}) \\ &\in M_{k+\frac{g}{2}}(\Gamma_0(bp^t), \psi\phi\chi_Q\alpha\omega^{-k}) \end{aligned}$$

with C a non-zero constant.

We first separate out a part of the proof:

Lemma 3.9 *Let β be a character defined mod p^{t_1} and let $Q \in symm_g^*(\mathbb{Z})$ with $Q > 0$ and $(\det Q, p^\infty) = p^{t_2}$. Then if $t = \max(t_1, t_2+1)$ and if $A = \sum A_h q^h$ is a formal q-expansion:*

$$\beta(p^{-t_2}\det(-Q))U_p^t(\theta_{Q,\psi\beta^2}A) = U_p^t(\theta_{Q,\psi}A_\beta)$$

where $A_\beta = \sum_{(\det h, p^\infty)=p^{t_2}} \psi(p^{-t_2}\det h)A_h q^h$.

<u>*Proof:*</u>We look at the coefficient of q^h on both sides. On the left hand side we get:

$$\sum_{g+{}^t nQn = p^t h} \beta^2\psi(\det n)A_g\beta(p^{-t_2}\det(-Q))$$

while on the right hand side we get:

$$\sum_{g+{}^t nQn = p^t h} \psi(\det n)A_g\beta(p^{-t_2}\det g)$$

This proves the lemma.

<u>*Proof of lemma three:*</u>

First note that:

$$\theta_{Q,\psi\alpha^2\omega^{-2k}} \in M_k(\Gamma_0(bp^t), \psi\chi_Q\alpha^2\omega^{-2k})$$

and that as $\phi^{(p)\,2} \neq 1$ and $k \geq \frac{g}{2}+1$ we have:

$$\tilde{E}_{k,\phi\alpha^{-1}\omega^k} \in M_k(\Gamma_0(bp^t), \phi\alpha^{-1}\omega^k)$$

Thus we need only show that the claimed identity holds (formally). However:

$$\begin{aligned} \alpha\omega^{-k}(\det Q)U_p^t(\theta_{Q,\psi\alpha^2\omega^{-2k}}\tilde{E}_{k,\phi\alpha^{-1}\omega^k}) &= U_p^t(\theta_{Q,\psi}E_1) \\ &= \epsilon(p^{-t_2}\det Q)U_p^t(\theta_{Q,\psi}E_2) \end{aligned}$$

where:

- $E_1 = \sum_{\substack{(\det h,\, p^\infty) \,=\, p^{t_2} \\ h \in symm_g^*(\mathbb{Z}) \\ h>0}} b_h q^h$
- $E_2 = \sum_{\substack{(\det h,\, p^\infty) \,=\, p^{t_2} \\ h \in symm_g^*(\mathbb{Z}) \\ h>0}} \epsilon(p^{-t_2}\det h) b_h q^h$
- $b_h = f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}}^{\frac{g}{2}-k}\phi^{(p)}(f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}})\tau(\widetilde{\phi_{(p)}^{-1}\xi\alpha\omega^{-k}})a_h$

  $$\cdot\begin{cases} 1 & f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}} = p^{2m} \\ \xi_h^{(p)}(p) & f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}} = p^{2m-1} \\ A_{\xi_h^{(p)}(p)} & \xi\phi_{(p)}\omega^k\alpha^{-1} = 1 \end{cases}$$

  with m a positive integer
- $A_\pm = \frac{1\mp\phi^{(p)}(p)p^{\frac{g}{2}-k}}{1\mp\phi^{(p)}(p)^{-1}p^{k-\frac{g}{2}-1}}$
- $\epsilon = \omega^{\frac{p-1}{2}}$

Here we have used the fact that if χ_1, χ_2 are primitive characters with coprime conductors then $\tau(\chi_1\chi_2) = \tau(\chi_1)\tau(\chi_2)\chi_1(f_{\chi_2})\chi_2(f_{\chi_1})$. Note that $A_\pm \neq \infty$ or 0 as $\phi^{(p)\,2} \neq 1$ and also that:

$$\xi_h^{(p)}(p) = \delta.\epsilon(p^{-t_2}\det h)$$

with $\delta = \epsilon(-1)^{\frac{g}{2}}$ if $p \equiv 1 \bmod 4$ and $\delta = \epsilon(-1)^{\frac{g}{2}+t_2}$ if $p \equiv 3 \bmod 4$.

Thus we see that:

- If $f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}}$ is an even power of p and is $\neq 1$ then:

$$E_1 = f^{\frac{g}{2}-k}_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}} \phi(p)(f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}})\tau(\phi_{(p)}\widetilde{\xi\alpha^{-1}}\omega^k)E_{\phi,b,t_2}(k,\alpha)$$

- If $f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}}$ is an odd power of p then:

$$E_2 = \delta f^{\frac{g}{2}-k}_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}} \phi(p)(f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}})\tau(\phi_{(p)}\widetilde{\xi\alpha^{-1}}\omega^k)E_{\phi,b,t_2}(k,\alpha)$$

- If $\phi_{(p)}\widetilde{\xi\alpha^{-1}}\omega^k = 1$ then:

$$E_1(A_\delta^{-1} + A_{-\delta}^{-1}) + E_2(A_\delta^{-1} - A_{-\delta}^{-1}) =$$
$$2f^{\frac{g}{2}-k}_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}} \phi(p)(f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}})\tau(\phi_{(p)}\widetilde{\xi\alpha^{-1}}\omega^k)E_{\phi,b,t_2}(k,\alpha)$$

In the first two cases we conclude at once the desired equality. In the last case we conclude that:

$$A^{-1}_{\epsilon(p^{-t_2}\det Q)\delta}\alpha\omega^{-k}(p^{-t_2}\det Q)U_p^t(\theta_{Q,\psi\alpha^2\omega^{-2k}}\tilde{E}_{k,\phi\alpha^{-1}\omega^k,bp}) =$$
$$f^{\frac{g}{2}-k}_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}} \phi(p)(f_{\widetilde{\phi_{(p)}\xi\alpha^{-1}\omega^k}})\tau(\phi_{(p)}\widetilde{\xi\alpha^{-1}}\omega^k)U_p^t(\theta_n Q,\psi E_{\phi,b,t_2}(k,\alpha))$$

and again we are done.

We shall now fix a cusp form f in $S_k(\Gamma_0(q),\chi)$. Assume that f has Fourier expansion $\sum_{h\in symm_g^*(\mathbb{Z})h>0} a_h(f)q^h$. Also if $Q \in symm_g^*(\mathbb{Z})$, $Q > 0$ and $\psi : (\mathbb{Z}/r\mathbb{Z})^\times \to \mathbb{Q}^{ac\,\times}$ we introduce a Dirichlet series:

$$\begin{aligned} D_{Q,\psi}(s) &= L(s+\tfrac{g}{2},\psi\chi\chi_Q)\prod_0^{\frac{g}{2}-1} L(2s+2i,\psi^2\chi^2) \\ &\quad \sum_{M\in SL_g(\mathbb{Z})\backslash M^+_{g\times g}(\mathbb{Z})} \psi(\det M)a_{MQ^tM}(f)(\det M)^{1-k-s} \end{aligned}$$

Lemma 3.10 *$D_{Q,\psi}(s) = \sum_{n=1}^\infty \frac{d_n}{n^s}$ with $|d_n| \leq Cn^{g+\frac{1}{2}}\log n$ for some constant C. In particular it is convergent for $\operatorname{Re} s > g + \frac{3}{2}$.*

<u>*Proof:*</u>We shall establish the following claims:

1. $|a_h(f)| \leq C_1(\det h)^{\frac{k}{2}}$

2. $\#\{[M] \in SL_g(\mathbb{Z})\backslash M^+_{g\times g}(\mathbb{Z}) | \det M = n\} \leq C_2 n^{g-\frac{1}{2}}\log n$

From these two it at once follows that:

$$\sum_{M\in SL_g(\mathbb{Z})\backslash M^+_{g\times g}(\mathbb{Z})} a_{MQ^tM}(f)(\det M)^{1-k-s} = \sum \frac{a_n}{n^s}$$

with $|a_n| \leq C_1C_2(\det Q)^{\frac{k}{2}} n^{g+\frac{1}{2}} \log n$, and hence that:

$$\begin{aligned}
|d_n| &\leq \textstyle\sum_{n=n_1n_2} |a_{n_1}| \sum_{n_2=n_3m_1^2\ldots m_{\frac{g}{2}}^2} 1 \\
&\leq C_1C_2(\det Q)^{\frac{k}{2}} n^{g+\frac{1}{2}} (\log n) \textstyle\sum_{n=n_1n_2} n_2^{-g} \sum_{n_2=n_3m_1^2\ldots m_{\frac{g}{2}}^2} 1 \\
&\leq C_1C_2(\det Q)^{\frac{k}{2}} n^{g+\frac{1}{2}} (\log n) \left(\textstyle\sum_{n_3=1}^n n_3^{-g}\right) \left(\sum_1^n m^{-2g}\right)^{\frac{g}{2}} \\
&\leq Cn^{g+\frac{1}{2}} \log n
\end{aligned}$$

We now prove the two claims that we made. For the first recall that:

$$a_h(f) = \int_{X\in symm_g(\mathbb{R})/symm_g(\mathbb{Z})} f(X+iY)\exp(-\pi i\ tr(zh))$$

where $z = X + iY$ and Y is constant. Thus from the bound given on page 335 of [Sm] we see that:

$$|a_h(f)| \leq C_3(\det Y)^{-\frac{k}{2}} \exp(\pi\ tr(Yh))$$

for any $Y \in symm_g(\mathbb{R})$ with $Y > 0$. In particular if $h^{\frac{1}{2}}$ denotes the positive definite square root of h, putting $U = h^{\frac{1}{2}} Y h^{\frac{1}{2}}$ we see that:

$$|a_h(f)| \leq C_3(\det h)^{\frac{k}{2}} (\det U)^{-\frac{k}{2}} \exp(\pi\ trU)$$

for any $U > 0$, as required.

For the second claim, we see that:

$$\begin{aligned}
\#\{[M] \in SL_g(\mathbb{Z})\backslash M^+_{g\times g}(\mathbb{Z}) | \det M = n\} &= \textstyle\sum_{n=n_1\ldots n_g} 1 n_2 n_3^2 \ldots n_g^{g-1} \\
&\leq n^{g-1} \textstyle\sum_{n=mn_g} \left(\prod_{i=1}^{g-1} \sum_{j=1}^m j^{-i}\right) \\
&\leq Cn^{g-1}(\log n)\#\{m \mid m|n,\ m>0\} \\
&\leq 2Cn^{g-1}(\log n)n^{\frac{1}{2}}
\end{aligned}$$

Lemma 3.11 *Let f, Q, ψ be as above with:*

- *ψ is primitive modulo r*

- $q|s(Q)r^2$
- *all prime divisors of* $qs(Q)$ *divide* r
- $\psi\chi(-1) = (-1)^k$
- $\det Q \neq 0$
- *either* $k \geq g+1$ *and* $(\psi\chi)^2 \neq 1$*, or* $k \geq g+3$

then:

$$D_{Q,\psi}(k-g) = C(f, \theta_{s(Q)Q^{-1},\psi^{-1}} \tilde{E}_{k-\frac{g}{2},\chi\psi\chi_Q,b})_b$$

where $C \neq 0$ *and* $b = s(Q)r^2$*.*

Proof:According to proposition 2.3 of [AK]:

$$D_{Q,\psi}(s) = c_1 c_2^s \gamma(s)^{-1} L(s) \langle f, \theta_{Q,\bar{\psi}} E(z, \frac{\bar{s}}{2} + \frac{g}{2} - \frac{k}{2}; k - \frac{g}{2}, \overline{\chi\psi\chi_Q}, b)\rangle_b$$

with $c_1 \neq 0 \neq c_2$, $\gamma(s) = \prod_1^g \Gamma(\frac{s+k-i}{2})$, $L(s) = L(s+\frac{g}{2}, \psi\chi_Q\chi) \prod_0^{\frac{g}{2}-1} L(2s+2i, \chi^2\psi^2)$ and Re s sufficiently large. By proposition 2.4 of [AK] the equation remains valid whenever all the terms remain defined. The result now follows from lemmas one and two and the facts that:

- $\langle A, B\rangle = \langle A|W_b, B|W_b\rangle$
- $\chi\psi\chi_Q(-1) = (-1)^{k+\frac{g}{2}}$
- $\tilde{E}^c_{k,\chi,b} = (-1)^{\frac{g}{2}} \chi(-1) \tilde{E}_{k,\chi^{-1},b}$ (as $\tau(\chi)^c = \chi(-1)\tau(\chi^{-1})$).

Lemma 3.12 *Let* f *be a cusp form in* $S_k(\Gamma_0(q),\chi)$ *with* $k \geq 2g+2$*, which is an eigenvalue of the Hecke operators and such that* $ef = f$*. Let* $Q \in symm_g^*(\mathbb{Z})$*,* $Q > 0$ *be such that* $a_Q(f) \neq 0$*. Let* $p^{t_2} = (p^\infty, s(Q)^g \det Q^{-1})$*. Choose* r *such that it is divisible by* p^2 *and by all prime divisors of* $s(Q)$ *and such that* $q|r^2 s(Q)$*. Write* $s(Q)r^2 = bp^{t_1}$ *with* b *prime to* p*. Let* $T \in \mathbb{T}_r$ *with* $Tf \neq 0$*.*

Then we can choose ψ a primitive character modulo r such that $\psi(-1) = (-1)^k\chi(-1)$ and $(\chi\psi)^2 \neq 1$. Write $\psi\chi\chi_Q = \phi\omega^k\alpha^{-1}$ with ϕ a character modulo bp, ω the Teichmuller character and $\alpha : (\mathbb{Z}/p^{t_1}\mathbb{Z})^\times \stackrel{\langle\rangle}{\to} (1+p\mathbb{Z})/(1+p^{t_1}\mathbb{Z}) \to \mathbb{Q}^{ac\times}$. Then:

$$(f, Te(\theta_{s(Q)Q^{-1},\psi^{-1}\omega^{2k}\alpha^{-2}}E_{\phi,b,t_2}(k-\frac{g}{2},\alpha)))_{bp^{t_1}} \neq 0$$

where $e(\theta E)$ is defined as $U_p^{-t}eU_p^t(\theta E)$ for $t \geq \max(t_1, t_2+1)$.

Proof: We know from Lemma 3 that the left hand side is a non-zero multiple of:

$$(f, Te(\theta_{s(Q)Q^{-1},\psi^{-1}}\tilde{E}_{k-\frac{g}{2},\chi\psi\chi_Q,bp}))_{bp^{t_1}}$$
$$(Tef, \theta_{s(Q)Q^{-1},\psi^{-1}}\tilde{E}_{k-\frac{g}{2},\chi\psi\chi_Q,bp})_{bp^{t_1}}$$

which is itself a non-zero multiple of $D_{Q,\psi}(k-g)$. However we know from [A1] that in this case $D_{Q,\psi}(s)$ has an Euler expansion of degree $2g+1$. The result thus follows from the following fact:

> Let $D(s) = \sum \frac{d_n}{n^s}$ be a Dirichlet series with $|d_n| \leq Cn^a$. Assume that for $\mathrm{Re}\, s$ sufficiently large $D(s) = \prod_p Q_p(p^{-s})^{-1}$ with Q_p a polynomial of bounded degree. Then $D(s)$ is non-zero and convergent for $\mathrm{Re}\, s > a+1$.

3.4 The General Strategy

This method of constructing Hida families is due to Wiles (see [Wi]).

Fix a positive integer N and a character $\chi : (\mathbb{Z}/Np\mathbb{Z})^\times \to \mathbb{Q}^{ac\times}$. Let $\mathcal{O}$ denote the integers in a finite extension of $\mathbb{Q}_p$ containing all the $\varphi(N)$ roots of unity (here φ is Euler's phi-function), and set $\Lambda = \mathcal{O}[[T]]$. Also let Υ denote the set of pairs (k, α) where k is an integer $\geq g+1$ and $\alpha : (1+p\mathbb{Z}_p) \to \mathbb{Q}^{ac\times}$ is a character of finite order, so that we may think of α as a character:

$$(\mathbb{Z}/p^r\mathbb{Z})^\times \stackrel{\langle\rangle}{\longrightarrow} (1+p\mathbb{Z})/(1+p^r\mathbb{Z}) \longrightarrow \mathbb{Q}^{ac\times}$$

We shall denote the smallest possible choice of r by $r(\alpha)$.

Now for I an infinite subset of Υ set:

$$\mathcal{M}_I(N,\chi,\Lambda) = \\ \{F \in \Lambda[[q]]_g \mid F|_{T=(\alpha(1+p)(1+p)^k-1)} \in M_k(\Gamma_0(Np^{r(\alpha)}), \chi\omega^{-k}\alpha, \mathcal{O}_\alpha) \text{ for } (k,\alpha) \in I\}$$

where $\mathcal{O}_\alpha$ denotes the integers of the field $F_{\mathcal{O}}(\alpha(1+p))$. Also set:

$$\mathcal{M}(N,\chi,\Lambda) = \\ \{F \in \Lambda[[q]]_g \mid F|_{T=(\alpha(1+p)(1+p)^k-1)} \in M_k(\Gamma_0(Np^{r(\alpha)}), \chi\omega^{-k}\alpha, \mathcal{O}_\alpha) \\ \text{for all but finitely many } (k,\alpha) \in \Upsilon\}$$

i.e.

$$\mathcal{M}(N,\chi) = \bigcup_{I \text{ of finite complement in } \Upsilon} \mathcal{M}_I(N,\chi)$$

Note that:

- $\lambda \in \Lambda$, $F \in \Lambda[[q]]_g$ and $\lambda F \in \mathcal{M}$ implies that $F \in \mathcal{M}$
- if $F \in \mathcal{M}$ then there is a $\lambda \in \Lambda$ such that $\lambda F \in \mathcal{M}_\Upsilon$.
- $\mathcal{M}_I/(1+T-\alpha(1+p)(1+p)^k)\mathcal{M} \hookrightarrow M_k(\Gamma_0(Np^{r(\alpha)}), \chi\omega^{-k}\alpha, \mathcal{O}_\alpha)$ if $\Upsilon - I$ is finite and $(k,\alpha) \in I$.

Before giving examples of elements of $\mathcal{M}(N,\chi)$, or as we shall write Λ-adic forms, we must introduce Hida's idempotent in this context. First we define an action of U_p on $\mathcal{M}$ compatible with specialisation by setting:

$$U_p(\sum_h A_h q^h) = \sum_h A_{ph} q^h$$

Then we have:

Lemma 3.13 *Let $I \subset \Upsilon$ be infinite. Then*

1. *there is a unique operator e on $\mathcal{M}_I$ compatible with Hida's idempotent under specialisation*
2. $eF = \lim_{r\to\infty} U_p^{r!} F$

3. *U_p is invertible on $e\mathcal{M}_I$.*

Proof: We have the specialisation map:

$$\phi : \mathcal{M}_I \hookrightarrow \bigoplus_{(k,\alpha)\in I} M_k(\Gamma_0(Np^{r(\alpha)}), \chi\omega^{-k}\alpha, \mathcal{O}_\alpha)$$

The image is certainly preserved by U_p. First we shall show that the image is also closed. For this let $F_j \in \mathcal{M}_I$, $\phi(F_j) \to \prod_I g_i$ as $j \to \infty$. Then if $\theta : \Lambda \to \prod_I \mathcal{O}_\alpha$ is the specialisation map we have that $\theta(a_h(F_j)) \to \prod a_h(g_i)$ as $j \to \infty$. However Λ is compact and so $\theta\Lambda$ is closed in $\prod_I \mathcal{O}_\alpha$. Thus we can find $b_h \in \Lambda$ such that $\theta b_h = \prod a_h(g_i)$. Then it is easily seen that $\sum b_h q^h \in \mathcal{M}_I$ and that $\phi(\sum b_h q^h) = \prod_I g_i$.

Thus the operator $e = \lim_{r\to\infty} U_p^{r!}$ preserves $\phi\mathcal{M}_I$ and this proves the first two parts. It is easily seen that U_p is injective on $e\mathcal{M}_I$. Finally if $F \in e\mathcal{M}_I$ then for $G = \lim_{r\to\infty} U_p^{r!-1} F \in \mathcal{M}_I$, which exists as $\lim_{r\to\infty} U_p^{r!-1}\phi F$ exists, $U_p G = eF = F$. This proves the last part also.

Corollary 3.1 *The Hida operator defined in the lemma satisfies the following properties:*

1. *$e^2 = e$*
2. *if t commutes with U_p then it also commutes with e*
3. *if $I \supset J$ the action of e is compatible with $\mathcal{M}_J \supset \mathcal{M}_I$*
4. *e extends to an operator on $\mathcal{M}$ with the same properties.*

These are all easy.

We are now in a position to construct some examples of Λ-adic forms:

Example 3.1 *Let*

- *b be prime to p*
- *$\phi : (\mathbb{Z}/bp\mathbb{Z})^\times \to \mathbb{Q}^{ac\times}$ be a character with:*
  - $\phi(-1) = 1$
  - *$\phi = \phi_{(p)}\phi^{(p)}$ its decomposition into "at p" and " away from p" parts*

  - $\phi^{(p)\,2} \neq 1$

- *$Q \in symm_g^*(\mathbb{Z})$, $Q > 0$*
- $(\det Q, p^\infty) = p^{t_2}$
- *N and r be such that $p|r$, $s(Q)r^2|N$ and $N = bp^{t_3}$*

Then there is a Λ-adic form $G_{N,Q,\psi,\phi} \in \mathcal{M}_\Upsilon(b, \psi\phi\chi_Q)$ such that:

$$G_{N,Q,\psi,\phi}|_{T=(\alpha(1+p)(1+p)^k-1)} = e(\theta_{Q,\psi}E_{\phi,b,t_2}(k-\frac{g}{2},\alpha))$$

in the notation of section two.

