Generalised Ramsey Theories

Master's thesis

Vedran Mihal

August 2020

Supervisors: Prof. Dr. Benjamin Sudakov, M. Math. Matija Bucić
Department of Mathematics, ETH Zürich

Contents

1 Introduction

Ramsey theory is a big and dynamic area of research in combinatorics. It is based on the idea that every large enough object contains some sort of order in its structure. The initiator of the field was Frank Plumpton Ramsey, a philosopher and mathematician, whose famous theorem from 1928 became a basis for extensive research.

We start by introducing the basic definitions in graph theory.

Definition 1.1. *A graph G is a pair (V, E), where V is a set of elements called vertices, and E is a set of unordered pairs of vertices, whose elements are called edges.*

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection $\Phi : V_1 \to V_2$ such that $(u, v) \in E_1$ if and only if $(\Phi(u), \Phi(v)) \in E_2$ for all $u, v \in V_1$.

A graph $H = (U, F)$ is called a subgraph of $G = (V, E)$ if $U \subseteq V$ and $F \subseteq E$. If H is a subgraph of G, we write $H \subseteq G$.

Definition 1.2. *A hypergraph G is a pair (V, E), where V is a set of elements called vertices, and E is a set of non-empty subsets of V, whose elements are called hyperedges.*

A hypergraph G is called k-uniform if all its hyperedges have size k. Observe that all graphs are 2-uniform hypergraphs.

Definition 1.3. *A graph G on n vertices is said to be complete and denoted by K_n if its edge set contains all pairs of distinct vertices.*

A clique in some graph G is a complete subgraph.

Definition 1.4. *A graph $G = (V, E)$ is called bipartite if there exists a partition $V = V_1 \cup V_2$ such that every edge $e \in E$ intersects both V_1 and V_2.*

A bipartite graph $G = (V_1 \cup V_2, E)$ is called a complete bipartite graph if E contains every possible edge between V_1 and V_2. For $n = |V_1|$ and $m = |V_2|$, we denote G by $K_{n,m}$.

Graph colourings comprise a broad area of graph theory that deals with assigning labels ("colours") to the elements of a graph (vertices and edges). We distinguish vertex and edge colourings. In the remainder of the section, we are going to define colourings and their generalisation, and briefly state some results and problems.

Definition 1.5. *Let G be a graph. A mapping $c : V(G) \to \{1, 2, \ldots, k\}$ is called a vertex k-colouring. We usually refer to numbers $\{1, 2, \ldots, k\}$ as colours.*

A colouring c is called proper if $c(v_1) \neq c(v_2)$ whenever (v_1, v_2) forms an edge.

The chromatic number of G is the least number of colours necessary for G to have a proper colouring, and it is denoted by $\chi(G)$.

Definition 1.6. *Let G be a graph. A mapping $c : E(G) \to \{1, 2, \dots, k\}$ is called an edge k-colouring. A colouring c is called proper if $c(e_1) \neq c(e_2)$ whenever e_1 and e_2 are adjacent edges (they share a vertex).*

The edge chromatic number of G is the least number of colours necessary for G to have a proper edge colouring, and it is denoted by $\chi'(G)$.

A famous result by Vizing states that for every graph G, its edge chromatic number is either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. Thus we can divide graphs into two classes: we usually refer to graphs with $\chi'(G) = \Delta(G)$ as class-1-graphs, and graphs with $\chi'(G) = \Delta(G) + 1$ as class-2-graphs.

Now we present a generalisation of vertex colourings.

Definition 1.7. *Let G be a graph. Fix an assignment of lists to the vertices of G, i.e., $L : V(G) \to 2^{\mathbb{N}}$. A list colouring is a proper vertex colouring such that the colour of the vertex v belongs to $L(v)$ for every $v \in V(G)$.*

A graph G is k-list-colourable if there exists a list colouring for any assignment of lists of size k to the vertices of G. The list-chromatic number of G is the least number k such that G is k-list-colourable, and it is denoted by $\chi_l(G)$.

There is also the edge variant of list colourings.

Definition 1.8. *Let G be a graph. Fix an assignment of lists to the edges of G, i.e., $L : E(G) \to 2^{\mathbb{N}}$. A list edge colouring is a proper edge colouring such that the colour of the edge e belongs to $L(e)$ for every $e \in E(G)$.*

The list chromatic index of G is the least number k such that G has a list edge colouring for any assignment of lists of size k to the edges of G, and it is denoted by $\chi'_l(G)$.

