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Combinatorial Perspective on the Log-rank Conjecture

Semester Paper

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Abstract

The log-rank conjecture states that the communication complexity of a 01-matrix is polylogarithmic in its rank. This is related to graph theory, since we can bound the logarithm of the chromatic number of a graph by the communication complexity of its adjacency matrix. This semester paper gives an overview of results in the area, focusing on the most recent upper bound by Lovett.

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Chapter 1

Introduction

This semester paper is about the relation between the chromatic number of graphs and the rank of their adjacency matrix. In this paper all graphs are simple, that is undirected with no loops and no multiple edges. Let us first quickly review some definitions.

Definition 1.1 (Chromatic Number) *Let G be a graph. Its chromatic number $\chi(G)$ is the number of colours needed to colour its vertices in such a way that no two adjacent vertices have the same colour.*

Definition 1.2 (Rank) *Let M be a real $n \times m$-matrix. We define* $\operatorname{rank}(M)$ *to be the maximal number of linearly independent columns over $\mathbb{R}$.*

We start by showing that the chromatic number is at most exponential in terms of the rank.

Lemma 1.3 *For every graph G with adjacency matrix A, it holds that $\chi(G) \leq 2^{\operatorname{rank}(A)}$.*

Proof We give two vertices the same colour when they have the same neighbours. In terms of adjacency matrices, this happens when the corresponding rows are equal. Observe that adjacent vertices always have different colours, as vertices of simple graphs aren't adjacent to themselves. It follows that $\chi(G)$ is at most the number of distinct rows in A.

We will now bound the number of distinct rows in A. First, we pick a basis

$$\{A_{\cdot c_1}, A_{\cdot c_2}, ..., A_{\cdot c_{\operatorname{rank}(A)}}\}$$

of the column space of A. We then consider the matrix B consisting of these columns, defined by $B_{ij} = A_{ic_j}$. Note that B can have at most $2^{\operatorname{rank}(A)}$ different rows, since the rows are 01-vectors of length $\operatorname{rank}(A)$. As the columns of B span the column space of A, there exists a real matrix C such that $A = BC$. Therefore A has at most $2^{\operatorname{rank}(A)}$ different rows as well. □

Now let's compare the chromatic number and the rank for some familiar graphs:

- E_n, the empty graph on n vertices, has $\chi(G) = 1$ and $\text{rank}(A) = 0$.
- K_n, the complete graph on n vertices, has $\chi(G) = n$ and $\text{rank}(A) = n$ for all $n \geq 2$.
- $K_{n,m}$, the complete bipartite graph, on $n + m$ vertices has $\chi(G) = 2$ and $\text{rank}(A) = 2$.
- In fact, any bipartite graph G has $\chi(G) \leq 2$, while the rank can get arbitrarily high.
- The Petersen graph has chromatic number $\chi(G) = 3$ and $\text{rank}(A) = 10$.
- For graphs of up to 8 vertices we have the following bounds on the chromatic number.

$\text{rank}(A)$	0	1	2	3	4	5	6	7	8
$\max \chi(G)$	1	-	2	3	4	5	6	7	8

Note that there are no adjacency matrices of rank 1.

In all of these cases, the chromatic number isn't much higher than the rank. It was conjectured by Van Nuffelen in 1976 that $\chi(G) \leq \text{rank}(A) + 1$. This was later proven wrong by Alon and Seymour in 1989.

From Lemma 1.3 and the K_n example, we now know the upper bound of the chromatic number lies somewhere between r and 2^r, but where exactly? It was proven by Raz and Spieker in 1993 that it is grows faster than polynomial, but it was only proven in 2013 by Lovett that it grows slower than exponential.

Lovett's proof, discussed in chapters 4 and 5, will be the main focus of this paper. It bounds the communication complexity of a matrix in terms of its rank, which in turn can be used to bound the chromatic number of the corresponding graph. The goal of this paper is to explain this proof from a slightly more graph theoretical perspective, leaving out the communication complexity part.

Nevertheless, we will briefly discuss the relation between the chromatic number and communication complexity in chapter 2. In chapter 3, we will give an overview of related works and describe the proofs of two previous bounds in detail.

Chapter 2

Communication Complexity and the Log-rank Conjecture

So what is communication complexity? Suppose that we have two players, Alice and Bob, say, and a 01-matrix M with m rows and n columns. Now Alice is assigned a row i and Bob is assigned a column j. Both players know what the matrix looks like, but they don't know which row/column the other player was assigned. What is the maximum number of bits $CC(M)$ they have to send to each other to figure out the value of M_{ij}?

This is a bit confusing, so let's illustrate it with an example of such protocol. Alice could just sends Bob her index i as a binary number using $\lceil \log_2(m) \rceil$ bits. Then Bob looks up the value of M_{ij} in the matrix and can send it to Alice using a single bit. This shows that the communication complexity is at most $O(\log(m))$.

This strategy is not necessarily optimal. The number of bits needed to send the choice of row is logarithmic in the number of rows. The same can be achieved for the columns. However if they work together, they might be able to use the information they get from each other to send information more efficiently, especially if the matrix has some 'nice' properties. In our case, we are interested in low-rank matrices. This leads us to the log-rank conjecture, which was first formulated by Lovász and Saks in 1988.

Conjecture 2.1 (Log-rank Conjecture) *There exists a polynomial P such that the communication complexity of a 01-matrix of rank r is bounded above by $P(\log(r))$.*

This is pretty interesting, but how does this relate to the chromatic number?

Lemma 2.2 *Let $G = (V, E)$ be a graph with adjacency matrix A. Then*

$$\chi(G) \leq 2^{CC(A)}.$$

Before we present the proof of the lemma, we need two more definitions.

Definition 2.3 (Combinatorial Rectangle) *A subset $X \subseteq V \times W$ is a combinatorial rectangle, if $X = A \times B$ for some $A \subseteq V, B \subseteq W$.*

Definition 2.4 (Monochromatic) *A matrix is monochromatic, if all its entries are equal.*

Proof There exists a scheme of length $CC(A)$ which gives for each input a stream of communication. We are going to colour the cells of the adjacency matrix, giving (i, j) and (i', j') the same colour if they result in the same stream of messages.

Suppose (i, j) and (i', j') have the same colour. Alice cannot distinguish (i, j') from (i, j), while Bob cannot distinguish (i, j') from (i', j'). Hence the same messages are sent and (i, j') has the same colour as (i, j) and (i', j'). The same argument works for (i', j) and it follows that the colour classes form combinatorial rectangles.

