Minimum saturated families of sets

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- Pairs A and $\bar{A}$ partition $2^{[n]}$.

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  $|\mathcal{F}| = \left(2^{s-1} - 1\right) 2^{n-s+1} = \left(1 - 1/2^{s-1}\right) 2^n$

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This implies $|\bar{\mathcal{F}}^c| \leq 2^{n-1} \Rightarrow |\mathcal{F}| = 2^n - |\bar{\mathcal{F}}^c| \geq 2^{n-1}$ □

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- If $\mathcal{F}$ is s-saturated then $\mathcal{F}_1 = \ldots = \mathcal{F}_s = \mathcal{F}$ is cross-saturated so Theorem 2 implies Theorem 1.

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- Let V be the vector space of functions from $\{0,1\}^n$ to $\mathbb{F}_2$, over $\mathbb{F}_2$.

Proof

$$\begin{aligned} |\mathcal{F}_1| + \ldots + |\mathcal{F}_s| \geq (s-1)2^n & & \Leftrightarrow \\ 2^n \geq 2^n - |\mathcal{F}_1| + \ldots + 2^n - |\mathcal{F}_s| & & \Leftrightarrow \\ 2^n \geq |\bar{\mathcal{F}}_1^c| + \ldots + |\bar{\mathcal{F}}_s^c| & & \end{aligned}$$

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We embed A to the following element of V:

$$Q_{i,A}(x) := \prod_{j \in B_{i,A}} x_j \prod_{j \in C_{i,A}} (1 - x_j)$$

Proof

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- $V_i = \{Q_{i,A} \mid A \in \bar{\mathcal{F}}_i^c\}$

Proof continued, overview

- $V_i = \{Q_{i,A} \mid A \in \bar{\mathcal{F}}_i^c\}$

Claim 1.

Given $i \neq j$, we have $V_i \perp V_j$.

Proof continued, overview

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Given $i \neq j$, we have $V_i \perp V_j$.

- where the orthogonality is with respect to the standard inner product on V, $\langle f, g \rangle = \sum_{x \in \{0,1\}^n} f(x)g(x)$.

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Elements of V_i are linearly independent over V.

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Elements of V_i are linearly independent over V.

- Claim 1 and 2 imply all $Q_{i,A}$ are independent.
- This implies $|\bar{\mathcal{F}}_1^c| + \ldots + |\bar{\mathcal{F}}_s^c| \leq 2^n$, as desired.

Proof continued, orthogonality between V_i's

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Given $i \neq j$, we have $V_i \perp V_j$.

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Proof.

□

Proof continued, orthogonality between V_i's

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Proof.

- W.l.o.g $i < j$,

$\square$

Proof continued, orthogonality between V_i's

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Proof.

- W.l.o.g $i < j$,
- Let $A \in \bar{\mathcal{F}}_i^c$ and $A' \in \bar{\mathcal{F}}_j^c$.

□

Proof continued, orthogonality between V_i's

Claim 1.

Given $i \neq j$, we have $V_i \perp V_j$.

Proof.

- W.l.o.g $i < j$,
- Let $A \in \bar{\mathcal{F}}_i^c$ and $A' \in \bar{\mathcal{F}}_j^c$.
- $B = B_{i,A}, C = C_{i,A}$ and $B' = B_{j,A'}, C' = C_{j,A'}$

□

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- $B' \cap C \neq \emptyset$

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  - Let $t \in B' \cap C$ then:
  - $0 \neq Q_{i,A}(x) = \prod_{k \in B} x_k \prod_{k \in C}(1 - x_k)$ implies $x_t = 0$ and

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Therefore, $\langle Q_{i,A}, Q_{j,A'} \rangle = \sum_{x \in \{0,1\}^n} Q_{i,A}(x) Q_{j,A'}(x) = 0$, as claimed. □

Proof continued, properties of V

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Proof continued, properties of V

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Proof continued, properties of V

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Lemma 1

The set $\{P_S \mid S \subseteq [n]\}$ is linearly independent over V.

Proof continued, independence of P_S's

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$\square$

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- $\sum_{S\subseteq[n]} \alpha_S P_S = 0$ and not all $\alpha_S = 0$.

□

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A contradiction. □

Proof continued, independence within V_i

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Elements of V_i are linearly independent over V.

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- $\sum_{A' \in \bar{\mathcal{F}}_i^c} \alpha_{A'} Q_{i,A'} = 0$ and not all $\alpha_{A'} = 0$.

□

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Elements of V_i are linearly independent over V.

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- $\sum_{A' \in \bar{\mathcal{F}}_i^c} \alpha_{A'} Q_{i,A'} = 0$ and not all $\alpha_{A'} = 0$.
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&= \alpha_A P_A + \sum_{|A'| \leq |A|, A' \neq A} \gamma_{A'} P_{A'}.
\end{aligned}
$$

So as P_S are independent we have $\alpha_A = 0$, a contradiction. □

Concluding remarks and open problems

Theorem 1

An s-saturated family $\mathcal{F} \subseteq 2^{[n]}$ *has size at least* $|\mathcal{F}| \geq \left(1 - \frac{1}{s}\right) 2^n$.

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- Further improvements?
- Examples with $|\mathcal{F}| = (1 - 2^{-(s-1)})2^n$:

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- Further improvements?
- Examples with $|\mathcal{F}| = (1 - 2^{-(s-1)})2^n$:
    - Let $[n] = I_1 \sqcup \ldots \sqcup I_{s-1}$

Concluding remarks and open problems

Theorem 1

An s-saturated family $\mathcal{F} \subseteq 2^{[n]}$ has size at least $|\mathcal{F}| \geq \left(1 - \frac{1}{s}\right) 2^n$.

- Further improvements?
- Examples with $|\mathcal{F}| = (1 - 2^{-(s-1)})2^n$:
  - Let $[n] = I_1 \sqcup \ldots \sqcup I_{s-1}$
  - Let $\mathcal{F}_i$ be any maximal intersecting family on I_i, for $i \in [s-1]$.

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Theorem 2

A cross-saturated sequence of families $\mathcal{F}_1, \ldots, \mathcal{F}_s \subseteq 2^{[n]}$ satisfies:
$|\mathcal{F}_1| + \ldots + |\mathcal{F}_s| \geq (s-1)2^n$

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- Are there other extremal examples?

Thank You!