Proof:For $t \geq \max(t_3, t_2+1)$ set $I_t = \{(k,\alpha) \in \Upsilon \mid r(\epsilon) \leq t\}$. Then we can define $G^{(t)} \in \mathcal{M}_{I_t}$ to be $U_p^t(\theta_{Q,\psi} \sum_{\substack{(\det h,\, p^\infty) = p^{t_2} \\ h \in symm_g^*(\mathbb{Z}) \\ h > 0}} B_h(T)q^h)$ where:

$$\begin{aligned} B_h(T) \;=\; & (\det h)^{-(g+\frac{1}{2})} f^g_{\widetilde{\phi^{(p)}\xi_h^{(p)}}} \phi_{(p)}\xi(f_{\widetilde{\phi^{(p)}\xi_h^{(p)}}})\tau(\widetilde{\phi^{(p)}\xi_h^{(p)}}) \\ & (1+T)^{\log_p(f^{-1}_{\widetilde{\phi^{(p)}\xi_h^{(p)}}} p^{-t_2}\det h)} G_{\phi^{-1}\xi_h\omega^{-\frac{g}{2}}}(\tfrac{(1+T)}{(1+p)^g}-1) \\ & \prod_{\substack{l \in P(h) \\ l \nmid bp}} M_{l,h}(\phi(l)\langle l\rangle^{\frac{g}{2}}(1+T)^{-\log_p l}) \\ & \prod_{\substack{l \nmid f_{\widetilde{\phi^{(p)}\xi_h^{(p)}}} \\ l|b}} (1 - l^{\frac{g}{2}}\langle l\rangle^{\frac{g}{2}}\widetilde{\phi\xi_h}(l)(1+T)^{-\log_p l}) \end{aligned}$$

and where:

- $x = \omega(x)(1+p)^{\log_p x}$ for $x \in \mathbb{Z}_p^\times$
- $G_\chi(\alpha(1+p)(1+p)^n - 1) = L_{(p)}(1-n, \chi\alpha\omega^{-n})$ for α as above, $n > 0$ an integer and $\chi(-1) = 1$.

That G^t is in fact in $\mathcal{M}_{I_t}$ follows from lemma 3.8 of section 3.3. Then $U_p^{-t}eG^{(t)} \in \mathcal{M}_{I_t}$ and $U_p^{-t}eG^{(t)}|_{T=(\alpha(1+p)(1+p)^k-1)} = e(\theta_{Q,\psi}E_{\phi,b,t_2}(k-\frac{g}{2},\alpha))$ again by lemma 3.8 of section

3.3. In particular if $t' > t$ then:

$$\begin{array}{ccc} U_p^{-t}eG^{(t)} & \in & \mathcal{M}_{I_t} \\ \| & & \cup \\ U_p^{-t'}eG^{(t')} & \in & \mathcal{M}_{I_{t'}} \end{array}$$

and so $G = U_p^{-t}eG^{(t)} \in \bigcap_{t' \geq t} \mathcal{M}_{I_{t'}} = \mathcal{M}_\Upsilon$ and has the desired specialisations.

Next we record two technical results about Λ-adic forms:

Lemma 3.14 *1. $e\mathcal{M}(Np,\chi) = e\mathcal{M}(N,\chi)$*

2. If $\mathcal{O}'$ denotes the integers of a finite extension of $\mathcal{O}$ and Λ' denotes $\mathcal{O}'[[T]]$ then:

$$\mathcal{M}(N,\chi,\Lambda) \otimes_\Lambda \Lambda' = \mathcal{M}(N,\chi,\Lambda')$$

These are both straightforward. We can use part two to define $\mathcal{M}(N,\chi,\mathcal{R})$ for any Λ-algebra $\mathcal{R}$ to be $\mathcal{M}(N,\chi,\Lambda) \otimes_\Lambda \mathcal{R}$.

We now prove one of the main theorems on Λ-adic forms:

Theorem 3.1 *$\mathcal{M}^\circ(N,\chi)$ is a finite free Λ-module.*

*Proof:*We may and shall assume that N is prime to p (by the last lemma). Note that $\mathcal{M}^\circ(N,\chi)$ is certainly torsion free over Λ.

Step 1 There is a constant C such that $\dim eM_k(\Gamma_1(Np)) \leq C$ for all k.

Let:

$$\Gamma^{(g)} = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \,|\, C \equiv 0 \bmod Np,\, B \equiv 0 \bmod N,$$
$$A \equiv D \equiv \bmod N,\, \det A \equiv 1 \bmod p\}$$

Then it will do to show that $\dim eM_k(\Gamma^{(g)})$ is bounded independently of k. However we know from proposition 2.1 that for each g' there is a constant $C_{g'}$ such that $\dim eS_k(\Gamma^{(g')}) < C_{g'}$. The result now follows by induction using lemma 3.5 of section 3.2.

Step 2 If C is as in step 1 then $\mathcal{M}^\circ(N,\chi)$ does not contain more than C Λ-linearly independent elements.

Assume that $F_0, ..., F_C$ were linearly independent elements of $\mathcal{M}^\circ(N,\chi)$. Then we can find $h_0, ..., h_C$ in $symm_g^*(\mathbb{Z})$ such that the matrix $(a_{h_i}(F_j))$ is non-singular. Then for k sufficiently large:

- $F_j|_{T=((1+p)^k-1)} \in eM_k(\Gamma_1(Np), \mathcal{O})$
- $\det(a_{h_i}(F_j|_{T=((1+p)^k-1)})) \neq 0$

For such k the $F_j|_{T=((1+p)^k-1)}$ are $(C+1)$ linearly independent elements of $eM_k(\Gamma_1(Np), \mathcal{O})$, which is a contradiction.

Step 3 $\mathcal{M}^\circ(N,\chi)$ is a compact finitely generated Λ-module.

We can choose $F_1, ..., F_r$ a maximal set of linearly independent elements of $\mathcal{M}^\circ(N,\chi)$, and we can choose $h_1, ..., h_r$ in $symm_g^*(\mathbb{Z})$ with $\lambda = \det(a_{h_i}(F_j)) \neq 0$. Then we claim that $\lambda\mathcal{M}^\circ(N,\chi) \subset \langle F_1, ..., F_r\rangle_\Lambda$. For if $F = \sum b_j F_j$ with $b_j \in F_\Lambda$ then we have the non-singular set of equations:

$$a_{h_i}(F) = \sum b_j a_{h_i}(F_j)$$

for "unknowns" b_j. As the $a_{h_i}(F)$ and the $a_{h_i}(F_j)$ are in Λ, $b_j \in \lambda^{-1}\Lambda$.

Step 4 In particular $\mathcal{M}^\circ(N,\chi) = \mathcal{M}_I^\circ(N,\chi)$ for some subset I of Υ with finite complement. Thus for almost all pairs $(k,\alpha) \in \Upsilon$ we have that:

$$\mathcal{M}^\circ(N,\chi)/(T - \alpha(1+p)(1+p)^k)\mathcal{M}^\circ(N,\chi) \hookrightarrow M_k(\Gamma_1(Np^{r(\alpha)}), \mathcal{O}_\alpha)$$

Thus the theorem follows from the following lemma:

Lemma 3.15 *Let M be a compact Λ-module and $\wp_i$ an infinite collection of height one primes such that $M/\wp_i M$ is a finite torsion free $\mathbb{Z}_p$-module then M is a finite free Λ-module.*

Proof:Let $\mathcal{O}_i = \Lambda/\wp_i$. Then $M/\wp_i M$ is a finite free $\mathcal{O}_i$-module. Let $M/\wp_i M \cong \mathcal{O}_i^{r_i}$. Let $r = r_{i_0} = \min r_i$. Then by Nakayama's Lemma $\Lambda^r \overset{\beta}{\twoheadrightarrow} M$, and $\mathcal{O}_i^r \twoheadrightarrow (M/\wp_i M)$ for all i. Thus $r = r_i$ for all i and $\mathcal{O}_i^r \overset{\sim}{\rightarrow} (M/\wp_i M)$. Thus if $\vec{\lambda} \in \Lambda^r$ is such that $\beta(\vec{\lambda}) = 0$ then $\vec{\lambda} \in \wp_i^r$ for all i and so $\vec{\lambda} = \vec{0}$. That is β is an isomorphism.

Corollary 3.2 *There exists $\lambda \in \Lambda$ such that:*

$$\mathcal{M}^\circ(N,\chi) \supset \mathcal{M}^\circ_\Upsilon(N,\chi) \supset \lambda\mathcal{M}^\circ(N,\chi)$$

Proof:If $F \in \mathcal{M}(N,\chi)$ then we know that we can find $\lambda \in \Lambda$ so that $\lambda F \in \mathcal{M}_\Upsilon(N,\chi)$, this is enough as $\mathcal{M}^\circ(N,\chi)$ is finitely generated over Λ.

We shall now define an action of the Hecke ring $\mathbb{T}_{Np}$ on $\mathcal{M}(n,\chi)$ (or $\mathcal{M}_I(N,\chi)$) by setting:

$$\begin{aligned} F|T(n) &= n^{-\frac{g(g+1)}{2}}\chi(n^g)(1+T)^{g\log_p n}\textstyle\sum_{d_1|d_2|\ldots|d_g|n} d_1^g \ldots d_g \\ &\quad \textstyle\sum_D \chi(\det D)^{-1}(1+T)^{-\log_p \det D} a_{n^{-1}Dh^tD} \end{aligned}$$

where the second sum is taken over a set of representatives for:

$$GL_g(\mathbb{Z})\backslash GL_g(\mathbb{Z})\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_g \end{pmatrix} GL_g(\mathbb{Z})$$

This is compatible with specialisation. We can similarly define an action of $\mathbb{T}^*_{Np}$ on $\mathcal{M}(N,\chi)\otimes_\mathcal{O}\mathbb{Q}_p^{ac}$ which is compatible with specialisation, by using the formula in section 3.2.

Lemma 3.16 $\mathbb{T}_{Np}$ *acts semi-simply on $\mathcal{M}(N,\chi,F_\Lambda)$.*

Proof:It will do to show that each $T(n) \in \mathbb{T}_{Np}$ acts semi-simply, because $\mathbb{T}_{Np}$ is commutative. Let $T(n)$ have characteristic polynomial $P(X)$, and let $Q(X) \in \Lambda[X]$ be the product of the distinct irreducible factors $P(X)$. It will do to show that $Q(T(n))$ acts as 0 on $\mathcal{M}$ or equivalently on $\mathcal{M}/(1+T-\alpha(1+p)(1+p)^k)\mathcal{M}$ for infinitely many (k,α). However for almost all $(k,\alpha) \in \Upsilon$, $\mathcal{M}/(1+T-\alpha(1+p)(1+p)^k)\mathcal{M} \hookrightarrow \mathcal{M}$. For such a pair (k,α) let $\bar{P}(X)$ and $\bar{Q}(X) \in \mathcal{O}_\alpha[X]$ denote the reduction of P and Q. Let q be defined from $\bar{P}$ the same way Q was defined from P. Then $q(X)|\bar{Q}(X)$. Moreover $q(T(n)) = 0$ as $\bar{P}(X)$ is the characteristic polynomial of $T(n)$ on a subspace of $M_k(Np^?,\chi\alpha\omega^{-k})$ on which $T(n)$ is known to act semi-simply. Thus $\bar{Q}(T(n)) = 0$ and we are done.

Before proving our second main theorem about lifting eigenforms we state and prove an algebraic lemma which we shall require:

Lemma 3.17 *Let R denote the integers of a finite extension of $\mathbb{Q}_p$. Let M be a finite torsion free R module. Let T be a commutative ring of operators on M. Let $M \otimes F_R = \bigoplus_I V_i$ where T acts on V_i by a character $\lambda_i : T \to R$. Let $a : M \to R$ be a linear form, $x \in M$ and $\lambda : T \to R$ be such that:*

- $(\lambda(t) - t)x \in p^A M$
- $\mathrm{val}_p\, a(x) \leq B$
- $\mathrm{rk}\, M \leq C$

Then for some i:

- $\mathrm{val}_p\, (\lambda(t) - \lambda_i(t)) \geq \frac{A-B}{C}$ *for all $t \in T$*
- $a \not\equiv 0$ *on V_i*

Proof:Let $J = \{i \in I \,|\, a|_{V_i} \not\equiv 0\}$. Let π denote the projection of M to $\bigoplus_J V_i$. Then

- $(\lambda(t) - t)\pi(x) \in p^A \pi M$
- $\mathrm{val}_p\, a(\pi(x)) \leq B$
- $\mathrm{rk}\, \pi M \leq C$

Thus we may assume that $a \not\equiv 0$ on all V_i. Let $M' = \bigoplus M \cap V_i$. Let $\mathcal{I} \lhd T$ and $\mathcal{I}' \lhd R$ be the annihilators of x in $M/p^A M$. Then $T/\mathcal{I} \cong R/\mathcal{I}'$ via λ. Then we have that:

$$\mathrm{Fitt}_T(M') = 0$$

and so $\mathrm{Fitt}_{T/\mathcal{I}}(M'/\mathcal{I}M') = 0$

and so $\mathrm{Fitt}_{R/\mathcal{I}'}(\bigoplus (M \cap V_i)/\lambda_i(\mathcal{I})(M \cap V_i)) = 0$

and so $\prod \lambda_i(\mathcal{I})^{\dim V_i} \subset \mathcal{I}' \subset p^{A-B}$

and so for some i, $\mathrm{val}_p\, \lambda_i(\mathcal{I}) \geq \frac{A-B}{C}$

as desired. (Here Fitt denotes the Fitting ideal. For some of its properties see (for example) the appendix of [MW1].)

Theorem 3.2 *Let $f \in eS_k(Np^s, \chi\alpha\omega^{-k}, \mathcal{O})$ be an eigenform of the Hecke algebra $\mathbb{T}_{Np}$, say $f|T(n) = \lambda(n)f$. Let $Q \in symm_g^*(\mathbb{Z})$, $Q > 0$ be chosen such that $a_Q(f) \neq 0$. Let M be chosen prime to p such that $N|M$ and such that if l ($\neq p$) is any prime dividing $s(Q)$, say $l^\beta||s(Q)$, then $l^{\beta+2i}||M$ with i some positive integer. Then there exists a finite extension of F_Λ, with integers $\mathcal{R}$ say, and a prime $\wp$ of $\mathcal{R}$ above $(1 + T - \alpha(1 + p)(1 + p)^k)$ and $F \in \mathcal{M}(M, \chi, \mathcal{R})$ such that:*

- $F|T(n) = \boldsymbol{\lambda}(n)F$ *with* $\boldsymbol{\lambda}(n) \in \mathcal{R}$
- $\boldsymbol{\lambda}(n) \equiv \lambda(n) \bmod \wp$

for all n such that $T(n) \in \mathbb{T}_{Mp}$.

Proof:We shall first show the result for sufficiently large k and then deduce it for all k. Choose k_0 such that:

- $k_0 \geq 2g + 2$
- $\mathcal{M} = \mathcal{M}_I$ for some I containing (k, α) for all $k \geq k_0$ and for all α.

Step 1 If $k \geq k_0$ then we can find a non-zero $f' = \mathcal{M}^\circ(M, \chi)/(1 + T - \alpha(1 + p)(1 + p)^k)\mathcal{M}^\circ(M, \chi)$ with $f'|T(n) = \lambda(n)f'$ for all $T(n) \in \mathbb{T}_{Mp}$.

We may assume f is an eigenvector for $\mathbb{T}^*_{Mp}$. Then we can find $T \in \mathbb{T}_{Mp}$ with $f|T = \mu f \neq 0$ and such that $TM_k(Mp^t, \chi\alpha\omega^{-k})$ is an eigenspace for $\mathbb{T}_{Mp}$. Then choose r and t such that $(Np^s)|s(Q)r^2 = Mp^{t_1}$ and $p^2|r$ and choose ψ a primitive character mod$r$ with $\psi(-1) = \chi(-1)$ and $(\chi^{(p)}\psi^{(p)})^2 \neq 1$. Set $\phi = \psi\alpha^2\omega^{-2k}\chi_Q$. Then we see from example 3.1 and lemma 3.12 and section 3.3 that:

$$0 \neq (f, TeG_{Mp^{t_1}, s(Q)Q^{-1}, \psi^{-1}\omega^{2k}\alpha^{-2}, \phi}|_{T=(\alpha(1+p)(1+p)^k-1)})_{Mp^{t_1}}$$

Thus $f' = TeG_{Mp^{t_1}, s(Q)Q^{-1}, \psi^{-1}\omega^{2k}\alpha^{-2}, \phi}|_{T=(\alpha(1+p)(1+p)^k-1)}$ is non-zero and will do.

Step 2 If $k \geq k_0$ then the theorem is true.

Choose $\mathcal{R}$ the integers in a finite extension of F_Λ such that $\mathbb{T}_{Mp}$ is diagonalisable on $\mathcal{M}^\circ(M, \chi, F_{\mathcal{R}})$. Let $e_1, ..., e_r \in \mathcal{M}^\circ(M, \chi, \mathcal{R})$ be eigenvectors of $\mathbb{T}_{Mp}$ which span

$\mathcal{M}^\circ(M,\chi,F_\mathcal{R})$. Let $\boldsymbol{\lambda}_i : \mathbb{T}_{Mp} \to \mathcal{R}$ be the eigenvalue corresponding to e_i. Let $U = \bigoplus \mathcal{R}e_i$ and let $\mathcal{T} \subset End_\mathcal{R}(\mathcal{M}(M,\chi,\mathcal{R}))$ be the $\mathcal{R}$-subalgebra generated by $\mathbb{T}_{Mp}$. Then U is a faithful $\mathcal{T}$ module. Let $\mathcal{I} \lhd \mathcal{T}$ be the annihilator of f'. Then arguing as in the proof of lemma 3.17 we see that there is a prime $\wp$ above $(1+T-\alpha(1+p)(1+p)^k)$ such that $\mathrm{Fitt}_{\mathcal{R}/\wp}(U/\mathcal{I}U) = 0$. Then $\prod \boldsymbol{\lambda}_i(\mathcal{I}) \subset \wp$ and so for some i $\boldsymbol{\lambda}_i(\mathcal{I}) \subset \wp$, i.e. for all $T(n) \in \mathbb{T}_{Mp}$ $\boldsymbol{\lambda}_i(T(n)) \equiv \lambda(T(n)) \bmod \wp$.

Step 3 The theorem is true for all k.

We have seen (proposition 2.1) that there is a constant C such that $\dim M_l^\circ(Np^t, \chi\alpha\omega^{-k}, F_\mathcal{O}) < C$ for all l. Let $\mathrm{val}_p(a_Q(f)) = B$. Let θ be the theta series defined in lemma 2.3. Then $\theta^{p^r} f \in M_{k+(p-1)p^r}(Np^t, \chi\alpha\omega^{-k}, \mathcal{O})$. Moreover $\theta^{p^r} f \equiv f \bmod p^{r+1}$, and so $e(\theta^{p^r} f) \equiv f \bmod p^{r+1}$ and:

$$\begin{aligned} (\theta^{p^r} f)|_{k+(p-1)p^r} T(n) &\equiv f|_k T(n) \\ &\equiv \mu(n) f \\ &\equiv \mu(n)(\theta^{p^r} f) \bmod p^{r+1} \end{aligned}$$

as $a^{k+(p-1)p^r} \equiv a^k \bmod p^{r+1}$ for all a. Thus also $((\theta^{p^r} f)|e)|T(n) \equiv \mu(n)((\theta^{p^r} f)|e) \bmod p^{r+1}$.

Thus by the lemma proved just before this theorem we see that for $r > B$ we can find $f_r \in M_{k+(p-1)p^r}(Np^t, \chi\alpha\omega^{-k}, \mathcal{O})$ such that:

- $f_r|T(n) = \lambda_r(n) f_r$
- $\lambda_r(n) \equiv \lambda(n) \bmod p^{\left[\frac{r+1-B}{C}\right]}$
- $a_Q(f_r) \neq 0$

Now for r sufficiently large this implies that we can find $F_r \in \mathcal{M}^\circ(M,\chi,\mathcal{R})$ (where $\mathcal{R}$ denotes the integers in an extension of Λ such that $\mathbb{T}_{Mp}$ is diagonalisable on $\mathcal{M}^\circ(M,\chi,F_\mathcal{R})$) such that:

- $F_r|T(n) \equiv \boldsymbol{\mu}_r(n) F_r$
- $\boldsymbol{\mu}_r(n) \equiv \lambda(n) \bmod (\wp_r, p^{\left[\frac{r+1-B}{C}\right]})$
- $\wp_r$ is a prime of $\mathcal{R}$ above $(1+T-\alpha(1+p)(1+p)^{k+(p-1)p^r})$.

Now there are only finitely many choices for $\boldsymbol{\mu}_r$ and so there is an infinite set S of positive integers such that for $r \in S$ $\boldsymbol{\mu}_r = \boldsymbol{\mu}$. Then:

$$\boldsymbol{\mu}(n) \equiv \lambda(n) \mod \bigcap_S (\wp_r,\, p^{\left[\frac{r+1-B}{C}\right]})$$

We claim that:

$$\bigcap_S (\wp_r,\, p^{\left[\frac{r+1-B}{C}\right]}) \cap \Lambda \subset (1 + T - \alpha(1+p)(1+p)^k)$$

from which the theorem would follow. However this inclusion follows from the two facts:

- $(\wp_r,\, p^{\left[\frac{r+1-B}{C}\right]}) \cap \Lambda = (p^{\left[\frac{r+1-B}{C}\right]},\, (1 + T - \alpha(1+p)(1+p)^{k+(p-1)p^r}))$
- $(p^{\left[\frac{r+1-B}{C}\right]},\, (1 + T - \alpha(1+p)(1+p)^{k+(p-1)p^r})) \subset (p^{\left[\frac{r+1-B}{C}\right]},\, (1 + T - \alpha(1+p)(1+p)^k))$

The first of these is easy and the second not much harder. (In general if $R_1 \subset R_2$ are two rings with prime ideals $\wp_1$ and $\wp_2$ where $\wp_1 = \wp_2^c$, and if $a \in R_1$ then $(a, \wp_2)^c = (a, \wp_1)$. To prove this one reduces at once to the case R_1, R_2 integral domains and $\wp_1 = \wp_2 = 0$, in which case it is obvious.)