Note that despite the fact that Alice cannot distinguish between (i, j) and (i, j'), she does know the correct matrix value in the end. Thus M_{ij} and $M_{ij'}$ must be equal. Analogously, we find $M_{ij} = M_{i'j}$. So all matrix entries in the rectangle must be the same.

In short, the colour classes partition the matrix into monochromatic rectangles, rectangles consisting of only 0's or only 1's.

We now colour vertex $i \in V$ in the colour of (i, i). Observe that if i and j have the same colour, then $M_{ij} = M_{ii} = 0$, and i and j are not adjacent. So the chromatic number is bounded by the number of colours of the matrix colouring, which equals the number of streams of messages. Since the messages have length at most $CC(A)$, it follows that

$$\chi(G) \leq 2^{CC(A)},$$

which completes our proof □

This lemma allows us to bound the chromatic number by the communication complexity. Substituting this into the log-rank conjecture gives

$$\log(\chi(G)) \leq P(\log \operatorname{rank}(A)),$$

where P is some polynomial in P.

The idea behind this proof will be needed again later on. The chromatic number $\chi(G)$ is the number of monochromatic rectangles needed to cover the diagonal of the adjacency matrix. This is bounded above by the number of monochromatic rectangles needed to partition the entire matrix, which we will call $\psi(A)$.

Definition 2.5 (Partition Number) *Let $\psi(M)$ be the minimal number such that we can partition the matrix into $\psi(M)$ monochromatic submatrices.*

Note that ψ is defined for all 01-matrices, not just adjacency matrices.

Chapter 3

Previous Results

In this chapter we will give a brief timeline of some **upper** and **lower** bounds as well as **other** related results.

1976 Van Nuffelen conjectures that $\chi(G) \leq \text{rank}(A) + 1$ [18].

1979 Yao introduces communiation complexity "to focus attention on the inherent cost of information transfer associated with a given distributed computation." [19].

1982 Mehlhorn and Schmidt prove that the communication complexity of a matrix A is always at least $\log_2 \text{rank}(A)$ [13].

1988 Fajtlowicz conjectures that $\chi(G) \leq \text{rank}(A) + 1$ based on computational results by his computer program Graffiti [4].

1988 Lovász and Saks formulate the log-rank conjecture [10].

1989 Alon and Seymour construct a graph with chromatic number 32 and rank 29, disproving Van Nuffelen's conjecture [1].

1992 Razborov proves that the chromatic number is supralinear in the rank. He shows that there exists a sequence of graphs with n^5 vertices, chromatic number $\Omega(n^4)$ and rank $O(n^3)$ [15]; see section 3.1 for a proof.

1993 Raz and Spieker construct a sequence of matrices whose communication complexity is superpolynomial in the rank [17].

1994 Kushilevitz writes a manuscript (cited in [14]) proving that the upper bound on the chromatic number is $2^{\Omega(\log \text{rank}(A))^{\alpha}}$, where $\alpha = \log_3 6$.

1996 Kotlov and Lovász prove that graphs with adjacency matrix A contain at most $O(2^{\text{rank}(A)/2})$ vertices with distinct neighbours [6]. This implies that $\chi(G) \leq O(2^{\text{rank}(A)/2})$, cf. Theorem 1.3.

1997 Kotlov proves that $\chi(G) \leq \text{rank}(A)(4/3)^{\text{rank}(A)}$ [5].

- **2009** Linial and Shraibman find a bound on the discrepancy of a matrix in terms of its rank [9], which will be a key lemma in the 2013 upper bound by Lovett.
- **2012** Ben Sasson, Lovett and Ron-Zewi prove, assuming the Freiman-Ruzsa conjecture, that $CC(A) = O(\text{rank}(A)/\log \text{rank}(A))$ [2].
- **2013** Lovett proves that $CC(A) \leq O(\sqrt{\text{rank}(A)} \log \text{rank}(A))$ [11]. We will discuss this proof in chapter 4.
- **2014** Rothvoß gives an alternative proof of Lovett's bound using a hyperplane rounding argument [16].

We will now give a proof of the 1992 lower bound by Razborov and a weaker version of the 1996 upper bound by Kotlov and Lovász.

3.1 Lower bound (Razborov 1992)

In this section we will show following the proof of Razborov [15] that the upper bound on the chromatic number is supralinear in r by presenting a sequence of graphs with rank $O(n^3)$ and chromatic number $\Omega(n^4)$.

Let G_n be the graph with vertex set $V := [n]^5$ and edge set

$$E := \{(v,w) \in V \times V \mid \delta(v,w) \notin \{(1,1,1,1,1),(0,0,0,1,1),(0,0,1,0,1),(0,0,1,1,0),(0,0,0,0,1),(0,0,0,1,0),(0,0,1,0,0),(1,1,0,0,0)\}\},$$

where δ is the pointwise delta function, i.e. $\delta(v,w) = (\delta(v_1,w_1),...,\delta(v_5,w_5))$.

This graph is known as the *incomplete extended p-sum (NEPS)* of five copies of K_n with the basis $B := \{(0,0,0,0,0),(1,1,1,0,0),(1,1,0,1,0),(1,1,0,0,1),(1,1,1,1,0),(1,1,1,0,1),(1,1,0,1,1),(0,0,1,1,1)\}$; see for example [3].

Claim The adjacency matrix of G_n has rank $O(n^3)$.

Proof of Claim Let A be the adjacency matrix of G_n. We will show that A has at most $O(n^3)$ non-zero eigenvalues (counted with multiplicity).