The next conjecture is a very famous open problem in graph theory. It has been proved for many classes of graphs, but it has not been proved in general, nor has a counterexample been found. It is called the List colouring conjecture.

Conjecture 1.9. (List colouring conjecture) $\chi'_l(G) = \chi'(G)$.

We will mention the List colouring conjecture again in Section 4.

Now we introduce Ramsey theory with the following motivating example.

Problem 1.10. *Can we colour the edges of K_6 in blue or red without creating a blue or a red triangle?*

Let us focus on a single vertex $v \in K_6$. Since v has five edges incident to it, at least three of them have to be coloured with the same colour. Now, if we observe the three respective vertices, we conclude that the edges between them have to be coloured with a different colour in order to avoid a monochromatic triangle. But then these three vertices form a monochromatic triangle! Therefore, the answer to the aforementioned problem is no.

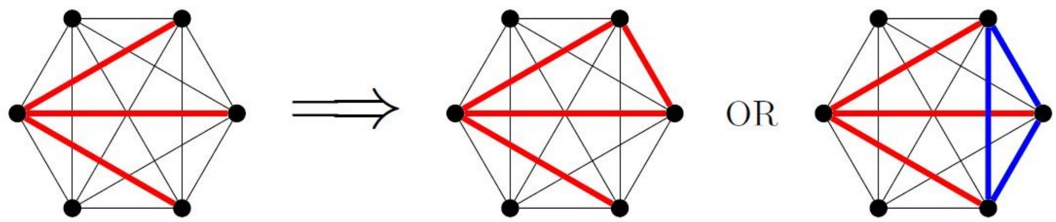

Figure 1: Ramsey number of K_6 [1]

This example raises a question: can we find an arbitrary complex monochromatic subgraph when we colour a graph that is large enough? The answer to this problem is also yes, as first proved by F. P. Ramsey in his 1928 paper *On a problem of Formal Logic*.

This and similar questions will be a subject of this thesis. The next section deals with classical Ramsey numbers. We prove their existence, state some upper and lower bounds, and mention the most important open problems in the area. Then we deal with some interesting variants of the classical Ramsey numbers. Finally, in the last section, we thoroughly study one variant of Ramsey numbers, namely list Ramsey numbers.

2 Classical Ramsey numbers

2.1 Definitions and main results

The classical Ramsey-type problem in graph theory is to find the smallest integer n such that every blue-red edge colouring of a complete graph on n vertices contains a blue s-clique or a red t-clique. It is not obvious that such integer always exists and it was first proved by F. P. Ramsey in the famous theorem we state below.

Theorem 2.1.1. [2] *There exists a smallest integer $R(s,t)$ such that every blue-red edge colouring of a complete graph on $R(s,t)$ vertices contains a blue s-clique or a red t-clique. We call numbers $R(s,t)$ the Ramsey numbers (or the two-colour Ramsey numbers). When $s = t$, we simply write $R(t) := R(t,t)$.*

s, t	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2		2	3	4	5	6	7	8	9	10
3			6	9	14	18	23	28	36	40-42
4				18	25	36-41	49-61	59-84	73-115	92-149
5					43-48	58-87	80-143	101-216	133-316	149-442
6						102-165	115-298	134-495	183-780	204-1171
7							205-540	217-1031	252-1713	292-2826
8								282-1870	329-3583	343-6090
9									565-6558	581-12677
10										798-23556

Table 1: Small Ramsey numbers

Proof. [1] We will prove a slightly stronger result, namely $R(s,t) \leq \binom{s+t-2}{s-1}$.

First we prove the auxiliary result: $R(s,t) \leq R(s-1,t) + R(s,t-1)$.

Inductively, we may assume that $R(s-1,t)$ and $R(s,t-1)$ are both finite. Now we define a complete graph G on $R(s-1,t)+R(s,t-1)$ vertices and take its any blue-red edge colouring. Fix some vertex v of the graph: v has $R(s-1,t)+R(s,t-1)-1$ edges incident to it, so v has either $R(s-1,t)$ blue incident edges or $R(s,t-1)$ red incident edges. Without loss of generality, we may assume that v has at least $R(s-1,t)$ blue incident edges. Now observe the subgraph H induced by these $R(s-1,t)$ adjacent vertices. If H contains a red t-clique, graph G also contains a red t-clique and we are done. Otherwise, H contains a blue $(s-1)$-clique, and together with vertex v it forms a blue s-clique in G. We can do a similar analysis for the case when v has at least red $R(s,t-1)$ incident edges.