3.5 Conjectural Applications

Throughout the rest of this paper we shall restrict to the case $g = 2$, i.e. to GSp_4. This is just for definiteness and because $g = 2$ was the case of interest for us. For other even g exactly similar results should hold with $\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \to GSp_4$ replaced with $\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \to GL_N$ for suitable N, and with a suitable characteristic polynomial. We shall also make free use of the following conjecture:

Conjecture 3.1 *Let $f \in M_k(N, \chi)$ be an eigenform of the Hecke algebra $\mathbb{T}_N$, say $f|T(n) = \lambda(n) f$. Assume $k \geq 3$. Then there is a continuous semi-simple representation:*

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GSp_4(\mathbb{Q}_p^{ac})$$

unramified outside pN and such that the characteristic polynomial of Frob_l *for $l \not| Np$ is given by:*

$$X^4 - \lambda(l)X^3 + (\lambda(l)^2 - \lambda(l^2) - l^{2(k-2)}\chi(l^2))X^2 - l^{2k-3}\chi(l^2)\lambda(l)X + l^{4k-6}\chi(l^4)$$

Our aim in this section is to show that these conjectures would imply similar results for ordinary Λ-adic forms and for ordinary eigenforms of weight two. We first prove a general result on lifting group representations:

Proposition 3.1 *Let R be an integral domain, $\{\wp_i\}_{i\in I}$ be an infinite set of prime ideals such that the intersection of any infinite subset is zero. Let $\mathcal{O}_i$ denote a finite extension of $R/\wp_i$ containing its integral closure, and assume that $F_{\mathcal{O}_i}$ is of characteristic zero. Let Γ be a group and $\rho_i : \Gamma \to GL_N(\mathcal{O}_i)$ a semi-simple representation. For each $x \in \Gamma$ suppose there exists $T_x \in R$ with $T_x \equiv tr(\rho_i x) \bmod \wp_i$.*

Then there is an infinite subset $J \subset I$, S an integral domain finite over $R[f^{-1}]$ for some $f \in R$, a semi-simple representation $\rho : \Gamma \to GL_N(S)$ and for each $i \in J$ a prime $\mathcal{P}_i$ over $\wp_i$, a field $L_i \supset F_{\mathcal{O}_i}$, a map $S/\mathcal{P}_i \hookrightarrow L_i$ such that:

$$\begin{array}{ccc} S/\mathcal{P}_i & \hookrightarrow & L_i \\ \cup & & \cup \\ R/\wp_i & \subset & \mathcal{O}_i \end{array}$$

commutes and such that $\rho \bmod \mathcal{P}_i$ is conjugate to ρ_i as a representation into $GL_N(L_i)$.

We break up the proof into a series of lemmas.

Lemma 3.18 *Let K be a field of characteristic zero, let $A \subset M_N(K)$ be a split semisimple sub-algebra and let $H \subset GL_N(K)$ be its normaliser. Let $\{a_1, ..., a_r\}$ and $\{b_1, ..., b_r\}$ be subsets of A each of which spans A and assume that for $s = 1, ..., 4$ we have that $tr(\prod_{j=1}^{s} a_{i_j}) = tr(\prod_{j=1}^{s} b_{i_j})$ for all s-tuples $(i_1, ..., i_s) \in \{1, ..., r\}^s$. Then there is an element $n \in H$ with $n a_i n^{-1} = b_i$ for all i.*

<u>*Proof:*</u>A is of the form $\bigoplus_\alpha M_{c_\alpha}(K)^{a_\alpha}$ and without loss of generality we may assume $(m_{\alpha\beta})_{\alpha;\,\beta=1,\dots,a_\alpha}$ embeds as $\bigoplus m_{\alpha\beta}^{\oplus b_\alpha}$, where $\sum_\alpha a_\alpha b_\alpha c_\alpha = N$. Also assume that the notation is such that for $\alpha \neq \alpha'$ $(b_\alpha, c_\alpha) \neq (b_{\alpha'}, c_{\alpha'})$.Let $e_{\alpha\beta}$ denote the idempotents corresponding to our decomposition of A into simple algebras. Let $e_{\alpha\beta} = \sum \lambda_{\alpha\beta i} a_i$ and set $f_{\alpha\beta} = \sum \lambda_{\alpha\beta i} b_i$ Then we see that:

- $$\begin{aligned} e_{\alpha\beta}^2 = e_{\alpha\beta} \quad &\Rightarrow \quad tr(e_{\alpha\beta}^2 - e_{\alpha\beta})a_j = 0 \;\; \forall j \\ &\Rightarrow \quad tr(f_{\alpha\beta}^2 - f_{\alpha\beta})b_j = 0 \;\; \forall j \\ &\Rightarrow \quad f_{\alpha\beta}^2 = f_{\alpha\beta} \end{aligned}$$

- $$\begin{aligned} e_{\alpha\beta}\,\text{central} \quad &\Rightarrow \quad tr(e_{\alpha\beta}a_ia_j) = tr(a_ie_{\alpha\beta}a_j) \;\; \forall i,j \\ &\Rightarrow \quad tr(f_{\alpha\beta}b_ib_j) = tr(b_if_{\alpha\beta}b_j) \;\; \forall i,j \\ &\Rightarrow \quad f_{\alpha\beta}\;\text{central} \end{aligned}$$

- $$\begin{aligned} c_\alpha = \text{rk}\,(tr(e_{\alpha\beta}a_ia_j))_{i,j} \quad &\Rightarrow \quad c_\alpha\text{rk}\,(tr(f_{\alpha\beta}b_ib_j))_{i,j} \\ &\Rightarrow \quad c_\alpha = \dim f_{\alpha\beta}A \end{aligned}$$

- $$tre_{\alpha\beta} = c_\alpha b_\alpha \quad \Rightarrow \quad trf_{\alpha\beta} = c_\alpha b_\alpha$$

From this we see that for each α $\{f_{\alpha\beta}\}$ is a permutation of $\{e_{\alpha\beta}\}$. We can conjugate the b_i's by an element of H such that for each α, β $f_{\alpha\beta} = e_{\alpha\beta}$. Thus we may without loss of generality assume that this equation holds. Now fix α, β and set $a_i' = e_{\alpha\beta}a_i$ and $b_i' = e_{\alpha\beta}b_i$. Then we can consider a_i' and $b_i' \in M_{c_\alpha}(K)$ such that for $s = 1$, 2 or 3 and $(i_1, ..., i_s) \in \{1, ..., r\}^s$ $tr(\prod_{j=1}^s a_{i_j}') = tr(\prod_{j=1}^s b_{i_j}')$. Moreover we need only show that we can find $n' \in GL_{c_\alpha}(K)$ with $n'a_i'n'^{-1} = b_i'$ for all i.

Let $\epsilon_{kl} = \sum \lambda_{kli}a_i'$ and set $\delta_{kl} = \sum \lambda_{kli}b_i'$. Then as above we can show that δ_{kk} form a set of commuting idempotents each of trace one and such that $\sum \delta_{kk} = 1$. Thus by conjugation by some element of GL_{c_α} we may assume that without loss of generality $\epsilon_{kk} = \delta_{kk}$. Then we also see that $\epsilon_{kk}\delta_{kl} = \delta_{kl}$ and $\delta_{kl}\epsilon_{ll} = \delta_{kl}$ so that $\delta_{kl} = \mu_{kl}\epsilon_{kl}$ for some $\mu_{kl} \in K$. Then $\mu_{kl}\mu_{lm} = \mu_{km}$ and $\mu_{kk} = 1$ so there exist $\nu_k \in K^\times$ such that $\mu_{kl} = \nu_k\nu_l^{-1}$. Thus we can find a diagonal matrix n' in GL_{c_α} such that after conjugation by n' we may assume $\epsilon_{kl} = \delta_{kl}$ for all k and l. Then it is easily seen that $a_i' = b_i'$ for all i.

Lemma 3.19 *Let the assumptions be as in the theorem, but assume further that Γ is generated by $x_1, ..., x_r$ and that for each i if A_i is the subalgebra of $GL_N(K_i)$ generated by $Im\,\rho_i$ then A_i is spanned by $\{\rho_i(x_j)\}$. Then the conclusions of the proposition hold, except that we do not yet claim that ρ is semi-simple.*

Proof: Without loss of generality we may assume that Γ is the free group on $x_1, ..., x_r$. A_i is semi-simple as it has a faithful semi-simple module. We may assume that A_i is split over $F_{\mathcal{O}_i}$. Then after conjugation and discarding some i we may assume that for all i $A_i = A \otimes F_{\mathcal{O}_i}$ where $A \cong \bigoplus M_{c_\alpha}(\mathbb{Z})^{a_\alpha}$ and $a \hookrightarrow M_N(\mathbb{Z})$ by:

$$(x_{\alpha\beta}) \longmapsto \oplus x_{\alpha\beta}^{b_\alpha}$$

where the pairs (b_α, c_α) are distinct for distinct α and where $\sum a_\alpha b_\alpha c_\alpha = N$.

Let $\Omega = \{(a_i) \in A^r \,|\, \text{the } a_i \text{ span } A\}$, so that Ω is a subvariety of $r \sum a_\alpha c_\alpha^2$ dimensional affine space defined over $\mathbb{Q}$. Then by the last lemma we have a map:

$$\Omega \longrightarrow \mathbb{A}^M$$

given by $(a_i) \mapsto (tr \prod_{j=1}^{s} a_{i_j})$ taken over $s = 1, ..., 4$ and all $(i_1, ..., i_s) \in \{1, ..., r\}^s$. Consider the point $T \in \mathbb{A}^M(R)$ defined by:

$$T = (T_{\prod x_{i_j}})$$

Then $T \bmod \wp_i$ is in the image of $\Omega(F_{\mathcal{O}_i})$ for all i. Thus T is in the image of $\Omega(F_R^{ac})$, because the image of Ω is defined by some polynomial equalities and inequalities. Thus we can find $X_1, ..., X_r$ in $A \otimes F_R^{ac}$ which span $A \otimes F_R^{ac}$ and which satisfy $tr \prod_1^s X_{i_j} = T_{\prod x_{i_j}}$ for all $(i_1, ..., i_s)$ as above. In fact all the X_i lie in some S/R as described in the proposition. Discard the finite number of $\wp_i$ which contain f and assume (as we may) that F_R/F_S is Galois. Choose $\mathcal{P}_i$ over $\wp_i$ and choose L_i such that $S/\mathcal{P}_i$ and K_i both embed in L_i over $R/\wp_i$. Then $(X_j \bmod \mathcal{P}_i) \in \Omega(S/\mathcal{P}_i) \subset \Omega(L_i)$ and $\phi(X_j \bmod \mathcal{P}_i) = (T_{\prod x_{i_j}} \bmod \wp_i)$. Thus by the last lemma there is n in the normaliser of A in $GL_N(L_i)$ with $n(X_j \bmod \mathcal{P}_i)n^{-1} = \rho_i(x_j)$ for all j and so we are done.

Proof of proposition 3.1

Let A be defined as in the proof of the last lemma. Pick a distinguished index, say 0. Let $x_1, ..., x_r \in \Gamma$ be such that $\{\rho_0(x_j)\}$ span $A \otimes F_{\mathcal{O}_0}$. Then $\text{rk}\,(T_{x_j x_k}) \geq \dim A$ and so with a finite number of exceptions which we may discard $\text{rk}\, tr \rho_i(x_j)\rho_i(x_j) \geq \dim A$. However if $\{a_j\}$ is a finite subset of $A \otimes F$ then $\text{rk}\, tr(a_i a_j) \leq \dim A$ with equality if and only if they span A. Thus we see that we may assume that the $\rho_i(x_j)$ span $A \otimes F_{\mathcal{O}_i}$ for all $i \in I$.

Now let Δ be the subgroup of Γ generated by the x_i. Then we by the last lemma that we have a map $\rho : \Delta \to (A \otimes S)^\times \subset GL_N(S)$ with the notation as in the proposition and $\rho_i|_\Delta \equiv \rho \bmod \mathcal{P}_i$. Now define:

$$A(S') \xrightarrow{\phi} S'^r$$

for S' any S algebra by:

$$a \longmapsto (tr\, a\rho(x_i))$$

This is a linear map and the image $\Omega'(S')$ is thus defined by the vanishing of certain linear forms with coefficients in F_S. By inverting some element of S we may assume that ϕ is injective and that Ω' is defined over S. Then for $\gamma \in \Gamma$ let $t_\gamma = (T_{\gamma x_i})$. By reduction mod $\wp_i$ for infinitely i we see that $t_\gamma \in \Omega'(S)$ and hence there exists a unique $\rho(\gamma)$ with $\phi\rho(\gamma) = t_\gamma$. Then $\rho(\gamma) \equiv \rho_i(\gamma) \bmod \mathcal{P}_i$ for all $i \in I$. Thus we have $\rho : \Gamma \to A \otimes S$, $\rho \equiv \rho_i \bmod \mathcal{P}_i$ for all $i \in I$. It follows at once that ρ is a representation.

Finally we replace ρ by ρ^{ss}, then at almost all $\wp_i$:

$$(\rho^{ss} \bmod \wp_i) = (\rho \bmod \wp_i)^{ss} = \rho_i$$

Corollary 3.3 *Let Λ be as in section 3.4. Let $\wp_i$ be an infinite set of distinct height one primes. For each i let:*

$$\rho_i : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GL_N(\mathbb{Q}_p^{ac})$$

be continuous representations, unramified outside some integer N. For $l \not| N$ let $c_l(X) \in \Lambda[X]$ be monic of degree N and such that:

$$c_l(X) \equiv char_{\rho_i(\mathrm{Frob}_l)}(X) \bmod \wp_i$$

where $char_a(X)$ denotes the characteristic polynomial of a. Then there exists $\mathcal{R}$ the integers of a finite extension of F_Λ and:

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GL_N(F_\mathcal{R})$$

such that $c(X) = char_{\mathrm{Frob}_l}(X)$ for all $l \not| N$.

Proof: We first show that the partial map from $\mathrm{Gal}(L/\mathbb{Q})$ where L denotes the maximal extension of $\mathbb{Q}$ unramified outside N:

$$\mathrm{Frob}_l \longmapsto c_l(X)$$

is continuous. Pick any positive integer n and pick some $l \not| N$. Then we would like to show that for l' with $\mathrm{Frob}_{l'}$ sufficiently close to Frob_l we have that $c_l \equiv c_{l'} \bmod \mathfrak{m}^n$ where $\mathfrak{m}$ is the maximal ideal of Λ. But $\bigcap_{I, \mathbb{Z}_{\geq 0}}(\wp_i, p^s) = 0$ and so by compactness there exist $\wp_1, ..., \wp_r$ and $s_1, ..., s_r$ such that $\bigcap_1^r(\wp_i, p^s) \subset \mathfrak{m}^n$. Then choose U_i open neighbourhoods of Frob_l such that for l' with $\mathrm{Frob}_{l'} \in U_i$ $\rho_i(\mathrm{Frob}_l) \equiv \rho_i(\mathrm{Frob}_{l'}) \bmod p^{s_i}$. Then for l' with $\mathrm{Frob}_{l'} \in U = \bigcap U_i$ $c_l \equiv c_{l'} \bmod \mathfrak{m}^n$ as desired.

Thus we can extend c uniquely to a continuous map:

$$\mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(L/\mathbb{Q}) \longrightarrow \Lambda[X]$$

Then $c_\sigma \equiv char_{\rho_i(\sigma)} \bmod \wp_i$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q})$ by continuity. Now apply the proposition and we find a representation $\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \to GL_N(F_{\mathcal{R}})$ for some appropriate $\mathcal{R}$ with $\rho \bmod \mathcal{P}_i$ conjugate to ρ_i for infinitely many i. Thus $char_{\rho(\mathrm{Frob}_l)} \equiv char_{\rho_i(\mathrm{Frob}_l)} \equiv c_l(X)$ for infinitely many height one primes, so ρ is the desired representation.

Before proving our last two main theorems we prove one further lemma.

Lemma 3.20 *Let F be a field, Γ a group, $\rho : \Gamma \to GSp_4(F) \subset GL_4(F)$. Then $\rho^{ss} : \Gamma \to GL_4(F)$ preserves a non-degenerate symplectic form, so we may consider $\rho^{ss} : \Gamma \to GSp_4(F)$.*

Proof: Let $0 \neq V \subset F^4$ be a simple Γ submodule. Let $\langle\rangle$ denote the symplectic form. Then we are in one of the following cases:

- $V = F^4$ and $\rho^{ss} = \rho$
- $\dim V = 2$, $F^4 = V \oplus V^\perp$. In this case we can choose a basis of F^4 with respect to

which $\langle\rangle$ is represented by $\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$ and ρ is either of the form:

$$\sigma \longmapsto \begin{pmatrix} a_\sigma & 0 & b_\sigma & 0 \\ 0 & a'_\sigma & 0 & b'_\sigma \\ c_\sigma & 0 & d_\sigma & \\ 0 & c'_\sigma & 0 & d'_\sigma \end{pmatrix}$$

in which case $\rho^{ss} = \rho$ or it is of the form:

$$\sigma \longmapsto \begin{pmatrix} a_\sigma & 0 & b_\sigma & * \\ * & e_\sigma & * & * \\ c_\sigma & 0 & d_\sigma & * \\ 0 & 0 & 0 & f_\sigma \end{pmatrix}$$

when ρ^{ss} is easily seen to preserve $\langle\rangle$.

- $\dim V = 2$, $V^\perp = V$. In this case we can choose a basis of F^4 such that $\langle\rangle$ is given by $\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$ and for all σ $\rho(\sigma) = \begin{pmatrix} A_\sigma & B_\sigma \\ 0 & D_\sigma \end{pmatrix}$. Then ρ^{ss} is given by $\sigma \mapsto \begin{pmatrix} A_\sigma & 0 \\ 0 & D_\sigma \end{pmatrix}$ and also preserves $\langle\rangle$.

- $\dim V = 1$, $V^\perp \supset V$, $\dim V^\perp = 3$. In this case we can choose a basis of F^4 with respect to which $\langle\rangle$ is represented by $\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$ and ρ has the form:

$$\sigma \longmapsto \begin{pmatrix} e_\sigma & * & * & * \\ 0 & a_\sigma & * & b_\sigma \\ 0 & 0 & f_\sigma & 0 \\ 0 & c_\sigma & * & d_\sigma \end{pmatrix}$$

Then ρ^{ss} also preserves $\langle\rangle$.

Theorem 3.3 *Assume the conjecture 3.1. Let $F \in \mathcal{M}^\circ(N, \chi, \mathcal{R})$, with N prime to p and $\mathcal{R}$ the integers of some finite extension of F_Λ, be an eigenform of the Hecke algebra $\mathbb{T}_N$, say $F|T(n) = \boldsymbol{\lambda}(n)F$. Then there is a finite extension L of $F_\mathcal{R}$ and a continuous representation:*

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GSp_4(L)$$

which is unramified outside Np and such that for $l \not| Np$, a prime, Frob_l *has characteristic polynomial:*

$$X^4 - \boldsymbol{\lambda} X^3 + (\boldsymbol{\lambda}(l)^2 - \boldsymbol{\lambda}(l^2) - l^{-1}\nu(l))X^2 - \nu(l)\boldsymbol{\lambda}(l)X + \nu(l)^2$$

where $\nu(l) = l^{-3}\chi(l^2)(1+T)^{2\log_p l}$.

Proof: Combining conjecture 3.1 and the corollary to the last proposition we at once deduce the existence of such a representation into $GL_4(L)$. We need only show that it preserves a non-degenerate symplectic form. Let G denote the Zariski closure of $Im\,\rho$, and G° the connected component of the identity. Let $G = \coprod \rho(\gamma_i)G^\circ$ and let $\Sigma = \{\gamma \in \Gamma \,|\, \rho(\gamma) \in G^\circ \,\mathrm{and}\, tr\rho(\gamma) \neq 0\}$. Then G° is the Zariski closure of Σ. Now what we require is a matrix $A \in M_4(L)$ and $\epsilon : \{1, ..., r\} \to \{\pm 1\}$ such that:

1. $(tr\rho(\gamma))\rho(\gamma)A^t\rho(\gamma) = (tr\rho(\gamma^{-1})A$ for all γ in Σ
2. $\rho(\gamma_i)A^t\rho(\gamma_i) = \epsilon_i\sqrt{\det\rho(\gamma_i)}A$
3. ${}^tA = -A$
4. $\det A \neq 0$

where $\sqrt{\det\rho(\gamma_i)}$ is some fixed square root of $\det\rho(\gamma_i)$ which we may assume lies in L. It is easy to check that a solution to these equations in $\mathcal{S}/\mathcal{P}_i$ for infinitely many height one primes $\mathcal{P}_i$ of some finitely generated extension of the integers of L contained in L implies the existence of such a solution in $\mathcal{S}$.