First, we rewrite A in terms of delta functions. To simplify our notation we use the complementary delta function $\eta_{ij} := 1 - \delta_{ij}$. Observe that η is also the adjacency matrix of the complete graph K_n.

$$\begin{aligned} A_{ijklm,abcde} &= 1 - \delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld}\delta_{me} - \eta_{ia}\eta_{jb}\eta_{kc}\delta_{ld}\delta_{me} - \eta_{ia}\eta_{jb}\delta_{kc}\eta_{ld}\delta_{me} \\ &\quad - \eta_{ia}\eta_{jb}\delta_{kc}\delta_{ld}\eta_{me} - \eta_{ia}\eta_{jb}\eta_{kc}\eta_{ld}\delta_{me} - \eta_{ia}\eta_{jb}\eta_{kc}\delta_{ld}\eta_{me} \\ &\quad - \eta_{ia}\eta_{jb}\delta_{kc}\eta_{ld}\eta_{me} - \delta_{ia}\delta_{jb}\eta_{kc}\eta_{ld}\eta_{me} \end{aligned}$$

Let v^1, v^2, v^3, v^4, v^5 be some eigenvectors of η with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ respectively. Let $v = v^1 \otimes v^2 \otimes v^3 \otimes v^4 \otimes v^5$ be their tensor product (with entries $v_{ijklm} = v^1_i \cdot v^2_j \cdot v^3_k \cdot v^4_l \cdot v^5_m$). Multiplying A by v gives

$$\begin{aligned} A_{ijklm,abcde}v_{abcde} &= ((1+\lambda_1)(1+\lambda_2)(1+\lambda_3)(1+\lambda_4)(1+\lambda_5) - 1 - \lambda_1\lambda_2\lambda_3 \\ &\quad - \lambda_1\lambda_2\lambda_4 - \lambda_1\lambda_2\lambda_5 - \lambda_1\lambda_2\lambda_3\lambda_4 - \lambda_1\lambda_2\lambda_3\lambda_5 - \lambda_1\lambda_2\lambda_4\lambda_5 \\ &\quad - \lambda_3\lambda_3\lambda_5)(v^1_i \cdot v^2_j \cdot v^3_k \cdot v^4_l \cdot v^5_m). \end{aligned}$$

This implies that v is always an eigenvalue of A. It follows that the vectors constructed in this way form an eigenbasis. Note that their eigenvalue is 0, whenever four or five of the λ_* are -1. Since the spectrum of K_n is $(n-1, -1, -1, \ldots, -1)$, we conclude that A has at most

$$\binom{5}{3}(n-1)^3 + \binom{5}{2}(n-1)^2 + \binom{5}{1}(n-1)^1 + \binom{5}{0}(n-1)^0 \leq 32n^3 = O(n^3)$$

non-zero eigenvalues, so $\text{rank}(A) = O(n^3)$.

Claim The adjacency matrix has chromatic number $\Omega(n^4)$.

Proof of Claim Consider a colouring of G_n. In independent sets, either both the first and the second coordinate differ or both the first and the second coordinate are the same. Therefore there are at most n possibilities for the first two coordinates. Note that when the first two coordinates are equal, the last three must be different. So there are at most n vertices in the independent set with the same first two coordinates.

Now suppose there are more than $3n$ vertices in the independent set. Then there are at least 4 vertices with the same first and second coordinate. There is also at least one vertex with different first and second coordinates. This vertex must share at least one of its last three coordinate with each of the 4 vertices. That is a contradiction.

Therefore independent sets have size at most $3n$. So the chromatic number is at least $n^5/3n = n^4/3 = \Omega(n^4)$. □

3.2 Upper bound

Kotlov and Lovász [6] proved that for a graph G with adjacency matrix A

$$\chi(G) \leq \sqrt{2}^{\operatorname{rank}(A)}$$

by showing that twin-free graphs of rank r have at most this many vertices. A twin-free graph is a graph whose adjacency matrix does not contain two equal rows. Note that this is sufficient, since we can give twin-vertices the same colour. In this section we will prove a slightly weaker result.

Theorem 3.1 *Let G be a graph whose adjacency matrix has rank $r \geq 1$. Then*

$$\chi(G) \leq 16r2^{r/2}.$$

Proof Let G_n be the graph with vertex set $V = \{1,2,3,4\}^n$ and edge set

$$E = \left\{ (u,v) \in V \times V \mid \sum_{i=1}^{n} A_{u_i v_i} \equiv 1 \mod 2 \right\}, \text{ where } A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Let G'_n be the graph consisting of two copies of G_n with the edges between both copies complemented; more formally $G'_n = (V', E')$, where $V' = \{0,1\} \times V$ and

$$E' = \left\{ (u,v) \in V' \times V' \mid u_1 + v_1 + \sum_{i=2}^{n+1} A_{u_i v_i} \equiv 1 \mod 2 \right\}.$$

We will now prove the following two lemmas:

- G'_n has chromatic number $\chi(G'_n) \leq 8n2^n$.
- G_n contains all twin-free graphs of rank $\leq 2n - 1$ as induced subgraphs.

Lemma 3.2 $\chi(G'_n) \leq 8n2^n$ *for all* $n \in \mathbb{N}$.

Proof We will first prove by induction on n that G'_n is transitive and then give a random colouring which colours G'_n in $8n2^n$ colours with positive probability.

Definition 3.3 (Transitive Graph) *A graph $G = (V, E)$ is transitive, if for all $u, v \in V$ there exists an automorphism of G which maps u to v.*

Induction Basis The graph G'_0 (two vertices and a single edge) is transitive.

Induction Hypothesis The graph G'_k is transitive for some $k \in \mathbb{Z}_{\geq 0}$.

Induction Step Let A be the adjacency matrix of G_k and let $B := J_{4^k} - A$. Then $C := \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$ is the adjacency matrix of transitive graph G'_k. The adjacency matrix of G'_{k+1} is given below.

$$A_{k+1} = \begin{pmatrix} A & A & A & A & B & B & B & B \\ A & A & B & B & B & B & A & A \\ A & B & A & B & B & A & B & A \\ A & B & B & A & B & A & A & B \\ B & B & B & B & A & A & A & A \\ B & B & A & A & A & A & B & B \\ B & A & B & A & A & B & A & B \\ B & A & A & B & A & B & B & A \end{pmatrix}$$

Applying the permutation $\left[\begin{smallmatrix} 1 & 5 & 2 & 6 & 3 & 7 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{smallmatrix}\right]$ to these 8 sets of 4^k vertices gives the following adjacency matrix of an automorphism of G'_{k+1}.

$$\left(\begin{array}{cc|cc|cc|cc} A & B & A & B & A & B & B & A \\ B & A & B & A & B & A & A & B \\ \hline A & B & A & B & B & A & A & B \\ B & A & B & A & A & B & B & A \\ \hline A & B & B & A & A & B & A & B \\ B & A & A & B & B & A & B & A \\ \hline B & A & A & B & A & B & A & B \\ A & B & B & A & B & A & B & A \end{array}\right) = \begin{pmatrix} C & C & C & D \\ C & C & D & C \\ C & D & C & C \\ D & C & C & C \end{pmatrix}.$$

Rewriting this matrix in terms of C and $D := J_{2\cdot 4^k} - C = \left(\begin{smallmatrix} B & A \\ A & B \end{smallmatrix}\right)$ shows that G'_{k+1} is transitive as well.