Using the inequality $R(s,t) \le R(s-1,t) + R(s,t-1)$, we prove $R(s,t) \le \binom{s+t-2}{s-1}$ by induction on $s+t$. The base case of $s=2$ and $t=2$ is trivial. We have:

$$R(s,t) \le R(s-1,t) + R(s,t-1) \le \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1},$$

where the second inequality follows by induction. □

A natural generalisation of Ramsey numbers is the Ramsey number for k-colour edge colourings (instead of blue-red edge colourings). For integers $s_1, \dots, s_k$, we want to find the smallest integer n such that every edge colouring of a complete graph on n vertices with colours $1, \dots, k$ contains a clique of size s_i in colour i for some $i \in \{1, \dots, k\}$. This smallest integer will be denoted by $R_k(s_1, \dots, s_k)$. We next prove that this number always exists.

Theorem 2.1.2. [1] *Denote by $R_k(s_1, \dots, s_k)$ the minimal number of vertices N such that any k-colour edge colouring of a complete graph on N vertices contains a monochromatic clique of size s_i in colour i for some $i \in \{1, \dots, k\}$. Then the following inequality holds:*

$$R_k(s_1, \dots, s_k) \le R_{k-1}(R(s_1, s_2), s_3, \dots, s_k).$$

We will call numbers of the form $R_k(s_1, \dots, s_k)$ the k-colour Ramsey numbers. In particular, since two-colour Ramsey numbers are finite by Theorem 2.1.1, k-colour Ramsey numbers are finite by induction on k for any $k \in \mathbb{N}$ and any choice of $s_1, \dots, s_k \in \mathbb{N}$.

Proof. Suppose $N = R_{k-1}(R(s_1, s_2), s_3, \dots, s_k)$ is finite. Fix any k-colour edge colouring of K_N. Now slightly modify this colouring by switching every occurrence of colour 2 into colour 1, to obtain a $(k-1)$-colouring. This colouring contains a clique of size $R(s_1, s_2)$ in colour 1, or a clique of size s_i in colour i for some $i \in \{3, \dots, k\}$. If it contains a clique of size s_i in colour i for some $i \in \{3, \dots, k\}$, the original colouring also contains the same clique in the same colour, so we are done. If it contains a clique of size $R(s_1, s_2)$ in colour 1, the original colouring contains a clique of size s_1 in colour 1 or a clique of size s_2 in colour 2.

Since two-colour Ramsey numbers are finite by Theorem 2.1.1, we can inductively conclude that $R_k(s_1, \dots, s_k)$ is finite for any $k, s_1, \dots, s_k \in \mathbb{N}$. □

Ramsey numbers can also be generalised to hypergraphs, i.e., to blue-red colourings of r-tuples of a finite set, in the following way:

Definition 2.1.3. *Let X be a finite set, and denote by $X^{(r)}$ the set of all r-tuples of X. Fix a blue-red colouring of $X^{(r)}$. A set $Y \subseteq X$ is called blue if all elements of $Y^{(r)}$ are coloured blue, and red if all elements of $Y^{(r)}$ are coloured red. Note that $Y \subseteq X$ is both red and blue if $|Y| < r$.*

Theorem 2.1.4. [1] *Denote by $R^{(r)}(s,t)$ the least integer N such that for any finite set X with $|X| = N$, every blue-red colouring of $X^{(r)}$ contains a blue set of size s or a red set of size t. Then the following inequality holds:*

$$R^{(r)}(s,t) \le R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1)) \; + \; 1.$$

In particular, every $R^{(r)}(s,t)$ is finite. Note that $R^{(2)}(s,t) = R(s,t)$.

Proof. Since numbers $R^{(2)}(s,t)$ are finite, we may suppose that numbers $R^{(r-1)}(s,t)$ are also finite. We also suppose that $R^{(r)}(s-1,t)$ and $R^{(r)}(s,t-1)$ are finite.