(More precisely let r_ϵ be the dimension of the space of matrices A satisfying conditions 1), 2) and 3). Let $e(\epsilon)_1, ..., e(\epsilon)_{r_\epsilon}$ be a basis of the space of such matrices. That

$\det(\sum e(\epsilon)_i\lambda_i) \not\equiv 0$ may be expressed as $F_{r_\epsilon}(e(\epsilon)_i)(X) \neq 0$ for some polynomial $F_{r_\epsilon}(e(\epsilon)_i(X)$. Thus we have a solution if and only if $r_\epsilon > 0$ and $F_{r_\epsilon}(e(\epsilon)_i)(X) \neq 0$. Then for some ϵ there exists a solution of 1)-4) for infinitely many height one primes $\mathcal{P}_i$. This implies $r_\epsilon > 0$ and $F_{r_\epsilon}(e(\epsilon))(X) \neq 0$.)

Theorem 3.4 *Assume conjecture 3.1. Let $f \in M_2^\circ(Np^r, \chi, \mathcal{O})$ be an eigenform of the Hecke algebra $\mathbb{T}_{Np}$, say $f|T(n) = \lambda(n)f$. ($\mathcal{O}$ the integers in a finite extension of $\mathbb{Q}_p$.) Then there is a multiple M of N and a continuous representation:*

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GSp_4(\mathbb{Q}_p^{ac})$$

which is unramified outside Mp and such that if $l \nmid Mp$ then Frob_l *has characteristic polynomial:*

$$X^4 - \lambda(l)X^3 + (\lambda(l)^2 - \lambda(l^2) - \chi(l^2))X^2 - l\chi(l^2)\lambda(l) + l^2\chi(l^4)$$

Proof: By theorem 3.2 we can find an M as above; characters ψ and α such that ψ is defined modulo Mp, α is of p power order and is defined modulo a power of p and $\chi = \psi\omega^{-2}\alpha$; $\mathcal{R}$ the integers of a finite extension of F_Λ; and $F \in \mathcal{M}^\circ(M, \psi, \mathcal{R})$ an eigenform for the ring of Hecke operators $\mathbb{T}_{Mp}$, say $F|T(n) = \boldsymbol{\lambda}(n)F$, such that $\boldsymbol{\lambda} \equiv \lambda(n) \bmod \mathcal{P}$ with $\mathcal{P}$ a prime of $\mathcal{R}$ above $(1 + T - \alpha(1+p)(1+p)^2))$. Then by the last theorem we can find a continuous representation:

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow GSp_4(L)$$

with L a finite extension of F_Λ. Moreover it is unramified outside Mp and for $l \nmid Mp$ the characteristic polynomial of Frob_l is congruent modulo a prime ($\mathcal{P}'$ say) above $(1+T-\alpha(1+p)(1+p)^2)$ to the polynomial described in the statement of the theorem. As $\mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q})$ is compact we can find a finite $\mathcal{R}'$ (the integers of L) $\mathcal{L} \subset L^4$ such that $\mathcal{L} \otimes_{\mathcal{R}'} L = L^4$ and which is preserved by the Galois action. Then $\mathcal{L}_{\mathcal{P}'}$ is free over $\mathcal{R}'_{\mathcal{P}'}$ and we can choose a basis with respect to which the symplectic form is given by:

$$\begin{pmatrix} 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & \mu_2 \\ -\mu_1 & 0 & 0 & 0 \\ 0 & -\mu_2 & 0 & 0 \end{pmatrix}$$

where $\mu_1|\mu_2$. We may assume that $\sqrt{\mu_1}$ and $\sqrt{\mu_2}$ lie in $\mathcal{R}'$, and that $\mu_1 = 1$. Let e_1, e_2, e_1', e_2' be the corresponding basis. We claim that $\mathcal{L}' = \langle e_1, \mu_2^{-\frac{1}{2}} e_2, e_1', \mu_2^{-\frac{1}{2}} e_2' \rangle$ is also preserved by $\mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q})$.

Let (a_{ij}) denote an element of $Im\ \rho$ with respect to the basis e_1, e_2, e_1', e_2'. Then:

$$(a_{ij}) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu_2 \\ 1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \end{pmatrix} (a_{ji}) = \nu \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu_2 \\ 1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \end{pmatrix}$$

with ν a unit in $\mathcal{R}'_{\mathcal{P}'}$. From this we see that:

- $a_{11}a_{33} - a_{13}a_{31} = \nu$
- $a_{11}a_{23} - a_{13}a_{21} \equiv 0 \bmod \mu_2$
- $a_{11}a_{43} - a_{13}a_{41} \equiv 0 \bmod \mu_2$
- $a_{21}a_{33} - a_{23}a_{31} \equiv 0 \bmod \mu_2$
- $a_{21}a_{43} - a_{23}a_{41} \equiv 0 \bmod \mu_2$
- $a_{31}a_{43} - a_{23}a_{41} \equiv 0 \bmod \mu_2$

and so:

$$a_{23}\nu = a_{33}(a_{11}a_{23} - a_{13}a_{21}) + a_{13}(a_{33}a_{21} - a_{23}a_{31}) \equiv 0 \bmod \mu_2$$

so that $a_{23} \equiv 0 \bmod \mu_2$. Similarly $a_{21} \equiv a_{43} \equiv a_{41} \equiv 0 \bmod \mu_2$. Thus (a_{ij}) also preserves the lattice $\mathcal{L}'$ as we wanted to show.

Now reduction gives us:

$$\rho : \mathrm{Gal}(\mathbb{Q}^{ac}/\mathbb{Q}) \longrightarrow Aut(\mathcal{L}' \otimes \mathcal{R}'_{\mathcal{P}'}/\mathcal{P}', \langle\rangle) \subset GSp_4(\mathbb{Q}_p^{ac})$$

a continuous representation unramified outside Mp with the characteristic polynomials of the desired forms.

Chapter 4

Modular Forms over an Imaginary Quadratic Field

4.1 Introduction

In this chapter we consider Hida families for GL_2 over an imaginary quadratic field. As a byproduct we are led to a method to exhibit torsion in the first homology groups of certain sheaves on the 3-manifolds associated to such forms.

To explain our results let K be an imaginary quadratic field which we shall assume has class number one (though this is almost certainly unnecessary) and let $\mathcal{O}$ denote its ring of integers. We fix $K^{ac} \subset \mathbb{C}$ and $K^{ac} \subset \mathbb{C}_p$. Let p be an odd rational prime which splits in K, and π the prime of K above p corresponding to $K^{ac} \subset \mathbb{C}_p$. By an ordinary cuspidal eigenform of "weight" n, level M, and character $\chi : (\mathcal{O}/M\mathcal{O})^\times \to (K^{ac})^\times$, we shall mean an eigenvalue of the Hecke operators T_n acting on the corresponding space of cusp forms (see section 4.2, but note that "weight n" is the "weight" corresponding to a sheaf of dimension $(n+1)^2$, and differs by a shift of two from the normal terminology in the case of elliptic modular forms.) with the eigenvalue of T_p prime to π. We shall write $M = Np^r$, with N

prime to p and decompose $\chi = \chi_{ac} \times \chi_{cy}$ corresponding to $(\mathcal{O}/M\mathcal{O})^\times = A \times B$ where:

$$A = \{\alpha \in ((1 + Np\mathcal{O})/(1 + Np^r\mathcal{O})) \mid \alpha\bar{\alpha} = 1\} \times (\mathcal{O}/Np\mathcal{O})^\times$$
$$B = \{\alpha \in ((1 + Np\mathcal{O})/(1 + Np^r\mathcal{O})) \mid \alpha = \bar{\alpha}\}$$

(the anti-cyclotomic and cyclotomic parts respectively). We shall consider Λ-adic eigenforms which interpolate ordinary cuspidal eigenforms. In the simplest case an Λ-adic eigenform of level N (prime to p) and character ϕ (an anti-cyclotomic character of defined modulo Np^r for some r) is a collection $a_m(T) \in \mathcal{O}_\pi[[T]]$ for each $m \in \mathcal{O}$ such that for each cyclotomic character ψ and non-negative rational integer n:

$$\{a_m((1 + T) - (1 + p)^{2n}\psi(1 + p))\}$$

are the eigenvalues of the Hecke operators T_m acting on an ordinary cuspidal eigenform of weight n, level Np^s (some s) and character $\phi\psi\omega^{-n}$ where ω is the Teichmuller lifting of the norm map $(\mathcal{O}/p\mathcal{O})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ to a map $(\mathcal{O}/p\mathcal{O})^\times \to \mathbb{Z}^\times$. In general we must allow the a_i to lie in a finite extension of $\mathcal{O}_\pi[[T]]$.

Our main theorem (theorem 4.1) states that if we fix n then there are only finitely many Λ-adic eigenforms of this level. If moreover we fix an anti-cyclotomic character ϕ then all but finitely many ordinary cuspidal eigenforms of level Np^s (any s), weight n (any n) and character $\phi\omega^n\psi$ (any cyclotomic character ψ) lift to an Λ-adic eigenform. It would be nice to strengthen this result to say that *any* ordinary cuspidal eigenform lifted to a unique Λ-adic eigenform. The problem here is torsion in the corresponding homology groups (see the comments following theorem 4.1).

Such results have been proved by Hida (and Wiles) for modular forms over totally real fields, however as our modular forms are not analytic we can not multiply them together to produce new ones with good congruence properties, so we have to rely completely on the cohomology. Our argument falls into two parts. In section 4.4 we use the inflation restriction long exact sequence to "change level". In section 4.5 we relate different weights. Both are achieved by developing ideas of Hida ([Hi1]).

For example in section 4.7 we prove that if p is as above and $p \nmid \tau(p)$ where τ is Ramanujam's function (i.e. $\Delta(z) = \sum \tau(n)e^{2n\pi iz}$ is the cuspidal elliptic modular function of weight

12 for $SL_2(\mathbb{Z})$), and if $n_1 \neq n_2$; $n_1, n_2 > 10$; $n_1+n_2 > 20+r$; and $n_1 \equiv n_2 \equiv 10 \bmod p^r(p-1)$ then

$$H_1(SL_2(\mathcal{O}), S_{n_1,n_2}) \quad \text{and} \quad H_{1\,cusp}(SL_2(\mathcal{O}), S_{n_1,n_2})$$

have torsion of exponent divisible by p^r. (Here S_{n_1,n_2} is the $SL_2(\mathcal{O})$-module which is the the tensor product of the n_1^{th} symmetric power of $\mathcal{O}^2$ with the natural $SL_2(\mathcal{O})$ action and the n_2^{th} symmetric power of $\mathcal{O}^2$ with $SL_2(\mathcal{O})$ action twisted by complex conjugation. Also see section 4.2 for the meaning of "*cusp*" in this context.)

The rest of the paper is organised as follows. Section 4.2 contains some analytic results we need. Section 4.3 is somewhat technical and is needed to show that our use of the inflation restriction sequence respects the cuspidal cohomology. (Torsion prevents us using the analytic theory of Eisenstein series which Hida used in [Hi1].) In section 4.6 we complete the proof of our main theorem and show how to construct some examples.

We should mention that while all the theory goes through in the case of a prime inert in K, we lack any examples in that case, so that theory may be vacuous.

Notation

Most of the notation used is either standard or explained in the text. If F is a field F^{ac} will denote its algebraic closure. $\mathbb{C}_p$ will denote the completion of $\mathbb{Q}_p^{ac}$ and $\mathbb{O}_p$ the ring of elements of non-negative valuation in $\mathbb{C}_p$. If π is a prime in a number field we shall use $\mathbb{C}_\pi$ and $\mathbb{O}_\pi$ for the corresponding notions. If A is a ring with ideal I then we shall set:

- $\Gamma(I) = \{\alpha \in SL_2(A) \mid \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod I\}$
- $\Gamma_1(I) = \{\alpha \in SL_2(A) \mid \alpha \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod I\}$
- $\Gamma_0(I) = \{\alpha \in SL_2(A) \mid \alpha \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod I\}$

We hope the context will always make it clear which ring we are talking about.

By a *cofinite* $\mathbb{Z}_p$ module we shall mean the Pontriagin dual of a finite $\mathbb{Z}_p$ module.

4.2 Review of Cohomology Groups and Automorphic Forms

Throughout let K be an imaginary quadratic field of class number one and let $\mathcal{O}$ denote its ring of integers. The assumption that the class number is one is almost certainly unnecessary but it simplifies the notation. Fix also an odd rational prime p, which is unramified in K and a prime π of $\mathcal{O}$ lying above p. Note that these conditions imply that the only unit of $\mathcal{O}$ congruent to 1 modulo π is 1 itself. Let $^-$ denote complex conjugation.

For any pair of non-negative integers n_1, n_2 we have a free $(n_1+1)(n_2+1)$ -dimensional $\mathcal{O}$ module with an action of $GL_2(\mathcal{O})$ (or in fact of $M_2(\mathcal{O})$).It may be explicitly described as $S^{n_1}(\mathcal{O}^2)\times S^{n_2}(\mathcal{O}^2)$ where S^n denotes the n-th symmetric power (i.e. the maximal symmetric quotient of the n-th tensor power) and where $\gamma\in GL_2(\mathcal{O})$ acts on the first $\mathcal{O}^2$ in the natural fashion and on the second via $\bar{\gamma}$. We will denote this module S_{n_1,n_2}. If A is an $\mathcal{O}$ module $S_{n_1,n_2}(A)$ will denote $S_{n_1,n_2}\otimes_{\mathcal{O}}A$. In particular $S_{n_1,n_2}(\mathbb{C})$ can be thought of as the irreducible finite dimensional representations of Lie group $SL_2(\mathbb{C})$. When we need to take an $\mathcal{O}$-basis we shall always take the natural basis with respect to which $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O})$ acts as:

$$\begin{pmatrix} a^{n_1}M & n_1 a^{n_1-1}bM & \cdots & b^{n_1}M \\ a^{n_1-1}cM & (a^{n_1-1}d+(n_1-1)a^{n_1-2}c)M & \cdots & b^{n_1-1}dM \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c^{n_1}M & n_1 c^{n_1-1}dM & \cdots & d^{n_1}M \end{pmatrix}$$

where M denotes the block:

$$\begin{pmatrix} \bar{a}^{n_2} & n_2\bar{a}^{n_2-1}\bar{b} & . \; . \; . & \bar{b}^{n_2} \\ \bar{a}^{n_2-1}\bar{c} & (\bar{a}^{n_2-1}\bar{d} + (n_2-1)\bar{a}^{n_2-2}\bar{c}) & . \; . \; . & \bar{b}^{n_2-1}\bar{d} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \bar{c}^{n_2} & n_2\bar{c}^{n_2-1}\bar{d} & . \; . \; . & \bar{d}^{n_2} \end{pmatrix}$$

We shall be interested in the cohomology of congruence subgroups $\Gamma < SL_2(\mathcal{O})$ with coefficients in $S_{n_1,n_2}(A)$.When $A = \mathbb{C}$ these groups can be studied analytically. More precisely let $\mathcal{Z}$ denote "the quaternion upper half plane" or "hyperbolic 3-space", that is to say:

$$\{quaternions\, z = x + y\mathbf{k} | x \in \mathbb{C}\, and\, y \in R_{>0}\}$$

Then $SL_2(\mathbb{C})$ acts on $\mathcal{Z}$ and in fact on $\mathcal{Z} \times S_{n_1,n_2}(A)$ by:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z, v) \mapsto ((az+b)(cz+d)^{-1}, \gamma v)$$

If Γ is torsion free, $\Gamma\backslash\mathcal{Z}$ is a smooth manifold with a sheaf $\tilde{S}_{n_1,n_2}(A)$ consisting of Γ-invariant sections of $\mathcal{Z} \times S_{n_1,n_2}(A)$. Then it is known that $H^\bullet(\Gamma, S_{n_1,n_2}(A)) =$ $H^\bullet(\Gamma\backslash\mathcal{Z}, \tilde{S}_{n_1,n_2}(A))$. In the case $A = \mathbb{C}$ this group is well studied. See for example Harder [Ha1],[Ha2] for the following results.

There is a compact manifold with boundary $\overline{\Gamma\backslash\mathcal{Z}}$ and an embedding $\Gamma\backslash\mathcal{Z} \hookrightarrow \overline{\Gamma\backslash\mathcal{Z}}$ which is a homotopy equivalence (the Borel-Serre compactification). The sheaves $\tilde{S}_{n_1,n_2}(A)$ extend to $\overline{\Gamma\backslash\mathcal{Z}}$ in such a way that $H^\bullet(\Gamma\backslash\mathcal{Z}, \tilde{S}_{n_1,n_2}(A)) \cong H^\bullet(\overline{\Gamma\backslash\mathcal{Z}}, \tilde{S}_{n_1,n_2}(A))$. We have a natural map:

$$H^\bullet(\overline{\Gamma\backslash\mathcal{Z}}, \tilde{S}_{n_1,n_2}(A)) \longrightarrow H^\bullet(\partial(\overline{\Gamma\backslash\mathcal{Z}}), \tilde{S}_{n_1,n_2}(A))$$

We shall denote the kernel of this map by $H^\bullet_{cusp}(\Gamma\backslash\mathcal{Z}, \tilde{S}_{n_1,n_2}(A))$ and the image by $H^\bullet_{Eis}(\Gamma\backslash\mathcal{Z}, \tilde{S}_{n_1,n_2}(A))$.

We have that:

- $H^i_{cusp}(\Gamma\backslash\mathcal{Z},\tilde{S}_{n_1,n_2}(\mathbb{C}))=0$ unless $n_1=n_2$ and either $i=1$ or 2
- $H^1_{cusp}(\Gamma\backslash\mathcal{Z},\tilde{S}_{n,n}(\mathbb{C}))\cong H^2_{cusp}(\Gamma\backslash\mathcal{Z},\tilde{S}_{n,n}(\mathbb{C}))\cong S_n(\Gamma,\mathbb{C})$
- $H^0_{Eis}(\Gamma\backslash\mathcal{Z},\tilde{S}_{n_1,n_2}(\mathbb{C}))=0$ unless $n_1=n_2=0$ when it is equal to $\mathbb{C}$
- $\dim H^1_{Eis}(\Gamma\backslash\mathcal{Z},\tilde{S}_{n_1,n_2}(\mathbb{C}))=\frac{1}{2}\dim H^1(\partial(\overline{\Gamma\backslash\mathcal{Z}}),\tilde{S}_{n_1,n_2}(\mathbb{C}))$
- $H^2_{Eis}(\Gamma\backslash\mathcal{Z},\tilde{S}_{n_1,n_2}(\mathbb{C}))=H^2(\partial(\overline{\Gamma\backslash\mathcal{Z}}),\tilde{S}_{n_1,n_2}(\mathbb{C}))$ unless $n_1=n_2=0$ in which case it is of codimension one

Here $S_n(\Gamma,\mathbb{C})=\oplus\rho_f^{U(\Gamma)}$ where $U(\Gamma)$ is the closure of Γ in SL_2 of the finite adeles of K and where the sum is taken over all cuspidal automorphic representations $\rho=\rho_f\otimes\rho_\infty$ of SL_2 over K with ρ_∞ the principal series representation of $SL_2(\mathbb{C})$ corresponding to the character $\begin{pmatrix} a & * \\ 0 & a^{-1}\end{pmatrix}\mapsto\left(\frac{a}{|a|}\right)^{2(n+1)}$.

This set up may be described in terms of group cohomology as follows. Define a (Γ-) cusp to be a Γ-conjugacy class of Borel subgroups of SL_2(K). For B such a Borel we will denote by $[B]$ (or if necessary $[B]_\Gamma$) the corresponding cusp. Set $\Gamma_B=\Gamma\cap B$. The connected components of the Borel-Serre compactification are in one-to-one correspondence with the cusps. Set:

$$H^\bullet_\partial(\Gamma,S_{n_1,n_2}(A))=\bigoplus_{[B]}H^\bullet(\Gamma_B,S_{n_1,n_2}(A))$$

This appears to depend on the choice of Borel which represents each cusp, however if $\gamma\in\Gamma$ then $\Gamma_{\gamma B\gamma^{-1}}=\gamma\Gamma_B\gamma^{-1}$ and we get a canonical isomorphism

$$\gamma_*:H^\bullet(\Gamma_B,S_{n_1,n_2}(A))\xrightarrow{\sim}H^\bullet(\Gamma_{\gamma B\gamma^{-1}},S_{n_1,n_2}(A))$$

Using the facts that Γ_B is its own normaliser in Γ and that if M is a G-module and $g\in G$ then the map $g_*:H^\bullet(G,M)\to H^\bullet(G,M)$ induced by conjugation by g on G and by translation by g on M is the identity; we see further that:

- $\gamma_*:H^\bullet(\Gamma_B,S_{n_1,n_2}(A))\xrightarrow{\sim}H^\bullet(\Gamma_{\gamma B\gamma^{-1}},S_{n_1,n_2}(A))$ is independent of the choice of $\gamma\in\Gamma$ conjugating B to $\gamma B\gamma^{-1}$

- The diagram:

$$\begin{array}{ccc} & & H^\bullet(\Gamma_B, S_{n_1,n_2}(A)) \\ & res \nearrow & \\ H^\bullet(\Gamma, S_{n_1,n_2}(A)) & & \downarrow \gamma_* \\ & res \searrow & \\ & & H^\bullet(\Gamma_{\gamma B \gamma^{-1}}, S_{n_1,n_2}(A)) \end{array}$$

commutes.