Conclusion G'_n is transitive for all $n \in \mathbb{Z}_{\geq 0}$.

Note that $\{0\} \times \{1,2\}^n$ is an independent set in G'_n. Because of transitivity, we can pick $2 \cdot 4^n$ independent sets of size 2^n such that each vertex is in exactly 2^n of them. We sample $8n2^n$ of these independent sets uniformly at random. By the union bound, the probability that there is a vertex that is contained in none of the sampled independent sets is at most

$$2 \cdot 4^n \cdot (1 - 2^{-(n+1)})^{(8n2^n)} \leq 2 \cdot 4^n \cdot \exp(-4n) < 1.$$

Therefore there exist $8n2^n$ independent sets which cover all vertices of G'_n. Hence $\chi(G'_n) \leq 8n2^n$ for all $n \in \mathbb{N}$. $\square$

Lemma 3.4 *The graph G_n contains all twin-free graphs of rank at most $2n - 1$ as induced subgraphs.*

Proof The proof of this lemma consists of three parts:

1. The graph G_n contains all graphs of order $2n - 1$ as induced subgraphs.

2. The graph G_n contains all twin-free graphs with an adjacency matrix of rank $2n-1$ over $\mathbb{F}_2$ as induced subgraphs.

3. For all 01-matrices A we have $\text{rank}_{\mathbb{F}_2}(A) \leq \text{rank}_{\mathbb{R}}(A)$.

Claim 1 *G_n contains all graphs of order $2n-1$ as induced subgraphs.*

We will prove by induction on $n \in \mathbb{N}$ that $G_n \setminus v_1$ contains all possible induced subgraphs of order $2n-1$, where $v_1 = \{1\}^n$.

Induction Basis $G_1 \setminus v_1 = K_3$ contains all possible induced subgraphs of order 1.

Induction Hypothesis $G_k \setminus v_1$ contains all possible induced subgraphs of order $2k-1$.

Induction Step Let H be a graph order $2k+1$, where $k \in \mathbb{N}$. We will prove that H is an induced subgraph of $G_{k+1} \setminus v_1$. We distinguish two cases: H is empty and H contains at least one edge.

Case 1: $H = E_{2k+1}$.
We know that $\{1,2\}^{k+1}$ is an independent set in G_{k+1}. Since $2^{k+1} - 1 \geq 2k+1$ for all $k \in \mathbb{N}$, it follows that H is an induced subgraph of $G_{k+1} \setminus v_1$.

Case 2: H contains at least one edge.
Let (u, v) be an edge in H. We partition the other vertices into four sets:

- The set W of vertices which are adjacent to neither u nor v.
- The set X of vertices which are adjacent to u, but not to v.
- The set Y of vertices which are adjacent to v, but not to u.
- The set Z of vertices which are adjacent to both u and v.

We define a new graph H' on the vertices $V[H \setminus \{u, v\}]$ as follows. We take all edges from $E[H \setminus \{u, v\}]$ and then complement the edges between the sets X, Y and Z. That is, for all $e \in (X \times Y) \cup (Y \times Z) \cup (Z \times X)$ we have $e \in H' \iff e \notin H$, while the other edges remain the same.

Since H' contains only $2k-1$ vertices, it is an induced subgraph of $G_k \setminus v_1$. Thus there exists an injective homomorphism f which maps H' to $G_k \setminus v_1$. Now we define a second function which maps H to $G_{k+1} \setminus v_1$ as follows.

$$g(x) = \begin{cases} \{3\} \times \{1\}^k, & \text{if } x = u \\ \{2\} \times \{1\}^k, & \text{if } x = v \\ \{1\} \times f(x), & \text{if } x \in W \\ \{2\} \times f(x), & \text{if } x \in X \\ \{3\} \times f(x), & \text{if } x \in Y \\ \{4\} \times f(x), & \text{if } x \in Z \end{cases}$$

Note that g is injective and that its range does not contain $\{1\}^{k+1}$. Moreover, it can be checked that it preserves the edges of H. Therefore H is an induced subgraph of $G_{k+1} \setminus v_1$.

Conclusion $G_n \setminus v_1$ contains all possible induced subgraphs of order $2n - 1$.

Claim 2 *The graph G'_n contains all twin-free graphs H with* $\text{rank}_{\mathbb{F}_2}(H) \leq 2n - 1$ *as induced subgraphs.*

Let H be a twin-free graph with adjacency matrix A of rank $r \leq 2n - 1$. We can find r vertices in H, whose columns span the column space of A. Let H' be the subgraph corresponding to these vertices. We know that H' is an induced subgraph of G_n by Claim 1. Note that the columns of the adjacency matrix of G_n form a group under addition over $\mathbb{F}_2$ isomorphic to $(C_2 \times C_2)^n$. In particular, it is closed. So the adjacency matrix of G_n contains all linear combinations of its own columns. Considering the vertices corresponding to the appropriate linear combinations, we obtain H as an induced subgraph of G_n. Observe that we need H to be twin-free to ensure that every vertex of H corresponds to a different linear combination of columns and hence to a different vertex in G_n.

Claim 3 *For all 01-matrices A it holds that* $\text{rank}_{\mathbb{F}_2}(A) \leq \text{rank}_{\mathbb{R}}(A)$.

Let A have rank r over the reals and r' over $\mathbb{F}_2$. Suppose for the sake of contradiction that $r' > r$. We pick r' linearly independent columns of A with respect to $\mathbb{F}_2$. They form a matrix B of rank at most r over the reals. Using Gaussian elimination we can find a non-zero rational solution to $Bx = 0$. Multiplying x by the smallest common multiple of the denominators of its entries and dividing it by the greatest common divisor of their numerators, we obtain an integer solution with at least one odd entry. Note that this is a non-zero solution of the equation with respect to $\mathbb{F}_2$. This is a contradiction, because the columns of B are linearly independent over $\mathbb{F}_2$. Therefore $r' \leq r$.

