GALOIS REPRESENTATIONS

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RATIONAL NUMBERS:

$$\mathbf{Q} = \{a/b : a, b \text{ integers}, b \neq 0\}$$

FUNCTION FIELD:

$(\mathbf{Z}/p\mathbf{Z})(X)$ **r i n l func i ns in n ri- bl r fini fi l .**

LOCAL FIELDS:

.g. $\mathbf{Q}_p$

i nn Z un i n:

$$\zeta\ s\ =\ \sum_{n=1}^{\infty} n^{-s}$$

$$=\ \prod_p\ 1-\ ^{-s\ -1}\quad \text{Re}\ s>1$$

i nn (1860): $\zeta\ s$ **s n l i n in-u i n C x r n si l l** $s=1$ **n s is s un i n l qu i n**

$$\pi^{-s/2}\Gamma\ s\ 2\ \zeta\ s\ =\pi^{(s-1)/2}\Gamma\ \ 1-s\ \ 2\ \zeta\ 1-s$$

H r r n (1932)- i (1976): $I\ n$ **is s-i i n in r n**

$$\zeta\ 1-n\ \in$$

n

$$|\zeta\ 1-n\ \Longleftrightarrow Cl\quad [e^{2\pi i/p}]\ [\ ]_{1-n}=$$

lli i ur s:

$$E: \quad y^2 = x^3 + ax + b$$

r a, b **n** $4a^3 = 7b^2$**.**

r ll :

$E \cong {}^{r_E}\oplus$ **ni li n r u**

r $r_E \geq 0$ **is K** E**.**

L- **un i n:**

$$L \quad , s \quad = \prod_{p} 1 \quad a_p \qquad ^{-s} \qquad ^{1-2s \ -1}$$

r e s 3 **, r**

$$a_p \qquad = \# \quad x, \qquad : \ ^{2} \equiv x^{3} \quad ax \quad b \bmod$$

l in s : $L \quad , s \ = L \quad ', s$ i n nl i n ' r is n us.

r uil, nr , i n , l r :
$, s$ s n l i n inu i n n
s is s un i n l u i n

$$N^{s\ 2}\ \pi^{\ -s}\ \ s\ \ , s\ =$$
$$WN^{(2-s)\ 2}\ \ \pi^{\ s-2}\ \ -s\ \ ,\ \ -s$$

r $W = \pm 1$ n $N\ \ _{>0}$ r ns n s
n in n .

ir - inn r n- r nj ur ,
n r $, , :

r nk r u ls r r n-
is in $, s$ $s = 1$.

Gr ss- i r , l in :
ru i r r nis in is ≤ 1.

s r i ri i. . n
n in r i s l n -
i ls i r i n l nu r ffi i n s

I is

$$, s = \prod_{p} \prod_{x \in X \times \mathbf{Z}} \prod_{p\mathbf{Z}} 1 - {}^{-s\deg x} {}^{-1}$$

l s:

in $, s = s$

$, s = s \quad s-1 \quad , s$

In n r l , s

- r r i n inu i n ,
- s is un i n l u i n r l in lu s lu 1 i – s, n
- n i r n ri i in r - i n u .

r n i 's :

$$H^i \quad , \overline{\ }_l = H^i \quad , \quad \otimes_{\mathbf{Q}} \overline{\ }_l$$

s n inu us i n $G_{\mathbf{Q}} = Gal \overline{\ }$
n

$$, s = \quad H^i \quad , \overline{\ }_l \ , s \ ^{(-1)^i}$$
$$i$$

l :

$$, s = \quad s \quad s-1 \quad , s$$

I :

$= \ a \ b : a, b$ inte ers, $b =$

I :

$\overline{} = \alpha$ $: \alpha$ **s is s l n i l u i n i r i n l i n s**

I P :

$= \ Aut \ \overline{}$

$=$ **i i ns** $\overline{} \to \overline{}$ **r s r in** $, \times$

i s l r i s - ilis r r l r i nu r is n.

usu l r i n s lu lu
$| \, | = | \, |$ in u s ri n . -
l in i is ri i s l
$\mathbb{R}$ r l nu rs.

$$\overline{\mathbb{R}} = \text{ut}^{cts} = 1, c$$

r ri - i s lu
lu n :

$$|\alpha|_p = {}^{-r} \text{ i } \alpha = {}^{r}a \; b \text{ i } |ab$$

- i nu rs $_p =$ l i n r
$| \, |_p$.

$$_p = \text{ut}^{cts} \; {}_p$$

- i in rs $_p =$ l n s α $_p$ i $|\alpha_p|_p$ 1.

$$_p \quad _p =$$

$$_p \quad _p = \langle Frob_p \rangle$$

rn l $= I_p =$ in r i r u .