We will identify the groups $H^\bullet(\Gamma_B, S_{n_1,n_2}(A))$ for B representing a cusp $[B]$ and write simply $H^\bullet(\Gamma_{[B]}, S_{n_1,n_2}(A))$. Restriction gives a well defined map $H^\bullet(\Gamma, S_{n_1,n_2}(A)) \to H^\bullet_\partial(\Gamma, S_{n_1,n_2}(A))$ and we have a commutative diagram:

$$\begin{array}{ccc} H^\bullet(\Gamma, S_{n_1,n_2}(A)) & \stackrel{res}{\to} & H^\bullet_\partial(\Gamma, S_{n_1,n_2}(A)) \\ \| \wr & & \| \wr \\ H^\bullet(\Gamma\backslash\mathcal{Z}, \tilde{S}_{n_1,n_2}(A)) & \to & H^\bullet(\partial(\overline{\Gamma\backslash\mathcal{Z}}), \tilde{S}_{n_1,n_2}(A)) \end{array}$$

We shall use $H^\bullet_{cusp}(\Gamma, M)andH^\bullet_{Eis}(\Gamma, M)$ in the obvious way.

If we use the group cohomology the analytic description of $H^\bullet_{cusp}(\Gamma, S_{n_1,n_2}(\mathbb{C}))$ given above remains true for Γ with torsion. To see this choose $\Delta \lhd \Gamma$ of finite index and without torsion. The results for Γ follow from those for Δ because, as we are working over a field of characteristic 0, the Serre-Hoschild spectral sequence implies that the Γ cohomology is just the $\Delta\backslash\Gamma$ invariant part of the Δ cohomology.

We now want to describe the action of the Hecke operators on these various spaces. For this we will work in the category of $(M_2(\mathcal{O}) \cap GL_2(K))$-modules. It is easily seen by abstract nonsense that for Γ a group contained in the semi-group $(M_2(\mathcal{O}) \cap GL_2(K))$ that the cohomology functors $H^\bullet(\Gamma, \_)$ defined on $(M_2(\mathcal{O}) \cap GL_2(K))$-modules can be thought of as the right derived functors of the fixed point functor $M \mapsto M^\Gamma$ and as such are a universal δ-functors. This will be very helpful in checking that diagrams commutes. When we can consider the maps as special instances of natural transformations between such universal δ-functors, it will do to check the commutativity only in degree zero.

To describe the Hecke operators let $g \in M_2(\mathcal{O})$, $\det g \neq 0$ and let Γ_1, Γ_2 be congruence subgroups of $SL_2(\mathcal{O})$. Then we have $[\Gamma_2 : \Gamma_2 \cap g\Gamma_1 g^{-1}] < \infty$ and so we can define a map $[\Gamma_2 g \Gamma_1] : H^\bullet(\Gamma_1, M) \to H^\bullet(\Gamma_2, M)$,or more precisely a natural transformation between δ-functors $[\Gamma_2 g \Gamma_1] : H^\bullet(\Gamma_1, \_) \to H^\bullet(\Gamma_2, \_)$ as follows:

$$H^\bullet(\Gamma_1, M) \xrightarrow{g_*} H^\bullet(\Gamma_2 \cap g\Gamma_1 g^{-1}, M) \xrightarrow{cor} H^\bullet(\Gamma_2, M)$$

where the first map is induced by the compatible maps :

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ \Gamma_1 & \xleftarrow{conjugation} & \Gamma_2 \cap g\Gamma_1 g^{-1} \end{array}$$

and the second map is corestriction. One can check straightforwardly that $[\Gamma_2 g \Gamma_1]$ only depends on the double coset $\Gamma_2 g \Gamma_1$ and not on the particular choice of g. To describe explicitly the action of $[\Gamma_2 g \Gamma_1]$ in degree zero and one, assume that $\Gamma_2 = \amalg \gamma_i (\Gamma_2 \cap g\Gamma_1 g^{-1})$ (which is easily seen to be equivalent to $\Gamma_2 g \Gamma_1 = \amalg \gamma_i g \Gamma_1$). Then

- $[\Gamma_2 g \Gamma_1] : M^{\Gamma_1} \to M^{\Gamma_2}$ by $m \mapsto \sum (\gamma_i g) m$
- $[\Gamma_2 g \Gamma_1] : H^1(\Gamma_1, M) \to H^1(\Gamma_2, M)$ is induced by sending a Γ_1-cocycle ϕ to the Γ_2-cocycle $\delta \mapsto \sum (\gamma_i g) \phi((\gamma_i g)^{-1} \delta (\gamma_{j_i} g))$ where j_i is the unique index such that $\gamma_i^{-1} \delta \gamma_{j_i} \in g\Gamma_1 g^{-1}$.

We can describe the Hecke operators on a topological level by considering the diagram:

$$\begin{array}{ccccccc} & \Gamma_2 \cap g\Gamma_1 g^{-1} \backslash \mathcal{Z} & & & & \mathcal{Z} & \\ \swarrow & & \searrow & \text{induced by} & g^{-1} \swarrow & & \searrow \text{Id} \\ \Gamma_1 \backslash \mathcal{Z} & & \Gamma_2 \backslash \mathcal{Z} & & \mathcal{Z} & & \mathcal{Z} \end{array}$$

This gives rise to:

$$\begin{array}{ccccc} & & H^\bullet(\Gamma_2 \cap g\Gamma_1 g^{-1} \backslash \mathcal{Z}, \tilde{M}) & & \\ & \nearrow & & \searrow \text{transfer} & \\ H^\bullet(\Gamma_1 \backslash \mathcal{Z}, \tilde{M}) & & & & H^\bullet(\Gamma_2 \backslash \mathcal{Z}, \tilde{M}) \end{array}$$

which is exactly the Hecke operator $[\Gamma_2 g \Gamma_1]$. (Here $\tilde{M}$ is the sheaf constructed from M exactly as $\tilde{S}_{n_1,n_2}$ was from S_{n_1,n_2}.)

We also want to define their action on the cohomology of the boundary. The topological picture shows us how to do this. For each cusp [C] of $\Gamma_2 \cap g\Gamma_1 g^{-1}$ we have a map:

$$H^\bullet(\Gamma_{1\,[g^{-1}Cg]}, M) \xrightarrow{g_*} H^\bullet((\Gamma_2 \cap g\Gamma_1 g^{-1})_{[C]}, M) \xrightarrow{cor} H^\bullet(\Gamma_{2\,[C]}, M)$$

(the corestriction map is defined because $[\Gamma_{2\,[C]} : (\Gamma_2 \cap g\Gamma_1 g^{-1})_{[C]}]$ is less than $[\Gamma_2 : \Gamma_2 \cap g\Gamma_1 g^{-1}]$). Summing these maps over $[C]$ we get:

$$[\Gamma_2 g \Gamma_1] : H^\bullet_\partial(\Gamma_1, M) \longrightarrow H^\bullet_\partial(\Gamma_2, M)$$

A certain amount of care is needed to keep track of the identification we are making of $H^\bullet(\Gamma_B, M)$ for different Borels B representing the same cusp. One also has to check that it is well defined up to these identifications. Moreover one can easily check that it depends only on the double coset $\Gamma_1 g \Gamma_2$ and that it is compatible with the restrictions maps from $H^\bullet(\Gamma_i, M)$ and the previous definition at this level. From this we see that Hecke operators preserve the cuspidal and Eisenstein comohology. For example to check the compatibility with our previous definition we must check that

$$\begin{array}{ccccc} \bigoplus_{[B]} H^\bullet(\Gamma_{1\,[A]}, M) & \rightarrow & \bigoplus_{[C]} H^\bullet(\Gamma_{1\,[g^{-1}Cg]}, M) & \overset{g_*}{\rightarrow} & \bigoplus_{[C]} H^\bullet((\Gamma_2 \cap g\Gamma_1 g^{-1})_{[C]}, M) \\ & \nwarrow & \uparrow & & \uparrow \\ & & H^\bullet(\Gamma_1, M) & \overset{g_*}{\rightarrow} & H^\bullet((\Gamma_2 \cap g\Gamma_1 g^{-1}), M) \end{array}$$

and

$$\begin{array}{ccccc} \bigoplus_{[C]} H^\bullet((\Gamma_2 \cap g\Gamma_1 g^{-1})_{[C]}, M) & \overset{cor}{\rightarrow} & \bigoplus_{[C]} H^\bullet(\Gamma_{2\,[C]}, M) & \rightarrow & \bigoplus_{[B]} H^\bullet(\Gamma_{[B]}, M) \\ \uparrow & & & \nearrow & \\ H^\bullet((\Gamma_2 \cap g\Gamma_1 g^{-1}), M) & \overset{cor}{\rightarrow} & H^\bullet(\Gamma_2, M) & & \end{array}$$

commute. The first diagram is easy. That the pentagon commutes follows from the following fact,which it suffices to check in degree zero:

> Assume $\Gamma \supset \Delta$ with finite index and $\Gamma \supset C$ and M is a Γ-module. Let $C_1, ..., C_s$ be representatives of the Δ-conjugacy classes of Γ-conjugates of C. Then
>
> $$\begin{array}{ccccc} \bigoplus H^\bullet(\Delta \cap C_i, M) & \overset{cor}{\rightarrow} & \bigoplus H^\bullet(C_i, M) & \overset{conjugation}{\rightarrow} & H^\bullet(C, M) \\ \uparrow res & & & & \uparrow res \\ H^\bullet(\Delta, M) & & \overset{cor}{\longrightarrow} & & H^\bullet(\Gamma, M) \end{array}$$
>
> commutes.

To consider the action of the Hecke operators analytically we must modify our analytic description slightly. Let U be an open compact subgroup of $GL_2(\mathbf{A}_f)$ (here $\mathbf{A}_f$ denotes the finite adeles of K) such that $\Gamma = U \cap GL_2(K) \subset SL_2(\mathcal{O})$. Then we have:

- $S_n(\Gamma, \mathbb{C}) = \oplus \rho_f^U$ where the sum is taken over all cuspidal automorphic representations $\rho = \rho_f \otimes \rho_\infty$ of $GL_2(K)$ with ρ_∞ the principal series representation of $GL_2(\mathbb{C})$ corresponding to the character $\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto \left(\frac{a}{|a|}\right)^{n+1} \left(\frac{|b|}{b}\right)^{n+1} |ab|^{-n}$.

- $H^1_{Eis}(\Gamma \backslash \mathcal{Z}, \tilde{S}_{n_1,n_2}(\mathbb{C})) = G_{n_1,n_2}(\Gamma, \mathbb{C}) = \bigoplus \left(Ind_{B_0(\mathbf{A}_f)}^{GL_2(\mathbf{A}_f)} \psi_f\right)^U$ where the sum is taken over all Hecke characters:

$$\psi = \psi_f \psi_\infty : \begin{pmatrix} \mathbf{A}^\times & * \\ 0 & \mathbf{A}^\times \end{pmatrix} \to \mathbb{C}^\times$$

  for which $\psi_\infty : \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto a\bar{a}^{-n_2} b^{-(n_1+1)}$. (The induction here is not the usual unitary induction but that in Harder [Ha2], which explains the slight discrepancy from the cuspidal case above.)

Now assume U_1, U_2 are as above and that $g \in GL_2(K)$, then we get a map:

$$[U_1 g^{-1} U_2] : S_n(\Gamma_1, \mathbb{C}) = \oplus \rho_f^{U_1} \longrightarrow \oplus \rho_f^{U_2} = S_n(\Gamma_2, \mathbb{C})$$

by v $\mapsto \int_{(U_1 g^{-1} U_2)} \mathrm{v}\, \rho_f(x)\, dx$ where the Haar measure dx is normalised so that $\int_{U_1} dx = 1$. If $\Gamma_2 g \Gamma_1 = \amalg \gamma_u g \Gamma_1$ then $U_1 g^{-1} U_2 \supset \amalg U_1 g^{-1} \gamma_u^{-1}$. If this is in fact an equality then $[U_1 g^{-1} U_2]$ and $[\Gamma_2 g \Gamma_1] : S_n(\Gamma_1, \mathbb{C}) \to S_n(\Gamma_2, \mathbb{C})$ coincide. This follows for example from results in Harder [Ha2]. Similar results hold for the Eisenstein cohomology.

Finally in this section we consider some special Hecke operators. Let $\mathbb{T}$ be the abstract commutative ring over $\mathbb{Z}$ generated by the symbols T_n for $n \in \mathcal{O} \setminus \{0\}$. For the rest of this section we consider only congruent subgroups of the form:

$$\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \mid \; c \equiv 0 \bmod N \,, a \equiv d \equiv 1 \bmod M \;\}$$

where $M \mid N$. For such a group Γ, $T_n \mapsto [\Gamma \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \Gamma]$ gives an action of $\mathbb{T}$ on $H^\bullet(\Gamma, M)$. (To see this is a good definition one must check that these operators commute. By abstract nonsense we can check this in degree zero where the problem reduces to showing that if $\Gamma \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \amalg \alpha\Gamma$ and $\Gamma \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \amalg \beta\Gamma$ then the cosets $\alpha\beta\Gamma$ and $\beta\alpha\Gamma$ coincide. For this we refer the reader to Shimura [Sh2].) Note that we can apply the remarks of the last paragraph to describe these Hecke operators analytically, taking:

$$U = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \prod_v GL_2(\mathcal{O}_v) \mid c \equiv 0 \bmod N \, , d \equiv 1 \bmod M \}$$

(Atleast if 1 is the only unit of $\mathcal{O}$ congruent to 1 mod M.) Moreover we can factorise $[U_1 \begin{pmatrix} n^{-1} & 0 \\ 0 & 1 \end{pmatrix} U_2]$ into a product of local operators, which are the identity for primes not dividing n.

If $\Gamma_1 > \Gamma_2$ are both of the above form defined by $N_1, M_1; N_2, M_2$ and if n is not divisible by primes dividing N_1 but not N_2, then we have a commutative diagram:

$$\begin{array}{ccc} H^\bullet(\Gamma_1, M) & \xrightarrow{T_n} & H^\bullet(\Gamma_1, M) \\ res \downarrow & & \downarrow res \\ H^\bullet(\Gamma_2, M) & \xrightarrow{T_n} & H^\bullet(\Gamma_2, M) \end{array}$$

and so if N_1 and N_2 have the same prime factors we see that the restriction map is $\mathbb{T}$-equivariant (or as we shall write "Hecke equivariant"). (This need only be checked in degree zero where it follows because we can find γ_i such that $\Gamma_1 = \amalg \gamma_i (\Gamma_1 \cap g\Gamma_1 g^{-1})$ and $\Gamma_2 = \amalg \gamma_i (\Gamma_2 \cap g\Gamma_2 g^{-1})$ - see Shimura [Sh2].)

We have an action of $(\mathcal{O}/N\mathcal{O})^\times \cong \Gamma_0(N)/\Gamma_1(N)$ on $H^\bullet(\Gamma_1(N), M)$ by conjugation. We shall fix $\Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathcal{O}/N\mathcal{O})^\times$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$.Then it is again easily checked that this action commutes with that of $\mathbb{T}$, and that if $N \mid N'$ then $(\mathcal{O}/N\mathcal{O})^\times \twoheadrightarrow (\mathcal{O}/N'\mathcal{O})^\times$ is compatible with $res : H^\bullet(\Gamma_1(N), M) \to H^\bullet(\Gamma_1(N'), M)$.

We shall be particularly interested in the Hecke operators T_μ for μ dividing some power of p. Thus we recall the decomposition for Γ now of the form

$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \mid c \equiv 0 \bmod Np^r, \, a \equiv d \equiv 1 \bmod Np^s\}$ with $r \geq 1$ and $r \geq s$ and for μ as above:

$$\Gamma \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \amalg \begin{pmatrix} \mu & u \\ 0 & 1 \end{pmatrix} \Gamma$$

as u runs over any set of representatives for congruent classes of $\mathcal{O}$ mod μ. If ν also divides some power of p we have also:

$$[\Gamma \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix} \Gamma][\Gamma \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} \Gamma] = [\Gamma \begin{pmatrix} \mu\nu & 0 \\ 0 & 1 \end{pmatrix} \Gamma]$$

Most essentially we have Hida's idempotent. In general to an operator T on a finite (or cofinite) $\mathbb{Z}_p$-module H we can define an idempotent e_T in $End_{\mathbb{Z}_p}(H)$ commutes with T such that T is an automorphism of $e_T H$ and is topologically nilpotent on $(1 - e_T)H$ (see Mazur and Wiles [MW2]). In fact $e_T = \lim_{r \to \infty} T^{r!}$. If $S : H \to H'$ is a morphism between finite $\mathbb{Z}_p$-modules and T, T' are operators on H and H' respectively such that $ST = T'S$ then S restricts to a map $S : e_T H \to e_{T'} H'$. In particular for the Hecke operator T_p acting on a finite $\mathbb{Z}_p$-module H we will denote the corresponding idempotent simply e, call it *Hida's idempotent* and write H° for eH, which we will call the *ordinary part* of H. If $p = \pi\bar{\pi}$ in $\mathcal{O}$ then we can define e_π and $e_{\bar{\pi}}$ similarly corresponding to T_π and $T_{\bar{\pi}}$ and we have $e_\pi e_{\bar{\pi}} = e$. In most of the following we will restrict to the ordinary parts of modules, and this will be essential for our arguments. Any $v \in H$ which is an eigenvector of T_p with eigenvalue a p-adic unit will be preserved by e. We see that almost all of the discussion of this section goes over to ordinary parts in the obvious fashion, but one should be aware that restriction does not usually map (for example) $H^\bullet(\Gamma_1(N), M)^\circ$ to $H^\bullet(\Gamma_1(Np), M)^\circ$ when $p \not| N$. (This is "because we are going from no p in the level to p in the level".)

Although Hida's idempotent is initially defined only in the p-adic setting we can sometimes consider it in more general situations. Fix embeddings $K^{ac} \subset \mathbb{C}_\pi$ and $K^{ac} \subset \mathbb{C}$. In particular we have fixed an extension of the π-adic valuation to K^{ac}. Now if H is a finite torsion free $\mathcal{O}$-module with an action of T_p then we have:

$$\begin{array}{ccc} e \in \mathcal{O}_\pi[T_p] & \subset & End_{\mathcal{O}_\pi}(H_\pi) \\ \cap & & \cap \\ K_\pi[T_p] & \subset & End_{K_\pi}(H \otimes K_\pi) \\ \cup & & \cup \\ K[T_p] & \subset & End_K(H \otimes K) \end{array}$$

and there is a finite extension L/K contained in K_π such that $e \in L[T_p]$ and hence $\in End_{\mathbb{C}}(H \otimes \mathbb{C})$.

Then we have that:

- $eS_n(\Gamma_1(Np^r), \mathbb{C}) = \bigoplus (e\rho_p^{U_p}) \otimes (\rho_f^p)^{U^p}$
- $eG_{n_1,n_2}(\Gamma_1(Np^r), \mathbb{C}) = \bigoplus \left(e\, Ind_{B_0(K_p)}^{GL_2(K_p)} \psi_p\right)^{U_p} \otimes V_{\psi_f^p}$

where:

$$U_p = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_p) \mid \ c \equiv 0 \bmod Np^r,\ d \equiv 1 \bmod Np^r \}$$

$$U^p = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \prod_{v \not| p} GL_2(\mathcal{O}_v) \mid \ c \equiv 0 \bmod N,\ d \equiv 1 \bmod N \}$$

here $\rho = \rho_p \otimes \rho_f^p \otimes \rho_\infty$ runs over cuspidal automorphic representations as before, where $\psi = \psi_p \times \psi_f^p \times \psi_\infty$ is also as described above, and where $V_{\psi_f^p}$ does not matter very much. If $p = \pi\bar{\pi}$ in $\mathcal{O}$ then we have moreover, $e\rho_p^{U_p} = e_\pi \rho_\pi^{U_\pi} \otimes e_{\bar{\pi}} \rho_{\bar{\pi}}^{U_{\bar{\pi}}}$. It is known that for v a prime above p with $\rho_v^{U_v} \neq 0$ and for $r > 0$:

- ρ_v supercuspidal implies that $e_v \rho_v^{U_v} = 0$
- $\rho_v = \rho(\psi, \psi|\,.\,|)$ implies that e_v is undefined if ψ_i is unramified and $(\mathbf{N}v)^{\frac{1}{2}}\psi(v)$ has positive π-adic valuation. Otherwise the dimension of $e_v\rho_v^{U_v}$ is 1 or 0 according to whether ψ is unramified and $(\mathbf{N}v)^{\frac{1}{2}}\psi(v)$ is a π-adic unit, or not.
- $\rho_v = \rho(\psi_1, \psi_2)$ principal series implies that e_v is undefined if ψ_i is unramified and $(\mathbf{N}v)^{\frac{1}{2}}\psi_i(v)$ has positive π-adic valuation for some i. Otherwise $e_v\rho_v^{U_v}$ has dimension

0, 1 or 2 corresponding to the number of ψ_i $(i = 1,\, 2)$ which are unramified and have $(\mathbf{N}v)^{\frac{1}{2}}\psi_i(v)$ a π-adic unit.

From this one can show that if p is inert in K then $eG_{n_1,n_2}(\Gamma_1(Np^r), \mathbb{C}) = 0$ and that if $p = \pi\bar{\pi}$ splits in K then:

$$eG_{n_1,n_2}(\Gamma_1(Np^r), \mathbb{C}) = \bigoplus V_{\psi_f^p}$$

where the sum is taken over characters $\psi = \psi_p \times \psi_f^p \times \psi_\infty$ as described above with the added condition that if $\psi_p : \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto \psi_1(a)\psi_2(b)$ then ψ_1 is unramified at π and ψ_2 at $\bar{\pi}$. In particular for $r \geq 1$, dim $eG_{n_1,n_2}(\Gamma_1(Np^r), \mathbb{C})$ is independent of n_1 and n_2 (as 1 is the only unit of $\mathcal{O}$ congruent to 1 mod (Np^r)), and $\dim eG_{n_1,n_2}(\Gamma_1(Np^r), \mathbb{C}) \geq \dim G_{n_1,n_2}(\Gamma_1(N), \mathbb{C})$.