It follows from Claim 2 and Claim 3 that G_n contains all twin-free graphs of rank $2n - 1$ over $\mathbb{R}$ as induced subgraphs. □

Combining both lemmas, we see that twin-free graphs of rank $\leq 2n - 1$ have chromatic number at most $8n2^n$. Again, since twin-vertices can be given the same colour and don't reduce the rank, all graphs of rank r have chromatic number at most $16r2^{r/2}$. □

Chapter 4

Lovett's Upper Bound

In this chapter we adapt the proof of the upper bound of the communication complexity by Lovett [12] to give an upper bound on the chromatic number of a graph. Recall that $\psi(A)$ is the minimal number of rectangles needed to partition 01-matrix A into monochromatic rectangles. We now define $\Psi(r, N)$ to be the maximal value of ψ over all possible 01-matrices of rank $\leq r$ and size $n \times m \leq N$.

Theorem 4.1 *Let A be a 01-matrix of rank $r \geq 1$, then $\psi(A) \leq 4r^{1000\sqrt{r}}$.*

Note in particular that picking A to be an adjacency matrix implies that $\chi(A) \leq \psi(A) \leq 4r^{1000\sqrt{r}}$.

Before proving theorem 4.1 we give an outline of the proof.

1. We first bound a parameter of 01-matrix A of rank r, called discrepancy (see section 4.1 for the definition) and denoted $d(A)$, below by $1/12\sqrt{r}$.

2. We use duality to obtain a distribution of rectangles in which each cell of A containing 1 is at least $2d(A)/3$ more likely to appear than each cell containing 0, or vice versa.

3. We consider the intersection of independently sampled rectangles from this distribution. We show that there exists such an intersection rectangle which is both sufficiently large and 'almost-monochromatic'. By almost-monochromatic, we mean that all but a fraction of $1/(4r)$ of its entries are equal.

4. We prove using linear algebraic techniques that each almost-monochromatic rectangle contains a fully monochromatic subrectangle of at least $1/8$ its size.

5. Given a sufficiently large monochromatic rectangle, we can split the matrix A into two submatrices: one of which has rank at most $\frac{r+1}{2}$ and one of which has size at most $N - \frac{N}{288} r^{-50\sqrt{r}}$. This gives us the inequality
$$\Psi\left(\frac{r+1}{2}, N\right) + \Psi\left(r, N - \frac{N}{288} r^{-50\sqrt{r}}\right).$$
6. Lastly, we use the induction on N and ...
7. ... induction on r to prove that
$$\Psi(r, N) \leq 4r^{1000\sqrt{r}}$$
for some constant D.

In the rest of the chapter we will discuss every step in detail.

4.1 Discrepancy

Let M be a $\{\pm 1\}$-matrix with rows indexed by X and columns indexed by Y and σ a matrix of non-negative reals such that $\sum_{i\in X, j\in Y} \sigma_{ij} = 1$. We can think of σ as a probability distribution on $X \times Y$, giving weights to the entries of M. We define a function $W_\sigma : \mathcal{P}(X) \times \mathcal{P}(Y) \to [-1, 1]$ to calculate the sum of the entries in a rectangle with respect to these weights σ

$$W_\sigma(A \times B) = \sum_{i\in A, j\in B} \sigma_{ij} M_{ij}.$$

The discrepancy with respect to a particular distribution σ is defined as the maximum of $|W_\sigma|$ over all possible rectangles.

$$d(M, \sigma) = \max_{A\subset X, B\subset Y} |W_\sigma(A \times B)|$$

The overall discrepancy of the matrix M defined as the minimum of $d(M, \sigma)$ over all possible distributions σ. This gives the following formula.

$$d(M) = \min_{\sigma} \max_{A\subset X, B\subset Y} \left| \sum_{i\in A, j\in B} \sigma_{ij} M_{ij} \right|$$

In this section we will prove the following lemma from [9] and [8].

Proposition 4.2 *For all $\{\pm 1\}$-matrices M of rank r we have $d(M) \geq 1/8\sqrt{r}$.*

We will use the following notation.

- Let x_i be the i-th row of X and y_j the j-th column of Y.
- Let $\gamma_2(M) := \min_{X,Y:XY=M} \max_{i,j} \|x_i\|_2 \|y_j\|_2$.
- Let $sp(XY)$ be the sign pattern matrix of XY, i.e. $sp(XY)_{ij} = \text{sign}(\sum_k X_{ik} Y_{kj})$.
- Let $m(M) := \max_{X,Y:sp(XY)=M} \min_{i,j} \frac{|x_i \cdot y_j|}{\|x_i\|_2 \|y_j\|_2}$.
- Let $K_G < 2$ denote Grothendieck's constant. It appears in Grothendieck's inequality, which is stated and used in the third part of the proof (see [7] for a proof of this upper bound).
- Let $\|Z\|_{N_1 \to N_2} := \max\{\|Zv\|_{N_2} \mid \|v\|_{N_1} = 1\}$ be the operator norm.

We break up the proof into three steps.

1. $r \geq \gamma_2(M)^2$
2. $\gamma_2(M) \geq 1/m(M)$
3. $m(M) \leq 4K_G d(M)$

Step 1

First, note that

$$\max_i \|x_i\|_2 = \|X\|_{l_2 \to l_\infty} \quad \text{and} \quad \max_j \|y_j\|_2 = \|Y\|_{l_1 \to l_2},$$

This way we can rewrite

$$\begin{aligned}
\gamma_2(M) &= \min_{X,Y:XY=M} \|X\|_{l_2 \to l_\infty} \|Y\|_{l_1 \to l_2} \\
&\leq \min_{X,Z:XZ=I} \|X\|_{l_2 \to l_\infty} \|ZM\|_{l_1 \to l_2} \\
&\leq \min_{X,Z:XZ=I} \|X\|_{l_2 \to l_\infty} \|Z\|_{l_\infty \to l_2} \|M\|_{l_1 \to l_\infty} \\
&= \min_{X,Z:XZ=I} \|X\|_{l_2 \to l_\infty} \|Z\|_{l_\infty \to l_2}, \text{since } M \text{ is a } \pm 1\text{-matrix.}
\end{aligned}$$

We will now use John's theorem.

Theorem 4.3 (John's Theorem) *For any norm E on an r-dimensional vector space, there exist matrices X and Z such that $XZ = I$ and*

$$\|X\|_{l_2 \to E} \|Z\|_{E \to l_2} \leq \sqrt{r}.$$

Choosing E to be the l_∞ norm on the rank(M)-dimensional vector space range(M), we get $\gamma_2(M) \leq \sqrt{r}$.

