Unit and distinct distances in typical norms

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based on joint work with Noga Alon and Lisa Sauermann

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Question

Is χ *of the unit distance graph of* $\mathbb{R}^d$ *subexponential for most norms?*


рахмат
danke
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
faafetai lava
mersi
kia ora
barka
welalin
tack
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vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
mahalo
tapadh leat
kiitos
dankie
nanni
nandri
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thank you
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хвала
asante
manana
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তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci
תודה רבה

Typical norms have small unit distance function

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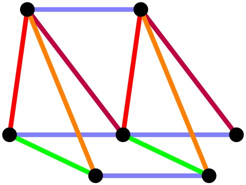

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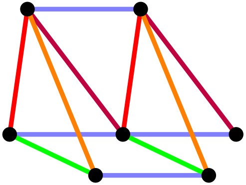

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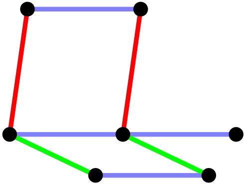

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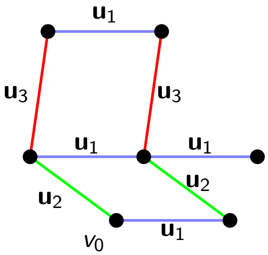

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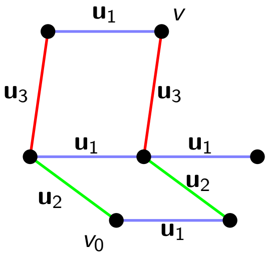

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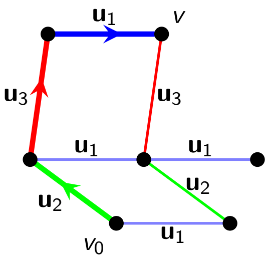

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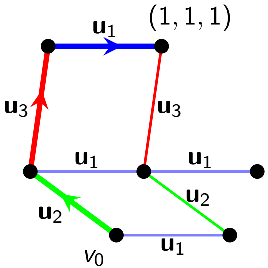

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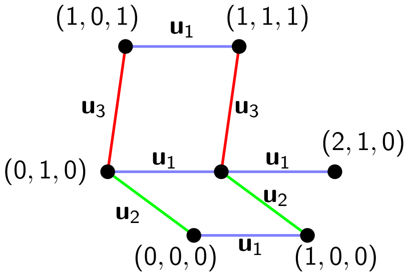

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- Approximate B by a convex 0-symmetric polytope with small facets

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- If there are $2f$ facets for each $\mathbf{x} \in [-\varepsilon, \varepsilon]^f$ we define $B(\mathbf{x})$ to be the polytope obtained by translating i-th facet pair by $\mathbf{x}_i$
- An (A, η)-special $B(\mathbf{x})$ must have its bad $\mathbf{u}_1, \ldots, \mathbf{u}_{kd+1}$ on different facets.

Special norms are meagre

- An $\mathbb{R}^d$-norm is (A, η)-special if $\exists$ non-parallel unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{kd+1}$ s.t.
  - $\mathbf{u}_j = \sum_{i=1}^{k} A_{ji} \mathbf{u}_i$ for all $j = 1, \ldots, dk + 1$.
  - $\angle(\mathbf{u}_i, \mathbf{u}_j) > \eta$, holds for all distinct i and j
- Fix a norm $\|.\|$ with unit ball B which is convex, 0-symetric polytope
- We need to find an open set close to $\|.\|$ not containing any (A, η)-special norm
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- If we fix which facets they belong to, this allows us to express $kd + 1$ of x_i's as linear functions of dk variables given by the coordinates of $\mathbf{u}_1, \ldots, \mathbf{u}_k$

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- A tiny open ball around the centre of the subbox has no (A, η)-special norms