Suppose $N = R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1)) + 1$ and X is a finite set with N elements. Fix a blue-red colouring c of $X^{(r)}$. Pick an element (vertex) $v \in X$ and fix the set of $(r-1)$-tuples of $X \setminus \{v\}$, i.e., $(X \setminus \{v\})^{(r-1)}$. Transfer the colouring c of $X^{(r)}$ to a colouring c' of $(X \setminus \{v\})^{(r-1)}$ in the following way:

$$c'(x_1, \ldots, x_{r-1}) := c(x_1, \ldots, x_{r-1}, v).$$

Since $|X \setminus \{v\}| = R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1))$, without loss of generality there exists a blue subset of $X \setminus \{v\}$ of size $R^{(r)}(s-1,t)$ with respect to the new colouring c', call this subset Y. This means that all elements of $Y^{(r-1)}$ are coloured blue. Since $|Y| = R^{(r)}(s-1,t)$, there exists a red subset of size t in original colouring c (in this case we are done), or there exists a blue subset $Z \subseteq Y$ of size $s-1$ with respect to the original colouring c. This means that all elements of $Z^{(r)}$ are coloured blue. Now observe that $Z \cup \{v\}$ is also a blue set, of size s, because every r-tuple of $Z \cup \{v\}$ containing v is coloured blue ($Z \subseteq Y$ is a blue set in c' and colours of c' were inherited from c). □

Combining two previous generalisations, we introduce a natural generalisation to multicolour Ramsey numbers of hypergraphs. The existence proof is analogous to the proof of Theorem 2.1.2, using Theorem 2.1.4 in place of Theorem 2.1.1.

Theorem 2.1.5. [1] *There exists a smallest integer $R^{(r)}(s_1, \ldots, s_k)$ such that every hyperedge colouring of a complete r-uniform hypergraph on $R^{(r)}(s_1, \ldots, s_k)$ vertices in colours $1, \ldots, k$ contains a monochromatic complete subgraph of size s_i in colour i for some $i \in \{1, \ldots, k\}$.*

We can generalise this problem of finding a monochromatic clique to a problem of finding arbitrary monochromatic graphs, which leads us to the following definitions:

Definition 2.1.6. *The k-colour ordinary Ramsey number of an r-uniform hypergraph G is:*

$$R(G,k) := \min\{N : \forall\, k\text{-colouring of } E(K_N^{(r)})\ \exists \text{ a monochromatic copy of } G\}.$$

Definition 2.1.7. *For r-uniform hypergraphs $G_1, \ldots, G_k$ we define:*

$$R(G_1, \ldots, G_k) := \min\{N : \forall\, k\text{-colouring of } E(K_N^{(r)})\ \exists \text{ a monochromatic copy of } G_i \text{ in colour } i, \text{ for some } 1 \le i \le k\ \}$$

We write $R(G,k) = R(\underbrace{G, \ldots, G}_{k \text{ times}})$.

2.2 Bounds on Ramsey numbers

There is a well-known lower bound on classical Ramsey numbers proved by Erdös in [3] in 1947, which is still essentially the best known. The proof incorporates a representative example of the probabilistic method.

Theorem 2.2.1. [3] *If* $\binom{n}{t}2^{1-\binom{t}{2}} < 1$, *then* $R(t) \geq n$.

Proof. Fix n and $t \leq n$, and suppose that $\binom{n}{t}2^{1-\binom{t}{2}} < 1$. To each edge of K_n assign the colour blue or the colour red with equal probability (one half). The probability that a subset of t vertices forms a monochromatic clique is $2 \cdot \left(\frac{1}{2}\right)^{\binom{t}{2}} = 2^{1-\binom{t}{2}}$. Since there are $\binom{n}{t}$ different subsets of t vertices in K_n, we obtain that the probability that at least one of these subsets forms a monochromatic clique is (by Boole's inequality) less than $\binom{n}{t}2^{1-\binom{t}{2}}$, which is less than 1 by assumption. This implies that there exists a blue-red colouring of the edges of K_n which contains a monochromatic t-clique, and subsequently, that $R(t) \geq n$. □

Applying Stirling's formula to this result gives us the following:

$$R(t) \geq t2^{\frac{t}{2}}\left(\frac{1}{e\sqrt{2}} + o(1)\right).$$

This bound turned out to be pretty hard to improve, and a slight improvement was given by Spencer in [4] with the following theorem:

Theorem 2.2.2. [4] *If* $4\binom{t}{2}\binom{n}{t-2}2^{1-\binom{t}{2}} < 1$, *then* $R(t) \leq n$.

Applying Stirling's formula to this result implies:

$$R(t) \geq t2^{\frac{t}{2}}\left(\frac{\sqrt{2}}{e} + o(1)\right),$$

which is an improvement of the previous result by a factor of 2.