Step 2

We will now manipulate the definitions of γ_2 and m to prove that $\gamma_2(M) \geq 1/m(M)$.

$$\begin{aligned}
\gamma_2(M) &= \min_{X,Y:XY=M} \max_{i,j} \|x_i\|_2 \|y_j\|_2 \\
&= \min_{X,Y:XY=M} \max_{i,j} \frac{\|x_i\|_2 \|y_j\|_2}{|x_i \cdot y_j|}, \text{ since } x_i \cdot y_j = M_{ij} \in \{\pm 1\} \\
&\geq \min_{X,Y:sp(XY)=M} \max_{i,j} \frac{\|x_i\|_2 \|y_j\|_2}{|x_i \cdot y_j|} \\
&= \left(\max_{X,Y:sp(XY)=M} \min_{i,j} \frac{|x_i \cdot y_j|}{\|x_i\|_2 \|y_j\|_2} \right)^{-1} = m(M)^{-1}
\end{aligned}$$

Step 3

The next step is an application of Grothendieck's inequality.

Theorem 4.4 (Grothendieck's Inequality) *There is a universal constant $K_G < 2$ such that for every real matrix B_{ij} and every $k \geq 1$,*

$$\max\left\{\sum B_{ij} u_i \cdot v_j | \forall i, j\ u_i, v_j \in \mathbb{R}^k\right\} \leq K_G \max\left\{\sum B_{ij} \varepsilon_i \delta_j | \forall i, j\ \varepsilon_i, \delta_j \in \{\pm 1\}\right\}.$$

Let σ be any probability measure on $X \times Y$. Then the following holds.

$$\begin{aligned}
m(A) &= \max_{X,Y:sp(XY)=M} \min_{i,j} \frac{|x_i \cdot y_j|}{\|x_i\|_2 \|y_j\|_2} \\
&= \max_{X,Y:sp(XY)=M} \min_{i,j} M_{ij} \frac{x_i}{\|x_i\|} \cdot \frac{y_j}{\|y_j\|}, && \text{since } sp(XY) = M \\
&\leq \max_{X,Y:sp(XY)=M} \sum_{i,j} \sigma_{ij} M_{ij} \frac{x_i}{\|x_i\|} \cdot \frac{y_j}{\|y_j\|}, && \text{since } \sum_{i,j} \sigma_{ij} = 1 \\
&\leq K_G \max_{\varepsilon,\delta} \sum_{i,j} \sigma_{ij} M_{ij} \varepsilon_i \delta_j && \text{by Grothendieck's inequality}
\end{aligned}$$

$$\begin{aligned}
&= K_G \max_{A,B} \left(\sum_{i \in A, j \in B} \sigma_{ij} M_{ij} - \sum_{i \in A, j \in B^C} \sigma_{ij} M_{ij} - \sum_{i \in A^C, j \in B} \sigma_{ij} M_{ij} + \sum_{i \in A^C, j \in B^C} \sigma_{ij} M_{ij} \right) \\
&\leq 4 K_G \max_{A,B} \left| \sum_{i \in A, j \in B} \sigma_{ij} M_{ij} \right| = 4 K_G d
\end{aligned}$$

Conclusion

Combining the three inequalities we conclude

$$d(M) \geq \frac{m(M)}{4K_G} \geq \frac{1}{4K_G \gamma_2(M)} \geq \frac{1}{4K_G \sqrt{r}} > \frac{1}{8\sqrt{r}}.$$

01-Matrices

As we are working with 01-matrices instead of $\{\pm 1\}$-matrices, we define the discrepancy of 01-matrix A to be the discrepancy of the corresponding $\{\pm 1\}$-matrix $M = 2A - J$, where J is the all-ones matrix. Since the rank of A and M differ by at most 1, we can bound the discrepancy of A with $\operatorname{rank}(A) \geq 1$ by

$$d(A) \geq \frac{1}{8\sqrt{\operatorname{rank}(M)}} \geq \frac{1}{8\sqrt{\operatorname{rank}(A) + 1}} \geq \frac{1}{8\sqrt{2 \operatorname{rank}(A)}} \geq \frac{1}{12\sqrt{\operatorname{rank}(A)}}.$$

4.2 Duality

In this section we will prove the following lemma.

Proposition 4.5 *There exists a distribution of rectangles ρ such that*

$$\mathbb{P}_{R\sim\rho}[(x,y) \in R] \geq \mathbb{P}_{R\sim\rho}[(x',y') \in R] + 2d(A)/3$$

for all (x,y) and (x',y') with $A_{xy} = 1$ and $A_{x'y'} = 0$.

Proof By the definition of discrepancy, there exists a rectangle $R = S \times Q$ with $|W_\sigma(R)| \geq d(A)$ for each distribution of weights σ, where W_σ is defined with respect to $M = 2A - J$ as in section 4.1. We know that

$$W_\sigma(X \times Y) = W_\sigma(S \times Q) + W_\sigma(S \times Q^C) + W_\sigma(S^C \times Q) + W_\sigma(S^C \times Q^C).$$

Thus if $W_\sigma(X \times Y) = 0$, then there is always a rectangle R' with

$$W_\sigma(R') \geq d(A)/3.$$

Since $\mathbb{E}_{R\sim\rho}[W_\sigma(R)]$ is linear in σ and ρ, Von Neumann's Minimax Theorem implies

$$\begin{aligned}
\max_\rho \min_\sigma \mathbb{E}_{R\sim\rho}[W_\sigma(R)] &= \min_\sigma \max_\rho \mathbb{E}_{R\sim\rho}[W_\sigma(R)] \\
&= \min_\sigma \max_R W_\sigma(R) \\
&\geq \min_\sigma d(A)/3 \\
&= d(A)/3,
\end{aligned}$$

where the minimum is taken over all σ with $W_\sigma(X \times Y) = 0$. Therefore there is a ρ such that

$$\mathbb{E}_{R\sim\rho}[W_\sigma(R)] \geq d(A)/3$$

for all distributions σ with $W_\sigma(X \times Y) = 0$. One such distribution σ is the one which assigns probability $1/2$ to both of (x,y) and (x',y'), where $A_{xy} = 1$ and $A_{x'y'} = 0$. This implies that

$$\mathbb{P}_{R\sim\rho}[(x,y) \in R] \cdot \frac{1}{2} \cdot 1 + \mathbb{P}_{R\sim\rho}[(x',y') \in R] \cdot \frac{1}{2} \cdot (-1) \geq d(A)/3,$$

as required. □

This lemma shows that there is a distribution of rectangles in which the cells containing 1 are $2d(A)/3$ more likely to appear than the cells containing 0. By symmetry, there also exists a distribution of rectangles in which the cells containing 0 are $2d(A)/3$ more likely to appear than the cells containing 1. In the next section we will amplify this result to find an almost-monochromatic rectangle.

