Turán numbers of sunflowers

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joint work with Domagoj Bradač and Benny Sudakov

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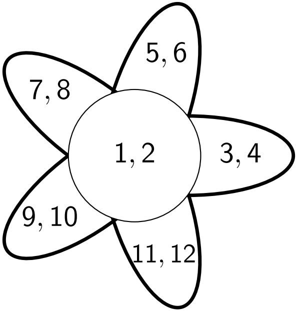

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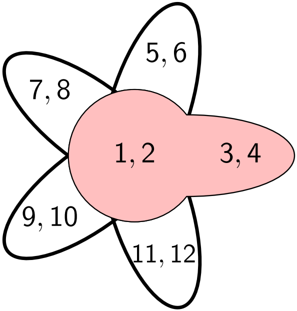

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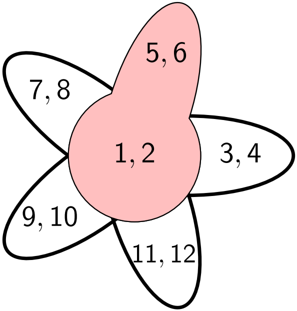

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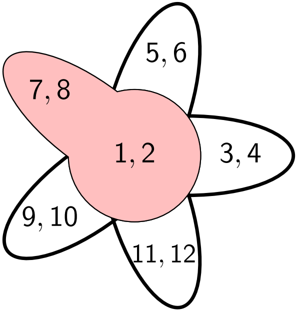

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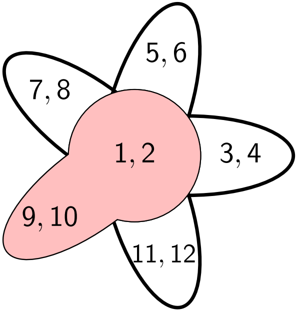

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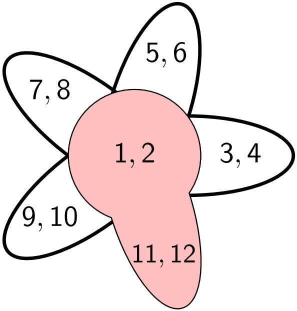

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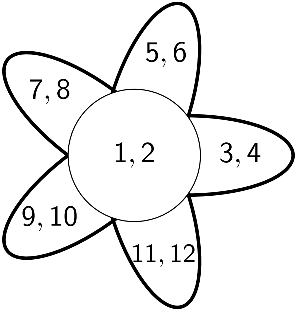

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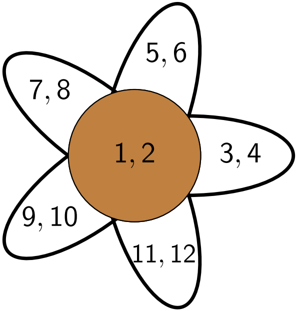

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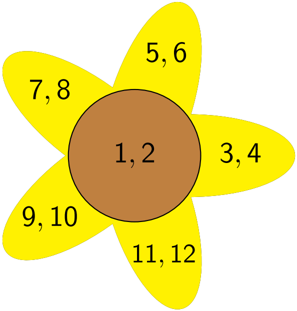

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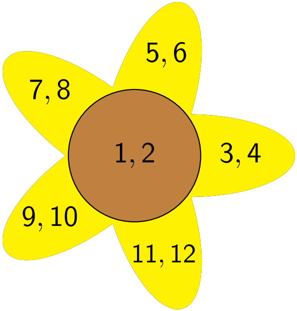

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- *r-uniform* if all sets have size r.

Erdős-Rado sunflower conjecture

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- Relations to many topics in computer science and probability theory.