$Frob_p =$ ri r nius l n : $Frob_p \ \alpha \ ^p = \alpha$.

$$_p \subset \quad \supset \ 1, c$$

IV I I :

s ri l n i $_p$, I_p, $Frob_p$. insi i .

r n i 's :

$$^{i}\ ,\ \overline{\ }_{l} = {}^{i}\ ,\ \overline{\ }_{l}$$

s n inu us i n n

$$,s = {}^{i}\ ,\ \overline{\ }_{l}\ ,s\ (-1)^{i}$$
$$_{i}$$

. r n i , ’s r ll u ni l
n in r i r u $_p$ s
ri i ll n $^i_{,\ \overline{l}}$ i. . is ‘un-
r i ’ .

. n in , ssin , l in s, ,
su i, J n ; ~ - 5
$^i_{,\ \overline{l}}$ is r r s n-
i n $_l$ n r l i is r s-
llin .

. li n , 4 r r -
ris i l n i l rob_p n $^i_{,\ \overline{l}}$
r $l =$ s i n s in $\overline{\ }$ n ll
i s r s in s lu lu $^{i\ 2}$
i. . is ‘ ur ’ i i .

I $V\ \overline{\ }_l$ **is ni i nsi n l r s n i**

$$r : \quad - \quad V$$

is n inu us r r s n i n s is in s r r r i s n n - un i n V, s **s**

$$_{p\neq l} \ \text{et} \ 1_V - \ ^{-s} \ rob_p \ |_{V^{Ip}}^{-1}$$

$\times$ **si il r r** l

in e s 1 i **.**

W n n r ll

$$\overline{\quad} \quad \overline{\quad}_{l\cdot}$$

:

u s is s r i ri-
. n r is si i n

$$^{i} \quad , \overline{\quad} = \bigoplus_{j} M_j$$

su r ri l **n -**
in $\iota : \overline{\quad} \circ \overline{\quad}_l$**,** $M_j \quad \overline{\quad}_{,\iota} \overline{\quad}_l$ **is n irr -**
u i l su r r s n i n $^{i} \quad , \overline{\quad}_l$ **.**
r r

$$M_j \quad \overline{\quad}_{,\iota} \overline{\quad}_l, s$$

is in n n l **n** ι**.**

s

$$1 \quad 2, s = 1, s \quad 2, s ,$$

is ul i ris i n

$$, s = j, s \pm 1$$

j

i $_j$ irr u i l r r s n i ns .

I -

u s

r : —

is n inu us irr u i l r r s n i n s is in r r i s . n . n:

- **is urs in s** i , $\overline{l}$ **.**
- **ls s is s r r .**
- $, s$ **is s n l i n inu i n ssi l r n si l l i i** $= 1$ **n s is s n li i un i n l u i n r l in** $, s$ $^*, 1 - s$ **.**

. r n i , ’s r ll u ni l
n in r i r u $_p$ s
ri i ll n ^{i} $_{,\ \overline{l}}$ i. . is ‘un-
r i ’ .

. n in , ssin , l in s, ,
su i, n ; ~ -
^{i} $_{,\ \overline{l}}$ is r r s n-
i n $_l$ n r l i is r s-
llin .

. li n , r r -
ris i l n i l rob_p n ^{i} $_{,\ \overline{l}}$
r $l =$ s i n s in $\overline{\ }$ n ll
i s r s in s lu lu $^{i\ 2}$
i. . is ‘ ur ’ i i .

i s l is r r –
ri i s s n i ns
r i. . s is in .
i s n .

n ir
– un i ns
i

l i l rin l s:

$$\mathbf{A} = \quad \times \qquad \qquad {}_{p} \quad {}_{p} \qquad \qquad \times \qquad {}_{p} \quad {}_{p}$$

A - is r n -

I Y:

$$rt_p : \quad {}^{\times}_{p} - \qquad {}^{ab}_{p} \quad \textbf{in} \quad \textbf{i} \quad , \quad \textbf{ns} \quad \textbf{i}$$

$$rt \quad : \quad {}^{\times} \quad {}^{\times}{}_{0} \, {}^{\sim}_{-}$$

$$rt = \qquad rt_x : \quad {}^{\times} \quad {}^{\times}{}_{0} \backslash \mathbf{A}^{\times} \, {}^{\sim}_{-} \qquad {}^{ab}$$

:

l r i us i- irr u i l n-
l u r i i nsi n l r -
r r s n i ns r s n i ns

n

Irr u i l r r s n i ns

$$\pi = \bigotimes_x{}' \pi_x$$

r I I i
ur in

$$\chi_{,0}^{2} \quad n \quad n \quad ,$$

r $gf\ \ h\ = f\ hg$.

n n n p $=$ n n
p

- π_x: π_p **r s . π is r r s n i n** n p **r s .** n **.**

- χ: z $= \chi\ z$ **r** z $^{\times}{}_0$.