Further lowering of this bound would be a very important result, so we state the following open problem:

Problem 2.2.3. [5] *Does there exist a positive constant* ϵ *such that*

$$R(t) \geq (1+\epsilon)\frac{\sqrt{2}t}{e}\sqrt{2}^t$$

for all sufficiently large t*?*

Now we turn our attention to the upper bounds. The upper bound from the proof of Theorem 2.1.1, namely $R(s,t) \leq \binom{s+t-2}{s-1}$, gives us the asymptotical behaviour of $R(t) = O\left(\frac{4^t}{\sqrt{t}}\right)$. This result was first improved by Graham and Rödl in [6] to $R(t) = o\left(\frac{4^t}{\sqrt{t}}\right)$, and then by Thomason in [7] to $R(t) = O\left(\frac{4^t}{t}\right)$. A significant improvement was given by Conlon in [8] with the following theorem:

Theorem 2.2.4. [8] *For all ϵ with $0 < \epsilon \leq 1$, there exists a constant C_ϵ such that, for $k \geq l \geq \epsilon k$,*

$$R(k+1, l+1) \leq k^{-C_\epsilon \frac{\log k}{\log \log k}} \binom{k+l}{k}.$$

In particular, there exists a constant C such that

$$R(k+1, k+1) \leq k^{-C \frac{\log k}{\log \log k}} \binom{2k}{k}.$$

This theorem implies in particular that $R(t) \leq R(t+1) \leq t^{-C \frac{\log t}{\log \log t}} \binom{2t}{t} \leq t^{-C \frac{\log t}{\log \log t}} 4^t$, for some constant C, which is an improvement of a previous bound by Thomason.

A very recent improvement of this bound was given by Ashwin Sah [9] in 2020 with the following theorems:

Theorem 2.2.5. [9] *There is an absolute constant $c > 0$ such that for $k \geq 3$,*

$$R(k+1, k+1) \leq e^{-c(\log k)^2} \binom{2k}{k}.$$

Theorem 2.2.6. [9] *For each $\epsilon \in \left(0, \frac{1}{2}\right)$ there is $c_\epsilon > 0$ such that*

$$R(k+1, l+1) \leq e^{-c_\epsilon (\log k)^2} \binom{k+l}{k}$$

whenever $\frac{l}{k} \in [\epsilon, 1]$ and $l \geq \frac{1}{c_\epsilon}$.

Now we turn our attention to hypergraphs. An interesting inductive bound was introduced by Erdös and Rado in [10].

Theorem 2.2.7. [10] $R^{(k)}(s,t) \leq 2^{\binom{R^{(k-1)}(s-1,t-1)}{k-1}} + k - 2.$

Using this theorem, we can establish an upper bound for diagonal two-colour Ramsey numbers of 3-uniform hypergraphs, namely $R^{(3)}(t,t)$.

Corollary 2.2.8. *There exists a positive constant c such that $R^{(3)}(t,t) \leq 2^{2^{ct}}$.*

Proof. By the proof of Theorem 2.1.1 we establish the following bound:

$$R(t-1,t-1) \leq \binom{2t-4}{t-2} \leq \left(\frac{e \cdot (2t-4)}{t-2}\right)^{t-2} \leq (2e)^{t-2}$$

Applying Theorem 2.2.7 gives us:

$$R^{(3)}(t,t) \leq 2^{\binom{R(t-1,t-1)}{2}} + 1 \leq 2^{(R(t-1,t-1))^2} \leq 2^{(2e)^{2t-4}} \leq 2^{2^{ct}}$$

for some constant c. $\square$

For the lower bound of $R^{(3)}(t,t)$, we present a result by Erdös, Hajnal and Rado [11].

Theorem 2.2.9. [11] *There is a positive real number c such that $R^{(3)}(t,t) \geq 2^{ct^2}$ for all t.*

They also conjectured the following:

Conjecture 2.2.10. [11] $R^{(3)}(t,t) \geq 2^{2^{ct}}$ *for some absolute real constant $c > 0$.*

For the case of three colours we present a lower bound given by Conlon, Fox and Sudakov in [12].

Theorem 2.2.11. [12] *There exists a constant c such that*

$$R^{(3)}(t,t,t) \geq 2^{t^{c \log t}}.$$

To conclude this section, we prove two results regarding Ramsey numbers of specific graphs.