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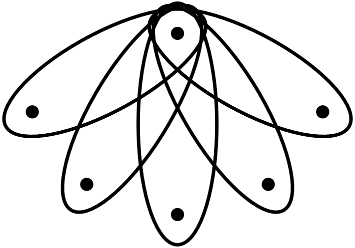

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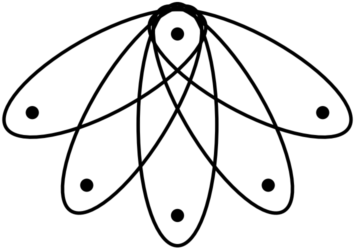

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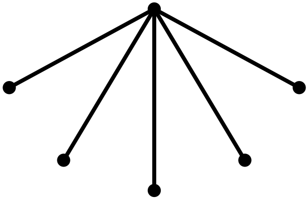

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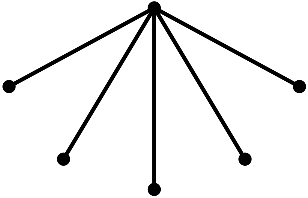

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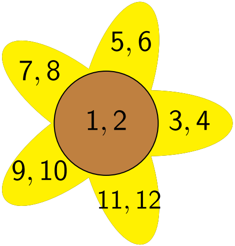

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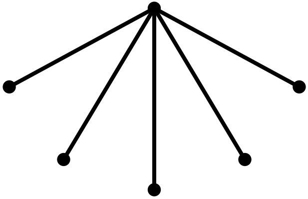

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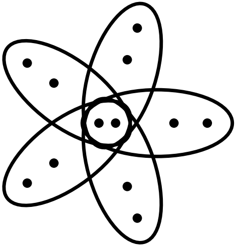

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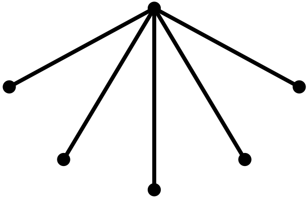

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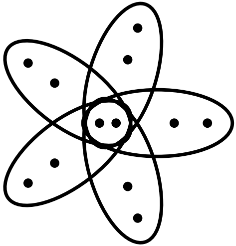

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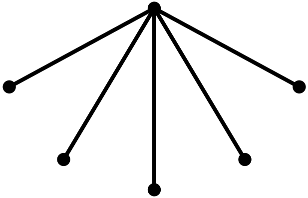

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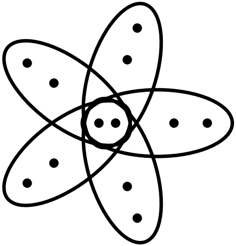

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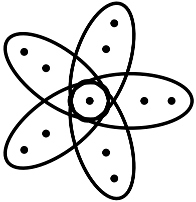

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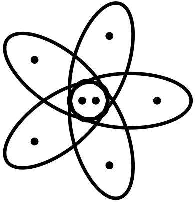

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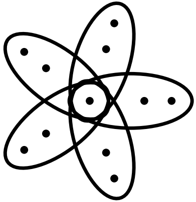

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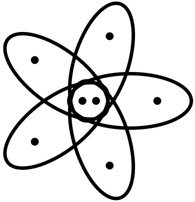

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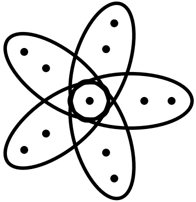

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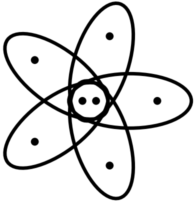

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Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\mathrm{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1,t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n?
- If $r = 2$ it is trivial to see $\mathrm{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
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- The $r = 4$ case solved approximately by B., Draganić, Sudakov and Tran.