This is because the p part of the representation corresponding to ψ is the principal series representation coming from χ_1 and χ_2 where in the inert case, up to roots of unity, $p\chi_1(p) = p^{n_2+1}$ and $p\chi_2(p) = p^{n_1+1}$. While in the split case:

- $p^{\frac{1}{2}}\chi_1(\pi) = \bar{\pi}^{n_2+1}$
- $p^{\frac{1}{2}}\chi_2(\pi) = \pi^{n_1+1}$
- $p^{\frac{1}{2}}\chi_1(\bar{\pi}) = \pi^{n_2+1}$
- $p^{\frac{1}{2}}\chi_2(\bar{\pi}) = \bar{\pi}^{n_1+1}$

4.3 Cohomology of the Boundary

In this section we want to describe the ordinary part of the cohomology of the boundary. We shall let M denote a finite or cofinite $\mathbb{Z}_p$-module with a continuous action of the multiplicative semi-group $M_2(\mathcal{O}) \cap GL_2(K)$ in the p-adic topology. Γ will denote a congruence subgroup of $SL_2(\mathcal{O})$ of the form:

$$\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \mid c \equiv 0 \bmod Np^r,\ d \equiv 0 \bmod Np^?\}$$

For B a Borel we shall let $\tilde{\Gamma}_B$ denote the unipotent radical of Γ_B. Then $\tilde{\Gamma}_B \lhd \Gamma_B$ and $W_B = \Gamma_B/\tilde{\Gamma}_B$ embeds into the roots of unity in K and so $[\Gamma_B : \tilde{\Gamma}_B]$ is prime to p. Thus we have:

$$res : H^\bullet(\Gamma_B, M) \xrightarrow{\sim} H^\bullet(\tilde{\Gamma}_B, M)^{W_B}$$

(from the Hoschild-Serre spectral sequence).

It is known that there is a bijection:

$$\begin{array}{rcl} \Gamma\backslash SL_2(\mathcal{O})/B_0 & \longleftrightarrow & \{\Gamma - \text{cusps}\} \\ \gamma & \longmapsto & \gamma B_0 \gamma^{-1} \end{array}$$

where $B_0 = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$. Then $\gamma^{-1}\tilde{\Gamma}_{\gamma B_0 \gamma^{-1}}\gamma = \gamma^{-1}\Gamma\gamma \cap \begin{pmatrix} 1 & \mathcal{O} \\ 0 & 1 \end{pmatrix}$ and we can think of $\tilde{\Gamma}_B$ as an ideal in $\mathcal{O}$. As B is its own normaliser we can easily deduce that this ideal depends only on the cusp $[B]$ and not on its representation $\gamma B_0 \gamma^{-1}$.We shall write $\tilde{\Gamma}_{[B]}$ for this ideal. Similarly we can legitimately write $W_{[B]}$ and $H^\bullet(\tilde{\Gamma}_{[B]}, M)$. Then we have canonically:

$$H^\bullet_\partial(\Gamma, M) \cong \bigoplus_{[B]} H^\bullet(\tilde{\Gamma}_{[B]}, M)^{W_{[B]}} \cong \bigoplus_{[B]} H^\bullet_{ct}(\tilde{\Gamma}_{[B],\, p}, M)^{W_{[B]}}$$

Here the " *ct* " indicates that we are using continuous cohomology, $\tilde{\Gamma}_{[B],\, p}$ denotes the closure of $\tilde{\Gamma}_{[B]}$ in $\mathcal{O}_p$ and the latter isomorphism follows because there is a bijection between $\tilde{\Gamma}_{[B]}$ cocycles (or coboundaries) and continuous $\tilde{\Gamma}_{[B],\, p}$ cocycles (coboundaries). We shall drop the " *ct* " from our notation in future. In the case $p = \pi\bar{\pi}$ is split in $\mathcal{O}$ we see that:

$$H^\bullet(\tilde{\Gamma}_{[B]}, M) = H^\bullet(\tilde{\Gamma}_{[B],\, \pi}, M) \oplus H^\bullet(\tilde{\Gamma}_{[B],\, \bar{\pi}}, M)$$

and that the $W_{[B]}$ action preserves this decomposition. Thus we can write:

$$H^\bullet_\partial(\Gamma, M) = H^\bullet_\partial(\Gamma, M)_\pi \oplus H^\bullet_\partial(\Gamma, M)_{\bar{\pi}}$$

We shall introduce the following definitions for a cusp $[B]$, and for$v$ a prime above p:

- $[B]$ is *v-unramified* if $\tilde{\Gamma}_{[B],\, v} = \mathcal{O}_v$.

- $[B]$ is *v-first class* if $[B]$ has a representative $\gamma B_0 \gamma^{-1}$ with $\gamma \in SL_2(\mathcal{O})$ and $\gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mod v^s for all s (or equivalently for $s = r)^{(1)}$.

- $[B]$ is *v- third class* if $[B]$ has a representative$\gamma B_0 \gamma^{-1}$ with $\gamma = \begin{pmatrix} a & * \\ c & * \end{pmatrix}$ where $(c, v) = 1$. In this case any representative has this form$^{(2)}$ and we can chose a representative with $a \equiv 0$ mod v^s for any $s^{(1)}$.

- $[B]$ is *v-second class* if it is neither v-first nor v-second class.

Here the results marked $^{(1)}$ follow as $\Gamma \supset \Gamma_1(Np^r)$ and those marked $^{(2)}$ as $\Gamma \subset \Gamma_0(p)$. Any first class cusp is unramified, as follows easily from the formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 - ac\alpha & a^2\alpha \\ -c^2\alpha & 1 + ac\alpha \end{pmatrix}$$

and the fact $^{(1)}$.

We shall prove:

Proposition 4.1 *If v is any prime above p and Γ and M are as above then we have an injection:*

$$e_v H^\bullet_\partial(\Gamma, M)_v \hookrightarrow \bigoplus_{[B]} H^\bullet(\tilde{\Gamma}_{[B],\,v}, M)$$

*where the sum is over v-*first class *cusps.*

Before proving this we shall draw the corollary that will be of use to us later:

Corollary 4.1 *If $\Gamma_1 \supset \Gamma_2$ are as Γ above and M is as above then the restriction map:*

$$e H^\bullet_\partial(\Gamma_1, M) \overset{res}{\hookrightarrow} H^\bullet_\partial(\Gamma_2, M)$$

is injective (on the ordinary part).

<u>Proof</u>: It will do to show that for each prime v above p

$$e_v H^\bullet_\partial(\Gamma_1, M)_v \overset{res}{\hookrightarrow} H^\bullet_\partial(\Gamma_2, M)_v$$

Then considering the commutative diagram:

$$\begin{array}{ccc} e_v H^\bullet_\partial(\Gamma_1, M)_v & \stackrel{res}{\longrightarrow} & H^\bullet_\partial(\Gamma_2, M) \\ res\downarrow & & \downarrow res \\ \bigoplus_{[B]\,v-class\,1} H^\bullet(\tilde{\Gamma}_{1[B],\,v}, M) & \stackrel{res}{\to} & \bigoplus_{[C]} H^\bullet(\tilde{\Gamma}_{2[C]}, M) \end{array}$$

the proposition tells us that the left hand vertical arrow is injective and so we need only show that the lower horizontal arrow is injective. But above each v-first class cusp $[B]$ of Γ_1 there lies a v-first class cusp $[C]$ of Γ_2, as we see at once from the definition, and

$$H^\bullet(\tilde{\Gamma}_{1[B],\,v}, M) \stackrel{\sim}{\longrightarrow} H^\bullet(\tilde{\Gamma}_{2[C],\,v}, M)$$

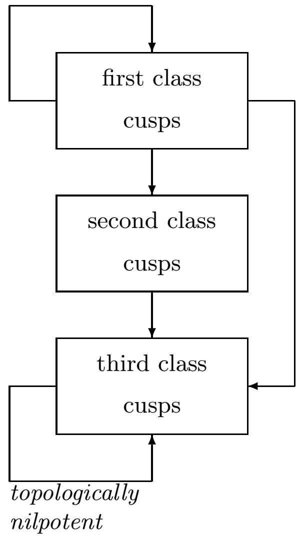
as $\tilde{\Gamma}_{1[B],\,v} = \mathcal{O}_v = \tilde{\Gamma}_{2[B],\,v}$.

We now turn to the proof of the proposition. We shall assume $p = \pi\bar{\pi}$ (which is the case of real interest and slightly harder than the inert case), and that $v = \pi$ ($v = \bar{\pi}$ is exactly the same).

Fix n such that $n > r$ and $\pi^n \equiv 1 \bmod N\bar{\pi}^r$ and set $g = \begin{pmatrix} \pi^n & 0 \\ 0 & 1 \end{pmatrix}$. We shall show that:

- If $x \in ker(H^\bullet_\partial(\tilde{\Gamma}, M)_\pi \to \bigoplus_{[B]\,class\,1} H^\bullet(\tilde{\Gamma}_{[B],\,\pi}, M))$ then $[\Gamma g\Gamma]x \in \bigoplus_{[B]\,class\,3} H^\bullet(\tilde{\Gamma}_{[B],\,\pi}, M)^{W_{[B]}}$ and this latter space is preserved by $[\Gamma g\Gamma]$.
- $[\Gamma g\Gamma] : \bigoplus_{[B]\,class\,3} H^\bullet(\tilde{\Gamma}_{[B],\,\pi}, M)^{W_{[B]}} \to \bigoplus_{[B]\,class\,3} H^\bullet(\tilde{\Gamma}_{[B],\,\pi}, M)^{W_{[B]}}$ is topologically nilpotent.

from which the proposition follows easily. To prove the second assertion it will do to take M of finite cardinality. Pictorially this all amounts to $[\Gamma g\Gamma]$ acting as follows:



Let $[B]$ be a Γ-cusp with $B = \delta B_0 \delta^{-1}$, $\delta = \begin{pmatrix} a & * \\ c & * \end{pmatrix} \in SL_2(\mathcal{O})$, we want to examine the $[B]$-component of $[\Gamma g \Gamma]$. Firstly the $\Gamma \cap g\Gamma g^{-1}$ cusps above $[B]$ are exactly $[\gamma_i^{-1} B \gamma_i]$ where $\Gamma = \coprod \Gamma_B \gamma_i (\Gamma \cap g\Gamma g^{-1})$ or (as we see after a small calculation) $\Gamma g \Gamma = \coprod \Gamma_B \gamma_i g \Gamma$. Thus the cusps $[C]$ such that $[\Gamma g \Gamma]$ gives a non-zero map from $H^\bullet(\Gamma_{[C]}, M)$ to $H^\bullet(\Gamma_{[B]}, M)$ are represented by $[g^{-1}\gamma_i^{-1} B \gamma_i g]$, or equivalently by $[g_u^{-1} B g_u]$ where $g_u = \begin{pmatrix} \pi^n & u \\ 0 & 1 \end{pmatrix}$ as u varies over congruence classes mod π^n and without loss of generality $u \equiv 0 \bmod N\bar{\pi}^r$. Explicitly $[C]$ is represented by $\varepsilon B_0 \varepsilon^{-1}$ with $\varepsilon = \begin{pmatrix} \frac{a-uc}{\pi^m} & * \\ \pi^{n-m}c & * \end{pmatrix} \in SL_2(\mathcal{O})$ for some m. Thus if $[B]$ is not third class (i.e. if $\pi | c$) then we see that $\pi \not| (a - uc)$ so that $m = 0$ and $[C]$ must be class one, which is our first claim.

For the second assertion consider $[B]$ a third class cusp, which we can write $[\delta B_0 \delta^{-1}]$ with $\delta = \begin{pmatrix} \pi^{2n}a & b \\ c & d \end{pmatrix}$ where $\pi \not| d$. Then if $[C]$ is a third class cusp giving rise to $[B]$ we must have $u \equiv 0 \bmod \pi^n$ (u as above) and so we may take $u = 0$ and $C = \varepsilon B_0 \varepsilon^{-1}$ with

$\varepsilon = \begin{pmatrix} \pi^n a & b \\ c & \pi^n d \end{pmatrix}$. Then $\lambda \equiv 0 \bmod N\bar{\pi}^r$ and $c\lambda \equiv d \bmod \pi^r$ implies that:

$$\varepsilon \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \Gamma_1(Np^r)$$

Thus $[B] = [C]$ and the map:

$$[\Gamma g \Gamma] : \bigoplus_{[B]\, class\, 3} H^{\bullet}(\tilde{\Gamma}_{[B],\,\pi}, M) \longrightarrow \bigoplus_{[B]\, class\, 3} H^{\bullet}(\tilde{\Gamma}_{[B],\,\pi}, M)$$

splits up as a direct sum of maps:

$$H^{\bullet}(\Delta_{1\,\pi}, M) \xrightarrow{g_*} H^{\bullet}(\Delta_{3\,\pi}, M) \xrightarrow{cor} H^{\bullet}(\Delta_{2\,\pi}, M)$$

where:

- $\Delta_1 = \tilde{\Gamma}_{\varepsilon B_0 \varepsilon^{-1}} = \left\{ \begin{pmatrix} 1 - \pi^n ac\alpha & \pi^{2n} a^2 \alpha \\ -c^2 \alpha & 1 + \pi^n ac\alpha \end{pmatrix} \in \Gamma \mid \alpha \in \mathcal{O} \right\}$
- $\Delta_2 = \tilde{\Gamma}_{\delta B_0 \delta^{-1}} = \left\{ \begin{pmatrix} 1 - \pi^{2n} ac\alpha & \pi^{4n} a^2 \alpha \\ -c^2 \alpha & 1 + \pi^{2n} ac\alpha \end{pmatrix} \in \Gamma \mid \alpha \in \mathcal{O} \right\}$
- $\Delta_3 = \Delta_2$ as $\Delta_2 \subset g\Gamma g^{-1}$

The map g_* is induced by the compatible maps $M \to M$, $m \mapsto gm$ and $\Delta_3 \hookrightarrow \Delta_1$ by:

$$\begin{pmatrix} 1 - \pi^{2n} ac\alpha & \pi^{4n} a^2 \alpha \\ -c^2 \alpha & 1 + \pi^{2n} ac\alpha \end{pmatrix} \longmapsto \begin{pmatrix} 1 - \pi^n ac(\pi^n \alpha) & \pi^{2n} a^2 (\pi^n \alpha) \\ -c^2 (\pi^n \alpha) & 1 + \pi^n ac(\pi^n \alpha) \end{pmatrix}$$

and *cor* reduces to the identity.

Thus we may describe $[\Gamma g \Gamma] : H^{\bullet}(\tilde{\Gamma}_{[B],\,\pi}, M) \to H^{\bullet}(\tilde{\Gamma}_{[B],\,\pi}, M)$ as the map induced by compatible maps $M \to M$ by $m \mapsto m$ and $\tilde{\Gamma}_{[B],\,\pi} \to \tilde{\Gamma}_{[B],\,\pi}$ by $\alpha \mapsto \pi^n \alpha$. Then if M is finite (as a set), for some a, $res : H^{\bullet}(\tilde{\Gamma}_{[B],\,\pi}, M) \to H^{\bullet}(\pi^{an}\tilde{\Gamma}_{[B],\,\pi}, M)$ is zero, and so by our above description $T^a_{\pi^n}$ is also zero, which is what we wanted to show.

4.4 Change of Level

The central result of this section is:

Proposition 4.2 *Let$N \in \mathcal{O}$ be prime to p, let $r \geq s \geq 1$ and let M be a $\mathbb{Z}_p$-module. Set $G_{r,s} = (1 + Np^s\mathbb{Z})/(1 + Np^r\mathbb{Z})$. Then*

i) $res : eH^\bullet(\Gamma_1(Np^s), M) \xrightarrow{\sim} eH^\bullet(\Gamma_1(Np^s) \cap \Gamma_0(Np^r), M)$

If further $eM^{\Gamma_1(Np^r)} = 0$ then

ii) $res : eH^1(\Gamma_1(Np^s) \cap \Gamma_0(Np^r), M) \xrightarrow{\sim} eH^1(\Gamma_1(Np^r), M)^{G_{r,s}}$

and hence

iii) $eH^1(\Gamma_1(Np^s), M) \xrightarrow{\sim} eH^1(\Gamma_1(Np^r), M)^{G_{r,s}}$

The modules $M = S_{n_1,n_2}(A)$ satisfy $eM^{\Gamma_1(Np^r)} = 0$ for A any $\mathcal{O}_\pi$-module.

Proof: Let us establish the notation:

$$\begin{array}{lcl} \Gamma_s & = & \Gamma_1(Np^s) \\ \bigcup & & \\ \Phi & = & \Gamma_1(Np^s) \cap \Gamma_0(Np^r) \\ \triangledown & & \\ \Gamma_r & = & \Gamma_1(Np^r) \end{array}$$

Then $\Phi/\Gamma_r \cong G_{r,s}$.

i) I claim the following diagram commutes:

$$\begin{array}{ccc} H^\bullet(\Gamma_s, M) & \xrightarrow{res} & H^\bullet(\Phi, M) \\ T_p^{r-s} \downarrow & \swarrow & \downarrow T_p^{r-s} \\ H^\bullet(\Gamma_s, M) & \xrightarrow{res} & H^\bullet(\Phi, M) \end{array}$$

where the diagonal arrow is given by the Hecke operator $[\Gamma_s \begin{pmatrix} p^{r-s} & 0 \\ 0 & 1 \end{pmatrix} \Phi]$. From this the result would follow at once as T_p is invertible on the ordinary part of any module. Further it suffices to check the commutativity in degree 0 by abstract nonsense, and here it follows from the existence of γ_u such that:

- $\Gamma_s \begin{pmatrix} p^{r-s} & 0 \\ 0 & 1 \end{pmatrix} \Gamma_s = \amalg \gamma_u \Gamma_s$

- $\Gamma_s \begin{pmatrix} p^{r-s} & 0 \\ 0 & 1 \end{pmatrix} \Phi = \amalg \gamma_u \Phi$

- $\Phi \begin{pmatrix} p^{r-s} & 0 \\ 0 & 1 \end{pmatrix} \Phi = \amalg \gamma_u \Phi$

(see Shimura [Sh2]).

ii) We look at the inflation-restriction sequence:

$$0 \to H^1(G_{r,s}, M^{\Gamma_r}) \stackrel{inf}{\to} H^1(\Phi, M) \stackrel{res}{\to} H^1(\Gamma_r, M)^{G_{r,s}} \stackrel{t}{\to} H^2(G_{r,s}, M^{\Gamma_r})$$

Let T_p and hence e act on these groups by giving them their normal action on the middle terms and letting them act on the outer terms through their action on $M^{\Gamma_r} = H^0(\Gamma_r, M)$. Assuming for a minute that these actions are compatible we see that there is an exact sequence:

$$0 \longrightarrow eH^1(\Phi, M) \stackrel{res}{\longrightarrow} eH^1(\Gamma_r, M)^{G_{r,s}} \longrightarrow 0$$

as desired.

To check the compatibility assertion let $^-$ denote the map:

$$\begin{array}{ccc} \Phi/\Gamma_r & \stackrel{\sim}{\longrightarrow} & G_{r,s} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & d \end{array}$$

Let also g_u be such that:

$$\Gamma_r \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_r = \amalg g_u \Gamma_r \text{ and } \Phi \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Phi = \amalg g_u \Phi$$

For $\gamma, \delta \in \Phi$ let $v = v(u)$ and $w = w(u)$ be the unique indices such that $g_u^{-1} \gamma g_v$ and $g_v^{-1} \delta g_w \in \Phi$. It is easily checked that $\overline{g_u^{-1} \gamma g_v} = \overline{\gamma}$ (and that $\overline{g_v^{-1} \delta g_w} = \overline{\delta}$). Then:

a) Let $\phi \in Z^1(G_{r,s}, M^{\Gamma_r})$. Then for $\gamma \in \Gamma_r$:

- $inf(T_p\phi)(\gamma) = (\sum g_u\phi)\,\overline{\gamma}$
- $(T_p\, inf\ \phi)(\gamma) = \sum g_u\phi(\overline{g_u^{-1}\gamma g_v})$

and so $inf(T_p\phi) = T_p(inf\ \phi)$.

b)Let $x \in H^1(\Gamma_r, M)^{G_{r,s}}$. Then there exists $\phi \in C^1(\Phi, M)$ such that:

- $\phi\mid_{\Gamma_r} \in Z^1(\Gamma_r, M)$ and represents x
- $d\phi \in Z^2(G_{r,s}, M^{\Gamma_r}) \subset C^2(\Phi, M)$ and represents $t(x)$

(see Hoschild-Serre [HS]).

$T_pt(x)$ is represented by $(\gamma, \delta) \mapsto \sum g_u(\phi(\gamma\delta) - \gamma\phi(\delta) - \phi(\gamma))$. Moreover consider $\psi \in C^1(\Phi, M)$ defined by $\psi(\gamma) = \sum g_u\phi(g_u^{-1}\gamma g_v)$. Then $\psi\mid_{\Gamma_r} \in Z^1(\Gamma_r, M)$ and represents T_px. Moreover

$$\begin{aligned}
(d\psi)(\gamma,\delta) &= \textstyle\sum_u g_u\phi(g_u^{-1}\gamma\delta g_w) - \gamma\sum_u g_v\phi(g_v^{-1}\delta g_w) - \sum_u g_u\phi(g_u^{-1}\gamma g_v) \\
&= \textstyle\sum_u g_u\left(\phi(g_u^{-1}\gamma g_v g_v^{-1}\delta g_w) - (g_u^{-1}\gamma g_v)\phi(g_v^{-1}\delta g_w) - \phi(g_u^{-1}\gamma g_v)\right) \\
&= \textstyle\sum_u g_u(d\phi)(g_u^{-1}\gamma g_v, g_v^{-1}\delta g_w) \\
&= \textstyle\sum_u g_u(d\phi)(\gamma,\delta) \quad \text{as } \overline{g_u^{-1}\gamma g_v} = \overline{\gamma} \text{ and } \overline{g_v^{-1}\delta g_w} = \overline{\delta} \\
&= \textstyle\sum_u g_u(\phi(\gamma\delta) - \gamma\phi(\delta) - \phi(\gamma))
\end{aligned}$$

Thus $(d\psi) \in Z^2(G_{r,s}, M^{\Gamma_r})$ and represents $T_pt(x)$, i.e. $t(T_px) = T_pt(x)$.