Theorem 2.2.12. [1] $R(K_3,k) \leq \lfloor e \cdot k! \rfloor + 1$.

Proof. [1] We prove this by induction on k.

The base case, namely $R(K_3,2) \leq \lfloor e \cdot 2! \rfloor + 1 = \lfloor 5.4366 \rfloor + 1 = 6$, was proved in the introductory section.

Now assume that $k \geq 3$ and that $R(K_3,k-1) \leq \lfloor e \cdot (k-1)! \rfloor + 1$, and observe any k-colouring of the edges of the complete graph on $\lfloor e \cdot k! \rfloor + 1$ vertices. Fix a vertex v in this graph. There are $\lfloor e \cdot k! \rfloor$ edges incident to v, so at least $\left\lceil \frac{\lfloor e \cdot k! \rfloor}{k} \right\rceil$ of them are coloured with the same colour, call it i. Let S be the set of vertices that are joined to v with an edge coloured with i.

If any edge between two vertices of S is coloured with i, there is a monochromatic triangle (K_3).

Now assume that no edge between two vertices of S is coloured with i, i.e., all the edges between vertices of S are coloured with one of $k-1$ different colours. Since

$$\left\lceil \frac{\lfloor e \cdot k! \rfloor}{k} \right\rceil = \left\lceil \frac{1}{k} \cdot \left\lfloor \sum_{j=1}^{\infty} \frac{k!}{j!} \right\rfloor \right\rceil = \left\lceil \frac{1}{k} \cdot \sum_{j=1}^{k} \frac{k!}{j!} \right\rceil = \left\lceil \frac{1}{k} + \sum_{j=1}^{k-1} \frac{(k-1)!}{j!} \right\rceil = 1 + \lfloor e \cdot (k-1)! \rfloor,$$

we can apply the assumption that $R(K_3, k-1) \leq \lfloor e \cdot (k-1)! \rfloor + 1$, and conclude that there is a monochromatic triangle in the subgraph induced by S. □

Theorem 2.2.13. [13] *If T is a tree on m vertices, then* $R(T, K_n) = (m-1)(n-1) + 1$.

Proof. [1] First we prove the lower bound: $R(T, K_n) > (m-1)(n-1)$.

We construct a blue-red edge colouring of $K_{(m-1)(n-1)}$ without a blue T and without a red K_n: divide the graph into $n-1$ cliques of size $m-1$, colour the cliques in blue, and colour all the other edges in red. Since every connected blue subgraph has at most $m-1$ vertices, there is no blue T. Since the red subgraph is $(n-1)$-partite, there is no red K_n.

Now we prove the upper bound: $R(T, K_n) \leq (m-1)(n-1) + 1$. We first prove the auxiliary result: if a graph has minimum degree at least $m-1$, then it contains every tree on m vertices as a subgraph.

We prove this auxiliary result by induction on m. The base case $m = 1$ is trivial. Now assume that every graph with minimum degree at least $m-2$ contains every tree on $m-1$ vertices as a subgraph. Fix any graph G with minimum degree at least $m-1$ and fix any tree T on m vertices. Delete a leaf v from T to obtain a tree $T \backslash v$ on $m-1$ vertices. By assumption, G contains $T \backslash v$. Since minimum degree of G is $m-1$, every vertex of $T \backslash v$ has a neighbour outside of $T \backslash v$. We can use one of these neighbours to obtain T as a subgraph.

Using this result, we prove $R(T, K_n) \leq (m-1)(n-1) + 1$ by induction on n. The base case $n = 1$ is trivial. Now assume that $R(T, K_{n-1}) \leq (m-1)(n-2) + 1$. Fix a blue-red edge colouring of $K_{(m-1)(n-1)+1}$. First assume that some vertex v has at least $(m-1)(n-2) + 1$ red edges incident to it. By assumption, among these neighbours there exists a blue T, or a red K_{n-1} that constitutes a red K_n together with v. Now assume that no vertex has more than $(m-1)(n-2)$ red edges incident to it. That implies that every vertex has at least $m-1$ blue edges incident to it, so minimum degree of the blue subgraph is at least $m-1$. Applying the result from above implies that there exists a blue T as a subgraph. □

3 Variants

In this section we are going to give a brief overview of some different variants of Ramsey numbers. In particular, we are going to define induced Ramsey numbers, Folkman numbers, size, chromatic, degree and bipartite Ramsey numbers. We will state the corresponding bounds and some open problems regarding these numbers. List Ramsey numbers will be analysed in more detail in the next section.