Main result

Theorem (Bradač, B. and Sudakov)

$$ex(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1}k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

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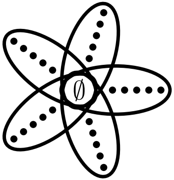

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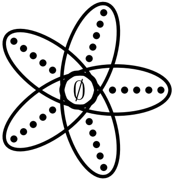

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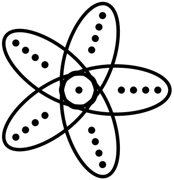

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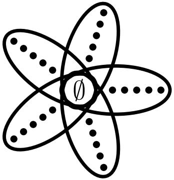

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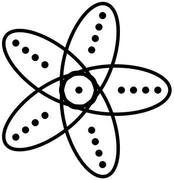

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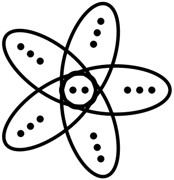

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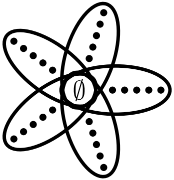

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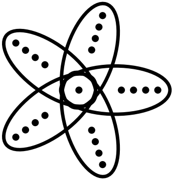

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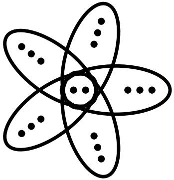

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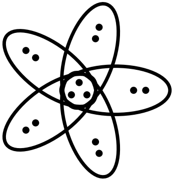

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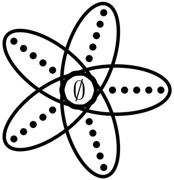

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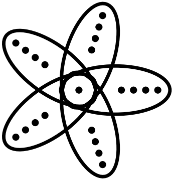

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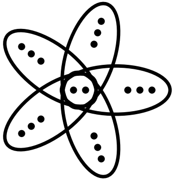

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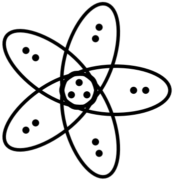

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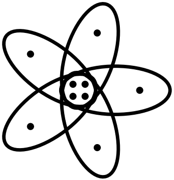

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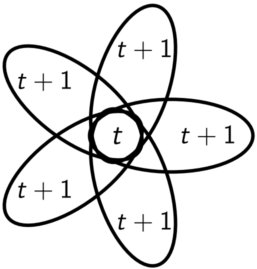

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- **Step 3:** Show there are no $(t+1, t)$-systems on ground set of size $N = 2t+1$

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- Known for $r \leq 4$, up to a constant factor.


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
спасибо
рахмат
Баярлалаа
bedankt
dziękuję
sukriya
kop khun krap
go raibh maith agat
mochchakkeram
terima kasih
grazie
arigatô
dakujem
мерси
tapadh leat
dank je
takk
taiku
sagolun
hvala
mauruuru
ありがとう
감사합니다
תודה רבה
ευχαριστώ
xièxie
rahmet
tanemirt
diolch
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shukriya
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sobodi
dėkuji
mèsi
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najis tuke
kam sah hamnida
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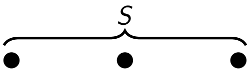

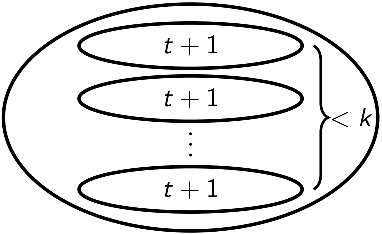

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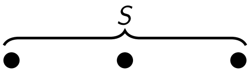
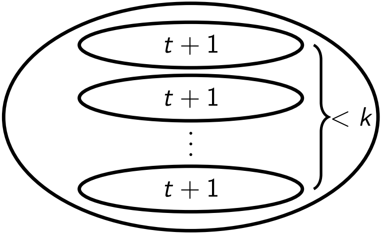
  - Taking the union of a maximal vertex disjoint collection gives τ_S.