4.3 Almost-monochromatic rectangle

In this section we prove the following proposition.

Proposition 4.6 *Let A be a 01-matrix with N entries, discrepancy d and at least as many 1's as 0's. Then A contains a rectangle of size at least $\frac{N}{36}r^{-50\sqrt{r}}$, in which the proportion of 0's is at most $1/(4r)$.*

Proof If all entries are 1, the proposition holds trivially, so suppose this is not the case.

We pick ρ as in Theorem 4.5. Let p be the minimum probability that (x, y) with $A_{xy} = 1$ is in R, and q the maximum probability that (x', y') with $A_{x'y'} = 0$ is in R. Then $p \geq q + 2d/3$.

Since there are more 1's than 0's and 1's are a bit more likely to be included, we expect the rectangles to contain slightly more 1's than 0's. However, since we are looking for almost entirely monochromatic rectangles, we need something much stronger. The trick is to look at the intersection $R = R_1 \cap ... \cap R_t$ of multiple rectangles independently sampled from ρ. That way we amplify ratio of 1's to 0's in R.

Now each 1 is included with probability at least p^t and each 0 is included with probability at most q^t. We define T to be the number of 1's in R minus $1/\varepsilon$ times the number of 0's in R. Note that if $\mathbb{E}[T] \geq M$, then there is a rectangle with at least M entries, at most an ε-fraction of which are 0.

$$\begin{aligned} \mathbb{E}[T] &\geq N(p^t - q^t/\varepsilon) \\ &\geq Np^t(1 - (1 - 2d/(3p))^t/\varepsilon) \end{aligned}$$

Now it is just a question of finding the right $t \in \mathbb{N}$ for the required parameters M and $\varepsilon = 1/(4r)$. Picking t an integer satisfying:

$$\ln(\varepsilon/2)/\ln(1 - 2d/(3p)) \leq t \leq \ln(\varepsilon/2)/\ln(1 - 2d/(3p)) + 1,$$

gives us

$$\begin{aligned} \mathbb{E}[T] \geq \frac{1}{2}Np^t &\geq \frac{p}{2}N \exp\left(\frac{\ln(p)\ln(\varepsilon/2)}{\ln(1 - 2d/(3p))}\right) \\ &\geq \frac{p}{2}N \exp\left(3p/(2d) \cdot \ln(p)\ln(8r)\right) \\ &\geq \frac{d}{3}N \exp\left(-\ln(8r)/d\right) \\ &\geq \frac{N}{36\sqrt{r}} \cdot (8r)^{-12\sqrt{r}} \geq \frac{N}{36}r^{-50\sqrt{r}}, \end{aligned}$$

which completes our proof. □

Note that we can similarly prove this for matrices with more 0 entries than 1 entries. Hence, any matrix contains a rectangle of size at least $\frac{N}{36}r^{-50\sqrt{r}}$ in which all but a fraction of at most $1/(4r)$ entries are equal.

4.4 Monochromatic rectangle

In the previous section we have seen that that every 01-matrix M of rank r and size N contains a submatrix M_{XY} of size $|X \times Y|$ at least

$$\frac{N}{36} r^{-50\sqrt{r}}$$

in which at most $|X \times Y|/4r$ entries are 0 or in which at most $|X \times Y|/4r$ entries are 1. In this chapter we will show that M_{XY} contains monochromatic submatirx of size at least $\frac{N}{288} r^{-50\sqrt{r}}$.

Lemma 4.7 *Let M be a 01-matrix of rank r in which at most $|X \times Y|/4r$ entries are 0. Then there exist $A \subset X, B \subset Y$ with $|A \times B| \geq |X \times Y|/8$ and M_{AB} monochromatic.*

Proof Let A correspond to the rows of M_{XY} in which at most $|Y|/2r$ entries are 0. There are at least $|X|/2$ such rows, since the total number of 0's in the matrix is bounded above by $|X \times Y|/4r$. So $|A| \geq |X|/2$.

We pick a largest subset $A' \subset A$ such that the corresponding row vectors form a linearly independent set. The size of this set equals $\text{rank}(M_{AY}) \leq \text{rank}(M_{XY}) = r$.

In each of these rows 0 appears at most $|Y|/2r$ times. So there are at least $|Y| - r|Y|/2r = |Y|/2$ columns that have a 1 in each row corresponding to A'. Let's call the set corresponding to those columns B. We have $|B| = |Y|/2$.

Since all rows of A are linear combinations of rows corresponding to A', each of $A \times B$ either contains only 1's or only 0's. Taking only the rows with the most frequent symbol, we can get a rectangle of size at least

$$|A \times B|/2 \leq |X \times Y|/8,$$

as required. □

Note that a similar proof works in the case that at most $|X \times Y|/4r$ entries of the matrix are 1.

4.5 Recursion

In the previous sections we have seen that every 01-matrix M of rank r and size N contains a monochromatic rectangle of size at least

$$\frac{N}{288} r^{-50\sqrt{r}}.$$

We will now use this fact to find a recursive formula for $\Psi(r, N)$

Let M be a 01-matrix of rank r and size N such that $\psi(M) = \Psi(r, N)$. We call the set of rows X and the set of columns Y. Let M_{AB} be a largest monochromatic rectangle in M.

First we pick $\text{rank}(M_{AY})$ linearly independent rows from M_{AY}. We then extend this to a basis of the rows of M by picking $r - \text{rank}(M_{AY})$ more rows from M. Restricting these rows to B gives a spanning set of M_{XB}. Since all rows from M_{AB} are monochromatic, this spanning set contains at most $r - \text{rank}(M_{AY}) + 1$ linearly independent rows. Hence

$$\text{rank}(M_{AY}) + \text{rank}(M_{XB}) \leq r + 1.$$

Without loss of generality we assume that $\text{rank}(M_{AY}) \leq \frac{r+1}{2}$.