For the final assertion that $eS_{n_1,n_2}(A)^{\Gamma_r} = 0$ we shall make an arbitrary extension of T_p to all of $S_{n_1,n_2}(A)$ and show that:

$$T_pS_{n_1,n_2}(A) \subset pS_{n_1,n_2}(A)$$

from which it follows that $eS_{n_1,n_2}(A)$ and hence $eS_{n_1,n_2}(A)^{\Gamma_r}$ vanish. Choose a set of representatives $\{u\}$ for congruence classes of $\mathcal{O}$ mod p, and set:

$$T_pm = \sum_u \begin{pmatrix} p & u \\ 0 & 1 \end{pmatrix}.m$$

Then:

$$
\begin{aligned}
T_p m \quad &\equiv \quad \sum_u \begin{pmatrix}
0 & . & . & . & . & . & . & . & 0 & u^{n_1}\bar{u}^{n_2} \\
0 & . & . & . & . & . & . & . & 0 & u^{n_1}\bar{u}^{n_2-1} \\
. & & & & & & & & . & . \\
. & & & & & & & & . & . \\
0 & . & . & . & . & . & . & . & 0 & u^{n_1} \\
0 & . & . & . & . & . & . & . & 0 & u^{n_1-1}\bar{u}^{n_2} \\
. & & & & & & & & . & . \\
. & & & & & & & & . & . \\
0 & . & . & . & . & . & . & . & 0 & u^{n_1-1} \\
. & & & & & & & & . & . \\
. & & & & & & & & . & . \\
. & & & & & & & & . & . \\
0 & . & . & . & . & . & . & . & 0 & 1
\end{pmatrix} m \mod \pi \\
&\equiv \quad 0m \mod \pi
\end{aligned}
$$

as for any character $\chi : (\mathcal{O}/\pi\mathcal{O})^\times \to (\mathcal{O}/\pi\mathcal{O})^\times$ one has:

$$\sum_{u \in (\mathcal{O}/\pi\mathcal{O})^\times} \chi(u) = 0$$

. Q.E.D.

Now define $\mathcal{H}^\circ_{n_1,n_2} = \lim_{\to\, r} eH^1(\Gamma_1(Np^r), S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi))$. Then $\mathcal{H}^\circ_{n_1,n_2}$ is an $\mathcal{O}_\pi[[G]]$-module where $G = \lim_{\leftarrow} G_r$ with $G_r = (\mathcal{O}/Np^r\mathcal{O})^\times$. Let $H_r = ker(G \twoheadrightarrow G_r)$ then what we have just shown amounts to:

$$(\mathcal{H}^\circ_{n_1,n_2})^{H_r} = eH^1(\Gamma_1(Np^r), S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi)) \quad (r \geq 1)$$

If $\check{}$ denotes the Pontriagin dual this is the same as:

$$(\check{\mathcal{H}}^\circ_{n_1,n_2})_{H_r} = eH_1(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}_\pi))$$

We have the following decompositions for G:

$$G = G^{tor} \times H_1 = (\mathcal{O}/N\mathcal{O})^\times \times H_0 = (\mathcal{O}/N\mathcal{O})^\times \times (\mathcal{O}/p\mathcal{O})^\times \times H_1$$

Moreover complex conjugation acts on H_1 and we have $H_1 = H_1^+ \times H_1^-$ where $\pm$ refer to the corresponding eigenspaces for complex conjugation. We can choose isomorphisms $\mathbb{Z}_p \to H_1^\pm$

by $\lambda \mapsto u_{\pm}^{\lambda}$. Then

$$\mathcal{O}_\pi[[G]] \cong \mathcal{O}_\pi[[H_1]][G^{tor}] \cong \mathcal{O}_\pi[[T_+, T_-]][G^{tor}]$$

the latter map being given by $u_\pm \leftrightarrow (T_\pm + 1)$. $\mathcal{O}_\pi[[H_1]]$ is a complete local noetherian ring, its maximal ideal corresponds to $(\pi, T_+, T_-) \lhd \mathcal{O}_\pi[[T_+, T_-]]$.

Corollary 4.2 *$\check{\mathcal{H}}^\circ_{n_1,n_2}$ is a finitely generated $\mathcal{O}_\pi[[H_1]]$-module.*

Proof: By Nakayama's lemma it will do to show that $\check{\mathcal{H}}^\circ_{n_1,n_2}$ is compact and that $(\check{\mathcal{H}}^\circ_{n_1,n_2})_{H_1} = (\check{\mathcal{H}}^\circ_{n_1,n_2}/(T_+, T_-)\check{\mathcal{H}}^\circ_{n_1,n_2})$ is finitely generated over $\mathcal{O}_\pi$. The proposition and the fact that $\check{\mathcal{H}}^\circ_{n_1,n_2} = \varprojlim eH_1(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}_\pi))$ reduces this to the well known fact that $H_1(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}_\pi))$ is a finitely generated $\mathcal{O}_\pi$-module.

From the corollary of section 4.3 we have for $r \geq s \geq 1$ a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & eH^1_{cusp}(\Gamma_1(Np^s), M) & \to & eH^1(\Gamma_1(Np^s), M) & \to & eH^1_{Eis}(\Gamma_1(Np^s), M) & \to & 0 \\ & & \downarrow & & \downarrow\wr & & \downarrow & & \\ 0 & \to & eH^1_{cusp}(\Gamma_1(Np^r), M)^{G_{r,s}} & \to & eH^1(\Gamma_1(Np^r), M)^{G_{r,s}} & \to & eH^1_{Eis}(\Gamma_1(Np^r), M)^{G_{r,s}} & & \end{array}$$

where the right hand vertical arrow is an injection and where $M = S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi)$. Thus the left hand vertical arrow is an isomorphism, and if we set $\mathcal{H}^\circ_{n_1,n_2\,cusp}$ to be $\varinjlim eH^1_{cusp}(\Gamma_1(Np^r), S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi))$ then we see that:

- $\mathcal{H}^\circ_{n_1,n_2\,cusp} \hookrightarrow \mathcal{H}^\circ_{n_1,n_2}$
- $\mathcal{H}^\circ_{n_1,n_2\,cusp}$ is a finitely generated $\mathcal{O}_\pi[[H_1]]$-module.
- $(\mathcal{H}^\circ_{n_1,n_2\,cusp})^{H_r} = eH^1_{cusp}(\Gamma_1(Np^r), S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi))$ for $r \geq 1$.

4.5 Change of Weight

For n_1, n_2 integers set:

$$\begin{array}{rcl} \mu_{n_1,n_2} : G = (\mathcal{O}/N\mathcal{O})^\times \times H_0 \to H_0 = \mathcal{O}_p^\times & \to & \mathcal{O}_p^\times \to \mathcal{O}_\pi^\times \\ \alpha & \mapsto & \alpha^{n_1}\bar{\alpha}^{n_2} \end{array}$$

This extends to a homomorphism $\mu_{n_1,n_2} : \mathcal{O}_\pi[[G]] \to \mathcal{O}_\pi$ and to a homomorphism $\tilde{\mu}_{n_1,n_2}$: $\mathcal{O}_\pi[[G]] \to \mathcal{O}_\pi[[G]]$ defined by sending $h \in G$ to $\mu_{n_1,n_2}(h)h$. Then

$$\begin{array}{ccc} \mathcal{O}_\pi[[G]] & \stackrel{\tilde{\mu}_{n_1,n_2}}{\longrightarrow} & \mathcal{O}_\pi[[G]] \\ \mu_{n_1,n_2} \searrow & & \swarrow \mu_{0,0} \\ & \mathcal{O}_\pi & \end{array}$$

commutes. For M an $\mathcal{O}_\pi[[G]]$-module we define a twist $M(\mu_{n_1,n_2})$ to be the same underlying topological abelian group but with a new $\mathcal{O}_\pi[[H_0]]$ action defined by:

$$h.m = \tilde{\mu}_{n_1,n_2}(h)m$$

Our aim is to prove:

Proposition 4.3 *There is a canonical Hecke equivariant isomorphism:*

$$\mathcal{H}^\circ_{n_1,n_2} \xrightarrow{\sim} \mathcal{H}^\circ_{0,0}(\mu_{n_1,n_2})$$

It also restricts to an isomorphism on the cuspidal parts.

Before proving this we note a couple of corollaries. From now on we will use simply $\mathcal{H}^\circ$ to denote $\mathcal{H}^\circ_{0,0}$.

Corollary 4.3 *Let I be the closed ideal of $\mathcal{O}_\pi[[H_1]]$ generated by $\{h - \mu_{n_1,n_2}(h) \mid h \in H_r\}$ then $\check{\mathcal{H}}^\circ / I\check{\mathcal{H}}^\circ \cong eH_1(\Gamma_1(np^r), S_{n_1,n_2}(\mathcal{O}_\pi))$, and a similar statement holds for the cuspidal part.*

This is clear.

Corollary 4.4 *$\check{\mathcal{H}}^\circ_{cusp}$ is a torsion $\mathcal{O}_\pi[[H_1]]$-module.*

<u>*Proof:*</u> Let I_r denote the closed ideal generated by $\{h - \mu_{0,1}(h) \mid h \in H_r\}$ and let $\mathcal{R}$ denote $\mathcal{O}_\pi[[H_1]]$ and M denote $\check{\mathcal{H}}^\circ$. Then we have that M/I_rM is p-torsion and that $\mathcal{R}/I_r$ is free of p-torsion. If M were not a torsion $\mathcal{R}$-module we would have an injection $\mathcal{R} \stackrel{\phi}{\hookrightarrow} M$. Let N denote the submodule $\{n \in M \mid \exists r \in \mathcal{R} \,\text{with}\, rn \in Im\,\phi\}$. Then we can define $\theta : N \to F_{\mathcal{R}}$

such that $\theta \circ \phi = \mathrm{Id}_{\mathcal{R}}$ and there is then a non-zero constant $\lambda \in \mathcal{R}$ such that $\lambda Im\ \theta \subset \mathcal{R}$. Then we have for each r:

$$M/I_rM \hookleftarrow N/I_rN \twoheadrightarrow \lambda\, Im\, \theta/I_r\lambda\, Im\, \theta \twoheadrightarrow (\lambda\, Im\, \theta + I_r)/I_r \hookleftarrow ((\lambda) + I_r)/I_r \hookrightarrow \mathcal{R}/I_r$$

Now M/I_rM is p-torsion and thus $((\lambda) + I_r)/I_r$ is p-torsion and so in fact zero. Thus $\lambda \in I_r\ \forall r$ which implies $\lambda = 0$ a contradiction.

We now turn to the proof of the proposition. We have that:

$$\mathcal{H}^{\circ}_{n_1,n_2} = \varinjlim eH^1(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O}))$$

where the i^{th} term on the right hand side is an $\mathcal{O}/\pi^r\mathcal{O}[(\mathcal{O}/Np^r)^{\times}]$-module. Call this ring Λ_r, then $\mathcal{O}_{\pi}[[G]] = \varprojlim \Lambda_r$ and this is compatible with the above direct limit. μ_{n_1,n_2} reduces to a map $\Lambda_r \to \mathcal{O}/\pi^r\mathcal{O}$.

Let

$$\begin{array}{rcl} j : S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O}) & \longrightarrow & \mathcal{O}/\pi^r\mathcal{O} \\ \left(\begin{array}{c} x_0 \\ \cdot \\ \cdot \\ x_{n_1n_2} \end{array}\right) & \longmapsto & x_{n_1n_2} \end{array}$$

where we choose a basis of $S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O})$ as described at the start of section 4.2. It is a map of modules over the semi-group:

$$\Delta = \{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in M_2(\mathcal{O})\ |\ ad - bc \neq 0,\, c \equiv 0 \bmod p^r,\, d \equiv 1 \bmod p^r\}$$

It thus induces a Hecke equivariant map:

$$j_* : H^{\bullet}(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O})) \longrightarrow H^{\bullet}(\Gamma_1(Np^r), \mathcal{O}/\pi^r\mathcal{O})$$

and for $r \geq s \geq 1$:

$$\begin{array}{ccc} H^{\bullet}(\Gamma_1(Np^s), S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O})) & \xrightarrow{j_*} & H^{\bullet}(\Gamma_1(Np^s), \mathcal{O}/\pi^r\mathcal{O}) \\ \mathrm{res} \downarrow & & \downarrow \mathrm{res} \\ H^{\bullet}(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O})) & \xrightarrow{j_*} & H^{\bullet}(\Gamma_1(Np^r), \mathcal{O}/\pi^r\mathcal{O}) \end{array}$$

commutes. It is easily checked that j_* gives a map of Λ_r-modules:

$$H^\bullet(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O})) \longrightarrow H^\bullet(\Gamma_1(Np^r), \mathcal{O}/\pi^r\mathcal{O})(\mu_{n_1,n_2})$$

(In fact it suffices to check that for $\sigma \in (\mathcal{O}/p^r\mathcal{O})^\times$ we have $j_* \circ \sigma = \sigma^{n_1}\bar{\sigma}^{n_2}(\sigma \circ j_*)$ and such an equality need only be checked in degree zero where it really is easy.)

Finally we know from lemma 1.1 that j_* is an isomorphism on ordinary parts as $j : gS_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O}) \xrightarrow{\sim} g(\mathcal{O}/\pi^r\mathcal{O})$ for $g = \begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}$.

4.6 Cyclotomic Hida Families

First let us introduce a definition of an eigenform (of the Hecke operators having weight n, level Np^r, and character $\chi : (\mathcal{O}/Np^r\mathcal{O})^\times \to K^{ac\,\times}$) suited to our purposes. First note that it is easy to see that there are natural bijections between the homomorphisms of each of the following forms:

- $\lambda : \mathbb{T}(H^1(\Gamma_1(Np^r), S_{n,n}(K_\pi/\mathcal{O}_\pi))^\chi) \to \mathbb{O}_\pi$
- $\lambda : \mathbb{T}(H_1(\Gamma_1(Np^r), S_{n,n}(\mathcal{O}_\pi))_\chi) \to \mathbb{O}_\pi$
- $\lambda : \mathbb{T}(H_1(\Gamma_1(Np^r), S_{n,n}(K_\pi))_\chi) \to \mathbb{C}_\pi$
- $\lambda : \mathbb{T}(H^1(\Gamma_1(Np^r), S_{n,n}(K_\pi))^\chi) \to \mathbb{C}_\pi$
- $\lambda : \mathbb{T}(H^1(\Gamma_1(Np^r), S_{n,n}(\mathcal{O}_\pi))_\chi) \to \mathbb{O}_\pi$
- $\lambda : \mathbb{T}(H_1(\Gamma_1(Np^r), S_{n,n}(K_\pi/\mathcal{O}_\pi))^\chi) \to \mathbb{O}_\pi$
- $\lambda : \mathbb{T}(H^1(\Gamma_1(Np^r), S_{n,n}(\mathbb{C}))^\chi) \to \mathbb{C}$ (using our embedding of K^{ac} into $\mathbb{C}_\pi$ and $\mathbb{C}$ and the fact that the eigenvalues of the Hecke operators are algebraic)

We will call such a homomorphism an *eigenform* of weight n, level Np^r , and character χ. We will call it *ordinary* (respectively *cuspidal*) if it factors through the Hecke algebra acting on the ordinary (respectively cuspidal) part of the cohomology. Thus the cuspidal eigenforms correspond to homomorphisms:

- $\lambda : \mathbb{T}(S_n(\Gamma_1(Np^r), \mathbb{C})_\chi) \to \mathbb{C}$

Let G, H_1, etc. be as in the last section. Let $\Lambda = \mathcal{O}_\pi[[H_1^+]]$ and let $\mathcal{R}$ denote the integers in the algebraic closure of F_Λ. If $\chi : G^{tor} \times H^- \to \mathbb{O}_\pi^\times$ is a finite character we shall mean by an (ordinary) Λ-*adic eigenform* of level N and character χ a homomorphism:

$$\boldsymbol{\lambda} : \mathbb{T}(\mathcal{H}^{\circ\,\chi}) \longrightarrow \mathcal{R}$$

or equivalently:

$$\boldsymbol{\lambda} : \mathbb{T}(\check{\mathcal{H}}^{\circ\,\chi}) \longrightarrow \mathcal{R}$$

We shall call it cuspidal if it factors through $\mathcal{H}^{\circ\,\chi}_{cusp}$.

For $\psi : H_1 \to \mathbb{O}_\pi^\times$ a finite character and n a positive integer set:

$$\begin{array}{rcl} \psi_n : H_1 & \longrightarrow & \mathbb{O}_\pi^\times \\ h & \longmapsto & h^{2n}\psi(h) \end{array}$$

and denote also its extension to a map $\Lambda \to \mathbb{O}_\pi$ by ψ_n. Let $\wp_{\psi,n}$ be the kernel of ψ_n and $\mathcal{Q}$ a prime of $\mathcal{R}$ above $\wp_{\psi,n}$. Also let $\omega : (\mathcal{O}/p\mathcal{O})^\times \to \boldsymbol{\mu}(\mathcal{O}_\pi)$ denote the unique lifting of the norm map $(\mathcal{O}/p\mathcal{O})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ (i.e. the Teichmuller character). Then we have:

$$\begin{array}{rcccl} & \mathbb{T}(\check{\mathcal{H}}^\circ_\chi) & \longrightarrow & \mathcal{R} & \\ & \downarrow & & \downarrow & \\ \mathbb{T}(eH_1(\Gamma_1(Np^r), S_{n,n}(\mathcal{O}_\pi))_{\chi\psi\omega^{-n}}) = & \mathbb{T}(\check{\mathcal{H}}^\circ_\chi/\wp_{\psi,n}\check{\mathcal{H}}^\circ_\chi) & - \to & \mathcal{R}/\mathcal{Q} & \subset \mathbb{O}_\pi \end{array}$$

where the vertical arrows are surjective and the map across the bottom making the diagram commute exists and is unique. (It exists because if I is the kernel of the vertical map on the left then $I \subset \sqrt{\wp_{\psi,n}}$ (as $\check{\mathcal{H}}^\circ_\chi$ is finitely generated) and $\sqrt{\wp_{\psi,n}} \subset ker(\mathbb{T}(\check{\mathcal{H}}^\circ_\chi) \to \mathcal{R}/\mathcal{Q})$ because $\wp_{\psi,n}$ is contained in this kernel.) This just says that for each $\psi, n, \mathcal{Q}$ as above $\boldsymbol{\lambda}$ reduces modulo $\mathcal{Q}$ to a unique ordinary eigenform of weight n, level Np^r for sufficiently large r, and character $\chi\psi\omega^{-n}$. A similar statement is true for cuspidal Λ-adic eigenforms.

We will prove a partial converse to this.

Theorem 4.1 *Fix $N \in \mathcal{O}$. Then :*

1. *There are only finitely many cuspidal Λ-adic eigenforms of level N.*

2. *If we also fix a finite character $\chi : G^{tor} \times H^- \to \mathbb{O}_\pi^\times$ then for almost all ordinary eigenforms λ of level Np^r, weight n and character $\psi\chi$ for some r, n and ψ a cyclotomic character*
   (i.e. $\psi : (1 + p\mathcal{O}/1 + p^r\mathcal{O}) \to \mathbb{O}_\pi^\times$ and $\psi(\alpha) = \psi(\bar{\alpha})$) there is a Λ-adic eigenform $\boldsymbol{\lambda}$ of level N, a character $\chi\omega^n$ and a prime $\mathcal{Q}$ of $\mathcal{R}$ above $\wp_{\psi,n}$ such that $\boldsymbol{\lambda}$ reduces to λ modulo $\mathcal{Q}$.

3. *For all but finitely many finite characters $\chi : G^{tor} \times H^- \to \mathbb{O}_\pi^\times$ there are only finitely many cuspidal eigenforms of weight n, level Np^s, and character $\chi\psi$ as s, n and ψ vary, with ψ cyclotomic.*

Remark It would be very nice to be able to strengthen part 2 to assert that we could lift any ordinary cuspidal eigenform to a unique Λ-adic one. This is false if one does not restrict to the cuspidal part as the ordinary Eisenstein series come in two variable p-adic families. However the examples of cuspidal forms coming either by base change from $GL_2/\mathbb{Q}$ or by theta series from grossencharacters on a quadratic extension of K fit nicely into cyclotomic families. One could prove this strengthening if one knew that $\check{\mathcal{H}}^\circ_{cusp}$ was a free Λ-module. This latter statement is equivalent to $eH_{1\,cusp}(\Gamma_1(Np), S_{n,n}(\mathcal{O}_\pi))$ being torsion free for infinitely many n. This is a consequence of the following result, which in turn follows easily from Nakayama's lemma and unique factorisation:

> If $\pi_i \in \Lambda$ are infinitely many distinct primes (all different from p) and M is a compact Λ-module then the following are equivalent:
>
> - M is a free Λ-module
> - for all i $M/\pi_i M$ is p-torsion free.

Proof: Note that part 3 follows from 1 and 2.