- **:** $\int_{N(\ \ )\ N(\mathbf{A})}$ n $dn =$ **r su - r u**

$$\begin{pmatrix} m & * \\ & n-m \end{pmatrix} \quad {}_n.$$

π **is I r i** π **is.**

$$\pi, s = \prod_p \pi_p, s$$

x :

$_1$: us i l u r i r r s n i ns $\sim$ iri l r rs

$\times$ $\times$

$_2$: ul r l r i us i l u r- i r s $\sim$ us i l l r i ul r r s i r n r s.

i s

ri i s

r i. .

i s

l is r r -

s n i ns

s is in .

n .

l r i

us i l u-

r i

r r s n i ns

n ir

- un i ns

i

ni u s r I –
I = I –
I :

- l ss l r ’s
- s n n l n s , r ur-
  l z l
- n rs r s il , … ,
  ll- i s i- ir
- i in r s il s, l r- il s
  , …
- r u r’s r

I

rt : $\times$ $\times_{0}$ $\times$ $\_\sim$ ab

r in ll s r $_{1}$.

is n l u r u-
r i r s r s ri i n -
l is r r s n i n n n r l su -
r u i ni li u i n . s s
r r ul .

ss r -
un i ns i r r i s ris r
u r i r r s n i ns.

+ -
= i nsi n l r r s n -
i ns r i I I I -
r ss i u r i r -
r s n i ns $_2$ n r, s is l -
r i .

n l n s, unn ll

I I n r l r :

I r, r' : $_{n \quad l}$ **, i** $r \circ l = r' \circ l$**, i** r' **ris s r n u r i r r s n- i n n i ... n** r **ls ris s r n u r i r r s n i n.**

n l n s- unn ll = i n- si n l r r s n i ns i - i ... ris r u r i r r s n i ns.

= il s, r uil- nr - i n - l r i ur - ni n ur .

2 3 **is r -s lu l**

= r , i , il s r 's l s - r .

I :

- r s ul n s l - u l i. . r is r irin i $r\ \sigma\ x, r\ \sigma\ = \mu\ \sigma\ x,$ n $x, = \mu\ c\ , x$.
- r s ul i is in nu rs.
- r s ul r in r r r s llin i nu rs s ll r l i l.
- r o l s ul l r i ?
- i $r =$?

,

I r is r r s n i n ni r u
n r r nil n su r u s $_i$,
r rs $_i$ $_i$ n in rs n_i su

$$r = \sum_i n_i \; n \; {}^{G}_{H_i} \; {}_i .$$

I I I : I r is r r s n-
i n i I I I n
r, s s r r i n inu i n n
.

$$\begin{aligned} r, s &= {}_i \; n \; {}^{G}_{H_i} \; {}_i, s \; {}^{n_i} \\ &= {}_i \; {}_i, s \; {}^{n_i} \end{aligned}$$

: I i $r =$, $r_p = 1$ r
n r is r s llin i -
nu rs n_1, n_2 s is in

$$< |n_1 - n_2| < l - 1$$

n r is ur n r, s s r r-
i n inu i n n s is s
un i n l u i n.

: - un i n n r -
ul r is in nu rs r n -
i s r r i n inu i n
n s is s un i n l u -
i n.

:

I is ri i i. . ^{2,1} **, =**
l i- u - l n $, s$ **s r -**
r i n inu i n n s is s
. . r l in $, s$ **n** $,$ **—**
s **.**

I

: $r \circ l$ s r
n u r i r r s l is -
ll r l l . is n n us s
ll in r .

r si i n r - ill , : u s
is s n ri ll n-
n n $= \emptyset$. n r is
in x ll s n u s li in
.

Q I : n x s n in s l-
u l nsi n

: r s r n u-
r i r r s l is ll
r l l li in r .

: u r s n
r s r n u r i r n n
in r i l K_i i K_i s lu-
l .

: s r u r ri

$$1 = \sum_i n_i \; n \; \frac{al(F \quad)}{al(F \; K_i)} \; i$$

s

$$r = \sum_i n_i \; n \; \frac{\mathbf{Q}}{K_i} \; r\big|_{K_i} \qquad i$$

n

$$r, s \; = \sum_i r\big|_{K_i} \qquad {}_i, s \; {}^{n_i}.$$

n n **is r ni - lu**

n n n p $=$ n n

p

I : $\times_0$ $\times$ **n**

$\chi_{,0}^2$ n n $=$ π

is s i-si l r r s n i n

$=$

n **, s irr u i l ns i u n s**

$$\pi = \prod_x{}' \pi_x$$

r ll I I

I .