We can now split the matrix into two submatrices: one with small rank M_{AY} and one with small size M_{A^CY}. Colouring M will take at least as much colours as colouring M_{AY} and M_{A^CY} separately. This yields the following inequalities.

$$\begin{aligned} \psi(M) &\leq \psi(M_{AY}) + \psi(M_{A^CY}) \\ \Psi(r, N) &\leq \Psi\left(\frac{r+1}{2}, N\right) + \Psi\left(r, N - \frac{N}{288} r^{-50\sqrt{r}}\right) \end{aligned}$$

4.6 Induction on N

We will use the recursive formula found in the previous section to prove by induction on N that

$$\Psi(r,N) \quad \leq \quad 288 r^{50\sqrt{r}} \log N \cdot \Psi\left(\frac{r+1}{2}, N\right) + 4. \tag{4.1}$$

for all real numbers $r \geq 1, N > 0$.

Induction Basis
$\Psi(r,N) = 0 \leq 4 = 288 \cdot r^{50\sqrt{r}} \cdot \log N \cdot \Psi\left(\frac{r+1}{2}, N\right) + 4$ for all $0 < N < 1$.
$\Psi(r,N) = 1 \leq 4 \leq 288 \cdot r^{50\sqrt{r}} \cdot \log N \cdot \Psi\left(\frac{r+1}{2}, N\right) + 4$ for all $1 \leq N < 2$.

Induction Hypothesis For some $k \in \mathbb{N}_{\geq 2}$ and all real numbers r and N such that $r \geq 1$ and $0 < N < k$ we have $\Psi(r,N) \leq 288 r^{50\sqrt{r}} \log N \cdot \Psi\left(\frac{r+1}{2}, N\right) + 4$.

Induction Step Let $k \leq N < k+1$. There exists a matrix M with $\text{rank}(M) = s \leq r$, $\text{size}(M) = l \leq k$ and $\psi(M) = \Psi(r,N)$. The recursive formula gives us

$$\Psi(r,N) = \psi(M) \leq \Psi(s,l) \leq \Psi\left(\frac{s+1}{2}, l\right) + \Psi\left(s, l - \frac{l s^{-50\sqrt{s}}}{288}\right).$$

We now use induction hypothesis for $N = l - \frac{l s^{-50\sqrt{s}}}{288}$

$$\Psi\left(s, l - \frac{l s^{-50\sqrt{s}}}{288}\right) \leq 288 s^{50\sqrt{s}} \log\left(l - \frac{l s^{-50\sqrt{s}}}{288}\right) \Psi\left(\frac{s+1}{2}, l - \frac{l s^{-50\sqrt{s}}}{288}\right) + 4$$

Since Ψ in increasing and $\log\left(l - \frac{l s^{-50\sqrt{s}}}{288}\right) \leq \log l - \frac{s^{-50\sqrt{s}}}{288}$, we have

$$\Psi(r,k) \leq 288 r^{50\sqrt{r}} \log k \cdot \Psi\left(\frac{r+1}{2}, k\right) + 4.$$

Conclusion We conclude that

$$\Psi(r,N) \quad \leq \quad 288 r^{50\sqrt{r}} \log N \cdot \Psi\left(\frac{r+1}{2}, N\right) + 4.$$

for all real numbers $r \geq 1$ and $N > 0$.

4.7 Induction on r

In this section we will show by induction on r that

$$\Psi(r, N) \leq 4r^{1000\sqrt{r}}$$

for all $r, N \in \mathbb{N}$.

Induction Basis $\Psi(1, N) \leq 4 = 4 \cdot 1^{1000\sqrt{1}}$ and $\Psi(2, N) \leq 6 = 4 \cdot 2^{1000\sqrt{2}}$.

Induction Hypothesis $\Psi(s, N) \leq 4s^{1000\sqrt{s}}$ for all $1 \leq s \leq r$.

Induction Step Let $r \geq 3$. As we have seen before, a matrix of rank r can have at most 2^r distinct rows, so $\Psi(r, N) \leq \Psi(r, 2^{r+1})$. Substituting r and 2^{r+1} into the equation obtained in section 4.6 gives

$$\begin{aligned}
\Psi(r, N) &\leq \Psi(r, 2^{r+1}) \\
&\leq 288r^{50\sqrt{r}}(r+1)\Psi\left(\frac{r+1}{2}, 2^{r+1}\right) + 4 \\
&\leq 288r^{50\sqrt{r}}(r+1) \cdot 4\left(\frac{r+1}{2}\right)^{1000\sqrt{(r+1)/2}} + 4 \\
&\leq 4r^{1000\sqrt{r}}
\end{aligned}$$

Conclusion Therefore

$$\Psi(r, N) \leq 4r^{1000\sqrt{r}}$$

for all $r, N \in \mathbb{N}$.

Chapter 5

Conclusion

We have seen some bounds on the chromatic number in terms of the rank of its adjacency matrix, sometimes with a proof included. In this final chapter we will make some remarks from [11] and [12].

5.1 Communication complexity

The proof of the bound on the communication complexity

$$CC(A) = O\left(\sqrt{r}\log(r)\right)$$

given in [11] is very similar to the one we have seen in chapter 4. We still use monochromatic rectangles to split the matrix, but now we also build a protocol tree according to these splits. The leafs of this tree correspond to the final partition of the matrix into at most $4r^{1000\sqrt{r}}$ rectangles. Rebalancing the tree to one of depth $O\left(\log\left(4r^{1000\sqrt{r}}\right)\right)$ gives a protocol of length $O(\sqrt{r}\log r)$.

5.2 Equivalence to monochromatic rectangles

Every graph G on n vertices with adjacency matrix A of rank r contains an independent set of size at least $n/\chi(G)$. Independent sets correspond to all-zero rectangles in the adjacency matrix. So if the log-rank conjecture is true, then there exists a polynomial P such that every matrix of rank r contains a monochromatic rectangle of size $2^{-P(\log(r))}n^2$. Conversely, if every matrix contains a monochromatic rectangle of size $2^{-P(\log(r))}n^2$, then we can use methods similar to those used in chapter 3 to show that the logarithm of the chromatic number is polylogarithmic. This tells us that the existence of these large monochromatic rectangles is both necessary and sufficient.

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