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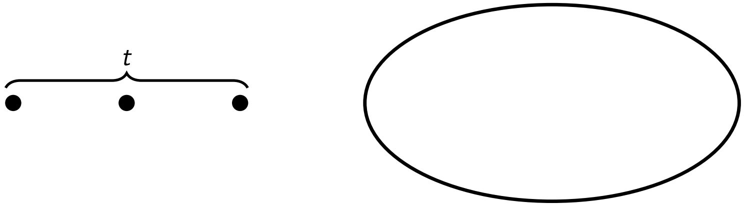

Reduction to the existence problem for a $(t+1,t)$-system

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No $(t+1,t)$-system on $2t+1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \le O(n^t k^{t+1})$

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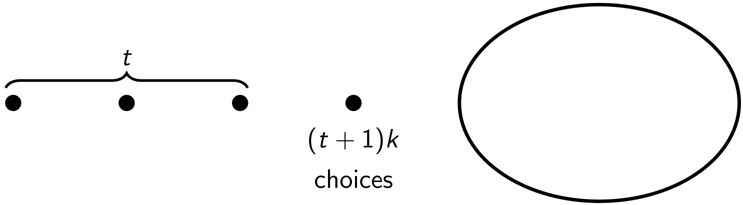

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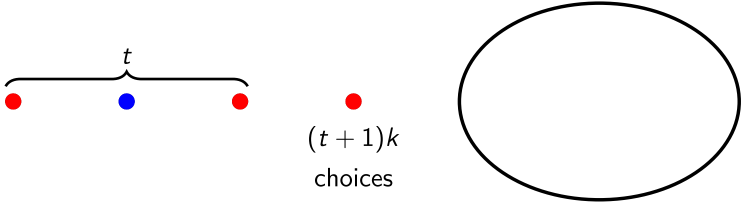

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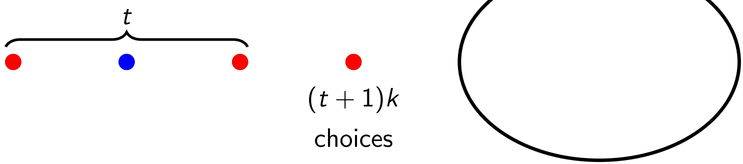

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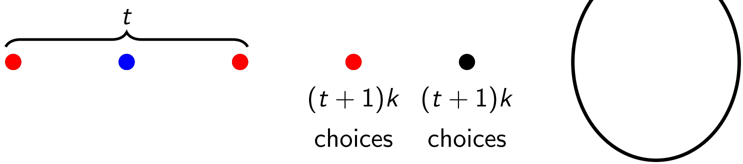

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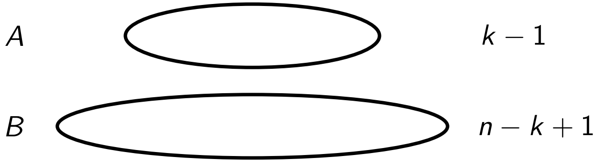

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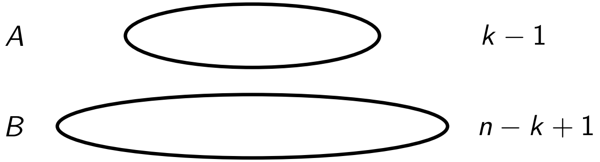

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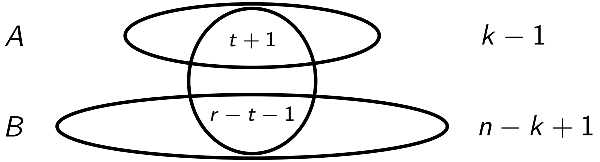

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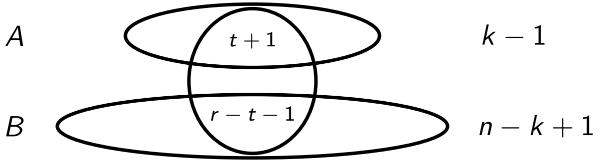

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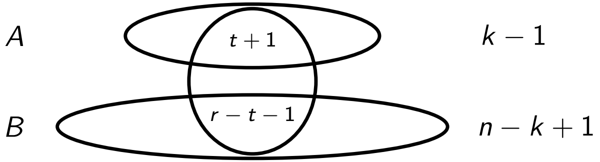

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