1) First note that $\check{\mathcal{H}}^\circ_{cusp\,\chi}$ is a finite torsion Λ-module for all but finitely many χ and that there are no Λ-adic eigenforms for such χ. (Infact $\mathcal{O}_\pi[[G]]$ has only finitely many height one primes $\mathcal{P}$ such that $\check{\mathcal{H}}^\circ_{cusp\,\mathcal{P}} \neq 0$ (such a prime is a minimal element of the support of $\check{\mathcal{H}}^\circ_{cusp}$) and so only finitely many such that $\mathcal{O}_\pi[[G]]/\mathcal{P} \hookrightarrow \check{\mathcal{H}}^\circ_{cusp}/\mathcal{P}\check{\mathcal{H}}^\circ_{cusp}$.) For the finite set of χ for which $\check{\mathcal{H}}^\circ_{cusp\,\chi}$ is not a torsion Λ-module it is finite and hence $\mathbb{T}(\check{\mathcal{H}}^\circ_{cusp\,\chi})$ is a finite Λ-algebra so there are only finitely many Λ-adic eigenforms of character χ.

2) Let $\mathcal{T} = \mathbb{T}(\check{\mathcal{H}}^\circ_{\chi\omega^m})$ for some $m = 0, \ldots p-2$. Then for $n \equiv m \bmod (p-1)$ $\mathcal{T} \twoheadrightarrow \mathbb{T}(H_1(\Gamma_1(Np^r), S_{n,n}(\mathcal{O}_\pi))_{\chi\psi})$, and an ordinary eigenform λ of weight n character $\chi\psi$ can be thought of as a map $\mathcal{T} \xrightarrow{\lambda} \mathbb{O}_\pi$. If $ker\,\lambda$ is not a minimal prime ideal then there is a prime $\mathcal{P}$ such that $ker\,\lambda$ strictly contains $\mathcal{P}$ and such that $\mathcal{T}/\mathcal{P} \hookrightarrow \mathcal{R}$ and so λ can be lifted. But $\mathcal{T}$ has only finitely many minimal prime ideals (as it is finite over an integral domain) and so there are only finitely many maps $\mathcal{T} \xrightarrow{\lambda} \mathbb{O}_\pi$ whose kernel is a minimal prime of depth 2. This implies that there are only finitely many eigenforms λ as in the theorem of weight $n \equiv m \bmod (p-1)$ which can not be lifted to a Λ-adic eigenform, which at once implies the result.

Finally in this section we would like to give some examples to show that our theory is not vacuous. Although one could describe these examples very precisely we shall content ourselves with an existence theorem.

Proposition 4.4 *Assume that $p = \pi\bar{\pi}$ is split in K then:*

1. *Let $N \in \mathbb{Z}$ be prime to p and such that there is an ordinary cuspidal eigenform of weight $k \geq 2$ and level Np^r for some r for $GL_2/\mathbb{Q}$. Then there is a cuspidal Λ-adic eigenform over K of level N.*

2. *If $\chi : (\mathcal{O}/p^r\mathcal{O})^\times \to \mathbb{C}_\pi^\times$ is an anticyclotomic character (i.e. $\chi(\bar{\alpha}) = \chi(\alpha)^{-1}$ for $\alpha \in (1 + p\mathcal{O})$) then there is an integer N prime to p and a character $\prime\chi : (\mathcal{O}/N\mathcal{O})^\times \to \mathbb{C}_\pi^\times$ and a Λ-adic eigenform over K of level N and character $\chi\prime\chi$.*

Proof: *1)* This comes from base change.

More precisely if the assumptions of the proposition hold then we know from the work of Hida (see for example [Hi1]) that as r varies there are infinitely many ordinary cuspidal eigenforms of level Np^r and weight k for $GL_2/\mathbb{Q}$. Hence there are infinitely many cuspidal automorphic representations $\rho = \rho_p \otimes \rho_f^p \otimes \rho_\infty$ of $GL_2/\mathbb{Q}$ such that:

- ρ_∞ is the discrete series representation whose infinitessimal character has Harish-Chandra parameter $(\frac{1}{2}, \frac{3}{2} - k) \in \mathbb{C}^2 = X_*(T)_\mathbb{C}$where we identify $X_*(T)$ with $\mathbb{Z}^2$ by $(n, m) : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto a^n b^m$.

- ρ_p is either a principal series or special representation $\rho(\psi_1, \psi_2)$ with (say) ψ_1 unramified and $\psi_1(p)$ a p-adic unit.

Let $\tilde{\rho} = \tilde{\rho}_p \otimes \tilde{\rho}_f^p \otimes \tilde{\rho}_\infty$ denote the base change of ρ to GL_2/K. Then $\tilde{\rho}_\infty$ is the irreducible principal series corresponding to the character:

$$\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \longmapsto \left(\frac{a}{|a|}\right)^{n+1} |a|^{-n} \left(\frac{\bar{b}}{|b|}\right)^{n+1} |b|^{-n}$$

(To check this one switches to the corresponding representations of the Weil groups $W_\mathbb{R}$ and $W_\mathbb{C}$. ρ_∞ corresponds to:

$$\text{by} \qquad \begin{array}{rcl} W_\mathbb{R} = \mathbb{C}^\times \rtimes \{1, c\} & \longrightarrow & GL_2(\mathbb{C}) \\ c & \longmapsto & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \mathbb{C}^\times \ni z & \longmapsto & \begin{pmatrix} z^{\frac{1}{2}} \bar{z}^{\frac{3}{2}-k} & 0 \\ 0 & z^{\frac{3}{2}-k} \bar{z}^{\frac{1}{2}} \end{pmatrix} \end{array}$$

and so $\tilde{\rho}_\infty$ is the representation corresponding to :

$$\begin{array}{rcl} W_\mathbb{C} = \mathbb{C}^\times & \longrightarrow & GL_2(\mathbb{C}) \\ z & \longmapsto & \begin{pmatrix} \left(\frac{z}{|z|}\right)^{k-1} |z|^{2-k} & 0 \\ 0 & \left(\frac{\bar{z}}{|z|}\right)^{k-1} |z|^{2-k} \end{pmatrix} \end{array}$$

which corresponds to the Langlands' quotient of the principal series described above, which is in fact irreducible.)

Thus $(e\tilde{\rho}_p^{U_p}) \times (\tilde{\rho}_f^p)^{U^p} \hookrightarrow eH_{1,\,cusp}(\Gamma_1(Np^r), S_{k-2,k-2}(\mathbb{C}))$ where:

$$U_p = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_p) \ | \ c \equiv 0 \bmod Np^r \,, \ \ d \equiv 1 \bmod Np^r \}$$

$$U^p = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \prod_{v \not| p} GL_2(\mathcal{O}_v) \ | \ c \equiv 0 \bmod N \,, \ \ d \equiv 1 \bmod N \}$$

Moreover $\rho_p^{U_p} \neq 0$ (resp. $(\rho_f^p)^{U^p} \neq 0$) implies that $\tilde{\rho}_p^{U_p} \neq 0$ (resp. $(\tilde{\rho}_f^p)^{U^p} \neq 0$). As p is split for $v \mid p$ $\tilde{\rho}_v = \rho_p$ and $e_v\tilde{\rho}_v = e\rho_p \neq 0$.

In summary we have for each $k \geq 2$ a cuspidal automorphic representation $\tilde{\rho} = \tilde{\rho}_p \otimes \tilde{\rho}_f^p \otimes \tilde{\rho}_\infty$ such that:

$$0 \neq (e\tilde{\rho}_p^{U_p}) \times (\tilde{\rho}_f^p)^{U^p} \hookrightarrow eH_{1,\,cusp}(\Gamma_1(Np^r), S_{k-2,k-2}(\mathbb{C}))$$

Thus by theorem 4.1 we are done.

2) This comes from using the Weil lifting.

More precisely let L/K be a quadratic extension which is Galois over $\mathbb{Q}$ and in which π and $\bar{\pi}$ split. Let σ_1, σ_2, $\bar{\sigma}_1$, $\bar{\sigma}_2$ be the embeddings $L \hookrightarrow K^{ac}$, with the first two extending the identity on K and with $\bar{\sigma}_i(\alpha) = \overline{\sigma_i(\alpha)}$. Let $\infty 1$, $\infty 2$ be the infinite places corresponding to σ_1 and σ_2; and let $v1$, $v2$, $\overline{v1}$, $\overline{v2}$ be the primes above p corresponding to σ_1, σ_2, $\bar{\sigma}_1$, $\bar{\sigma}_2$. We can find an integer M prime to p, all whose prime divisors split from K to L, and which is such that, for each $n \geq 0$, there are grossencharacters χ_1, χ_2 of l such that:

- $$\begin{array}{rcl} \chi_{1\infty} : \mathbb{C}^\times \times \mathbb{C}^\times & \longrightarrow & \mathbb{C}^\times \\ (a,b) & \longmapsto & \left(\frac{a}{|a|}\right)^{n+1} |a|^{-n} \left(\frac{\bar{b}}{|b|}\right)^{n+1} |b|^{-n} \end{array}$$
- conductor(χ_1) divides M
- χ_2 is a finite character
- conductor(χ_2) divides $M(v2)^\infty \overline{v1}^\infty$
- $\chi_{2\,v2} \times \chi_{2\,\overline{v1}} : (\mathbb{Z}_p^\times)^2 \to \mathbb{C}^\times$ equals $\chi_\pi \times \chi_{\overline{\pi}} : (\mathbb{Z}_p^\times)^2 \to \mathbb{C}^\times$

(This is possible because for ϵ in a subgroup of finite index in the units of L, $\sigma_i\epsilon \in \mathbb{R}_{>0}$ for $i = 1, 2$ and so $\chi_{1\,\infty}(\epsilon) = 1$.) In this case $p^{\frac{1}{2}}\chi_{1\,v1}(v1)$ and $p^{\frac{1}{2}}\chi_{1\,\overline{v2}}(\overline{v2})$ are $v1$-adic units.

Then let ϑ denote the Weil lifting of $\chi_1\chi_2$ to an automorphic representation of GL_2/K. Then:

- ϑ is cuspidal as $\chi_{1\,\infty1} \neq \chi_{1\,\infty2}$
- ϑ_∞ is the principal series representation corresponding to:

$$\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \longmapsto \left(\frac{a}{|a|}\right)^{n+1} |a|^{-n} \left(\frac{\bar{b}}{|b|}\right)^{n+1} |b|^{-n}$$

- ϑ_π is the principal series representation $\rho(\phi_1, \phi_2)$ with ϕ_1 unramified, $p^{\frac{1}{2}}\phi_1(\pi)$ a $v1$-adic unit and $\phi_2|_{\mathcal{O}_\pi^\times} = \chi_\pi|_{\mathcal{O}_\pi^\times}$
- $\vartheta_{\bar{\pi}}$ is the principal series representation $\rho(\phi_1', \phi_2')$ with ϕ_1' unramified, $p^{\frac{1}{2}}\phi_1'(\bar{\pi})$ a $v1$-adic unit and $\phi_2'|_{\mathcal{O}_{\bar{\pi}}^\times} = \chi_{\bar{\pi}}|_{\mathcal{O}_{\bar{\pi}}^\times}$
- $(\vartheta_f^p)^{U^p} \neq 0$ where U^p is as above with $N = M^2$

Thus:

$$0 \neq (e\vartheta_p^{U_p}) \otimes (\vartheta_f^p)^{U^p} \hookrightarrow eH^1(\Gamma_1(Np^r), S_{n,n}(\mathbb{C}))$$

with U_p also as above, for some r. As this is true for a single value of N and for all $n \geq 0$ we are done by theorem 4.1.

4.7 Torsion

Lastly we shall use our results to exhibit torsion in the first homology group of certain sheaves. First we shall recall briefly what is obvious. There are four things we might look at which fall by Pontriagin duality into two pairs:

1. torsion in H_1 with coefficients in $S_{n_1,n_2}(\mathcal{O}) \longleftrightarrow$ lack of p-divisibility in H^1 with coefficients in $S_{n_1,n_2}(K/\mathcal{O})$.

2. torsion in H^1 with coefficients in $S_{n_1,n_2}(\mathcal{O}) \longleftrightarrow$ lack of p-divisibility in H_1 with coefficients in $S_{n_1,n_2}(K/\mathcal{O})$

The obvious way to look for torsion is to consider the long exact sequence corresponding to:

$$0 \longrightarrow S_{n_1,n_2}(\mathcal{O}) \stackrel{\alpha}{\longrightarrow} S_{n_1,n_2}(\mathcal{O}) \longrightarrow S_{n_1,n_2}(\mathcal{O}/\alpha\mathcal{O}) \longrightarrow 0$$

In cohomology this gives for Γ a congruence subgroup:

$$0 \longrightarrow S_{n_1,n_2}(\mathcal{O}/\alpha\mathcal{O})^{\Gamma} \longrightarrow H^1(\Gamma, S_{n_1,n_2}(\mathcal{O}))_{\alpha} \longrightarrow 0$$

which in some sense describes explicitly the torsion in this case. However in homology we get:

$$H_2(\Gamma, S_{n_1,n_2}(\mathcal{O})) \longrightarrow H_2(\Gamma, S_{n_1,n_2}(\mathcal{O}/\alpha\mathcal{O})) \longrightarrow H_1(\Gamma, S_{n_1,n_2}(\mathcal{O}))_{\alpha} \longrightarrow 0$$

which seems to be very little help as H_2 is as mysterious as H_1. (Dually if we look at $0 \to \mathcal{O}/\alpha\mathcal{O} \to K/\mathcal{O} \stackrel{\alpha}{\to} K/\mathcal{O} \to 0$ we again see that we get an answer for ii) but not for i).) Our methods allow us to say something about case i) in the case $n_1 \neq n_2$.

Theorem 4.2 *Fix n_1, n_2 with $n_1 \neq n_2$, and suppose $\dim eH_{1,\,cusp}(\Gamma_1(Np^r), \mathcal{O}_\pi) > 0$, which is certainly the case if there is a Λ-adic eigenform of level N. Then for $t \geq s \geq r$ with $n_1 \equiv n_2 \equiv 0 \bmod p^{t-s}$, $H_1(\Gamma_1(Np^s), S_{n_1,n_2}(\mathcal{O}_\pi))$ and $H_{1,\,cusp}(\Gamma_1(Np^s), S_{n_1,n_2}(\mathcal{O}_\pi))$ have torsion of exponent at least p^t.*

Notes: 1) Examples of values of N and p for which we can apply the theorem are provided easily by proposition 4.4 of the last section.

2) If $H_{1,\,cusp}(\Gamma_1(Np^s), S_{n_1,n_2}(\mathcal{O}_\pi))$ has torsion of exponent of order p^t then the same is true for the relative homology $H_1(\overline{\Gamma_1(Np^s)\backslash\mathcal{Z}}, \partial(\overline{\Gamma_1(Np^s)\backslash\mathcal{Z}}), \tilde{S}_{n_1,n_2}(\mathcal{O}_\pi))$ as we see from the exact sequence:

$$\ldots \longrightarrow H_1(\partial(\overline{\Gamma\backslash\mathcal{Z}}), \tilde{M}) \longrightarrow H_1(\Gamma\backslash\mathcal{Z}, \tilde{M}) \longrightarrow H_1(\overline{\Gamma\backslash\mathcal{Z}}, \partial(\overline{\Gamma\backslash\mathcal{Z}}), \tilde{M}) \longrightarrow \ldots$$

3) These are not the most precise results that can be proved by these methods but give a good indication of the type of question that can be treated.

4) We could deduce a less precise theorem from our general theory using commutative algebra, but we shall go back to the method of proof as it yields more precise information more easily.

Proof: Without loss of generality $r = s \geq 1$.

We have from the proof of theorem 4.1 an isomorphism:

$$j_* : eH^1(\Gamma_1(Np^t), S_{n_1,n_2}(\mathcal{O}/\pi^t\mathcal{O})) \xrightarrow{\sim} eH^1(\Gamma_1(Np^t), \mathcal{O}/\pi^t\mathcal{O})$$

As $n_1 \equiv n_2 \equiv 0 \bmod p^{t-r}$ we have that $x^{n_1}\bar{x}^{n_2} \equiv 1 \bmod \pi^t$ $\forall x \in G_{t,r} = (1+\pi^r\mathcal{O})/(1+\pi^t\mathcal{O})$ then this map is $G_{t,r}$equivariant. In this case proposition 4.2 tells us that:

$$eH^1(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}/\pi^t\mathcal{O})) \xrightarrow{\sim} eH^1(\Gamma_1(Np^r), \mathcal{O}/\pi^t\mathcal{O})$$

The long exact sequence corresponding to:

$$0 \to S_{n_1,n_2}(\mathcal{O}/\pi^t\mathcal{O}) \to S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi) \stackrel{\pi^t}{\to} S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi) \to 0$$

and the fact that $eS_{n_1,n_2}(K_\pi/\mathcal{O}_\pi)^{\Gamma_1(Np^r)} = 0$ (see the last part of proposition 4.2) imply that:

$$eH^1(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}/\pi^t\mathcal{O})) \cong eH^1(\Gamma_1(Np^r), S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi))_{\pi^t}$$

and hence that:

$$eH^1(\Gamma_1(Np^r), S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi))_{\pi^t} \cong eH^1(\Gamma_1(Np^r), K_\pi/\mathcal{O}_\pi)_{\pi^t}$$

or dualising:

$$eH_1(\Gamma_1(Np^r), S_{n_1,n_2}(\mathcal{O}_\pi)) \otimes \mathcal{O}/\pi^t\mathcal{O} \cong eH_1(\Gamma_1(Np^r), \mathcal{O}_\pi) \otimes \mathcal{O}/\pi^t\mathcal{O}$$

The theorem now follows at once, recalling that $\dim eH_{1\,Eis}(\Gamma_1(Np^r), S_{n_1,n_2}(K_\pi)) = \dim eH_{1\,Eis}(\Gamma_1(Np^r), K_\pi)$. The same arguments apply to the cuspidal parts.

We can somewhat extend these results on torsion to the case in which p does not divide the level. The crucial result will be:

Proposition 4.5 *Let N be prime to p; n_1, n_2, n positive rational integers with $n_1, n_2 > n$; $n_1 + n_2 \geq 2n + r$; $n_1 \equiv n_2 \equiv n \bmod p^r$. Then we have a map of $\Gamma_1(Np^r)$-modules:*

$$j : S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O}) \twoheadrightarrow S_{n,n}(\mathcal{O}/\pi^r\mathcal{O})$$

such that if $I = [\Gamma_1(N)\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}\Gamma_1(Np^r)]_{j_*^{-1}}$ *then* $j_* \circ I = T_{p^r}$ *and so:*

$$I : eH^{\bullet}(\Gamma_1(Np^r), S_{n,n}(\mathcal{O}/\pi^r\mathcal{O})) \hookrightarrow H^{\bullet}(\Gamma_1(N), S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O}))$$

This map preserves the cuspidal parts.

Proof:This follows from lemma 1.1 of section 1.1 because if $g = \begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}$ then $\Gamma_1(N)g\Gamma_1(Np^r) = \Gamma_1(Np^r)g\Gamma_1(Np^r) \amalg \coprod \Gamma_1(Np^r)g_i\Gamma_1(Np^r)$ where each g_i is of the form $\begin{pmatrix} * & * \\ 0 & p* \end{pmatrix}$ (see Shimura [Sh2]). The fact that this preserves the cuspidal part can easilly be checked as in section 4.2. From this we deduce:

Corollary 4.5 *Under the same assumptions as the proposition we have:*

$$eH^{\bullet}(\Gamma_1(Np^r), S_{n,n}(\mathcal{O}/\pi^r\mathcal{O})) \hookrightarrow H^{\bullet}(\Gamma_1(N), S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi))$$

This map preserves cuspidal parts.

Proof:This follows from the long exact sequence corresponding to:

$$0 \to S_{n_1,n_2}(\mathcal{O}/\pi^r\mathcal{O}) \to S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi) \stackrel{\pi^r}{\to} S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi) \to 0$$

and the fact that:

$$S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi)^{\Gamma_1(N)} = S_{n_1,n_2}(K_\pi/\mathcal{O}_\pi)^{SL_2(\mathcal{O})} = 0$$

Corollary 4.6 *Let N be prime to p and $n_1 \neq n_2$, n positive rational integers with $n_1, n_2 > n$; $n_1 + n_2 \geq 2n + r$; $n_1 \equiv n_2 \equiv n \bmod p^r$. If $\dim eH^1_{cusp}(\Gamma_1(Np^r), S_{n,n}(\mathbb{C})) > 0$ then*

$$H_{1,\,cusp}(\Gamma_1(N), S_{n_1,n_2}(\mathcal{O}_\pi)) \text{ and } H_1(\Gamma_1(N), S_{n_1,n_2}(\mathcal{O}_\pi))$$

have torsion of exponent divisible by π^r.

Proof: This follows as in the proof of theorem 4.2 using the fact that $\dim eG_{n,n}(\Gamma_1(Np^r), \mathbb{C}) \geq \dim G_{n_1,n_2}(\Gamma_1(N), \mathbb{C})$ which was noted at the end of section 4.2.

For example taking $n = 10$ we find:

Example 4.1 *Assume that p splits in K and $p \nmid \tau(p)$ where τ is Ramanujam's function (i.e. p is ordinary for $\Delta(z) = \sum \tau(n) e^{2n\pi iz}$, the cuspidal elliptic modular function of weight* 12 *for $SL_2(\mathbb{Z})$), and if $n_1 \neq n_2$; $n_1, n_2 > 10$; $n_1 + n_2 > 20 + r$; and $n_1 \equiv n_2 \equiv 10 \bmod p^r(p-1)$ then*

$$H_1(SL_2(\mathcal{O}), S_{n_1,n_2}) \ \text{ and } \ H_{1\,cusp}(SL_2(\mathcal{O}), S_{n_1,n_2})$$

have torsion of exponent divisible by p^r.

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