3.1 Induced Ramsey numbers

Definition 3.1.1. *Let G be an r-uniform hypergraph. An r-uniform hypergraph H is called an induced subgraph of G if $V(H) \subseteq V(G)$ and if every r vertices of H form an edge if and only if the same vertices in G form an edge.*

Definition 3.1.2. *The induced k-colour Ramsey number $R_{ind}(H,k)$ of an r-uniform hypergraph H is the least integer N such that there exists an r-uniform hypergraph G on N vertices for which every k-colouring contains a monochromatic copy of H as an induced subgraph. For the two-colour case we simply write $R_{ind}(H) := R_{ind}(H,2)$.*

For r-uniform hypergraphs H_1 and H_2 we also define $R_{ind}(H_1,H_2)$ to be the least integer N such that there exists an r-uniform hypergraph G on N vertices for which every 2-colouring contains a monochromatic copy of H_1 or a monochromatic copy of H_2 as an induced subgraph.

We first turn our attention to ordinary graphs (edges connect two vertices). In 1984 in [14] Erdös conjectured that $R_{ind}(H) \leq 2^{cn}$, where n is the number of vertices of H and c is a constant. The most recent development in this direction was made by Conlon, Fox and Sudakov in [15], as an improvement of an earlier result by Kohayakawa, Prömel and Rödl in [16]. We present their results.

Theorem 3.1.3. [16] *Let G and H be graphs with $|V(G)| = k$ and $|V(H)| = t$, where $k \leq t$, and suppose $q = \chi(H) \geq 2$. Then*

$$R_{ind}(G,H) \leq t^{Ck\log(q)}$$

for some absolute constant C.

In particular, for a graph H on n vertices, this theorem implies that:

$$R_{ind}(H) \leq n^{C_1 \cdot n\log(\chi(H))} \leq 2^{\,C_2\log(n)\cdot n\log((\chi(H))} \leq 2^{\,C_2\, n\log^2(n)}$$

for some absolute constants C_1 and C_2. The next theorem is a refinement of this result by a factor of $\log(n)$ in the exponent.

Theorem 3.1.4. [15] *There exists a constant c such that every graph H with n vertices satisfies*

$$R_{ind}(H) \leq 2^{cn \log n}.$$

If we include more information about the graph in our analysis, there exist better bounds which are worth noting.

Theorem 3.1.5. [17] *There is an absolute constant c such that every d-degenerate graph H on n vertices with chromatic number $\chi \geq 2$ has induced Ramsey number $R_{ind}(H) \leq n^{cd \log(\chi)}$.*

For the lower bound, we state some interesting results by Gorgol in [18] that include independence and clique number.

Theorem 3.1.6. [18] *Let G be an arbitrary connected graph with $\alpha(G) = \alpha \geq 2$ and H be an arbitrary graph with $\omega(H) = \omega$. Then*

$$R_{ind}(G, H) \geq (\alpha - 1)\frac{\omega(\omega - 1)}{2} + \omega.$$

Theorem 3.1.7. [18] *Let G be an arbitrary isolates-free (without isolated vertices) graph with $\alpha(G) = \alpha \geq 2$ and H be an arbitrary graph with $\omega(H) = \omega \geq 3$. Then*

$$R_{ind}(G, H) \geq \alpha\omega.$$

For hypergraphs, it was shown by Conlon, Dellamonica, La Fleur, Rödl and Schacht in [19] that the induced Ramsey number is never significantly larger than the ordinary Ramsey number for the same graph.

Theorem 3.1.8. [19] *Let H be a r-uniform hypergraph with n vertices and t edges. Then there are positive constants c_1, c_2, and c_3 such that*

$$R_{ind}(H, k) \leq 2^{c_1 r t^3 \log(knt)} R^{c_2 r t^2 + c_3 nt}$$

where $R = R(H, k)$ is the ordinary k-colour Ramsey number of H.

Finally, we state two open problems regarding induced Ramsey numbers.

Problem 3.1.9. [5] *Show that if H is a graph on n vertices and $q \geq 3$ is a natural number, then $R_{ind}(H, q) \leq 2^{n^{1+o(1)}}$.*

Problem 3.1.10. [5] *Does there exist a constant d such that $R_{ind}(H) \leq c(\Delta)n^d$ for all graphs with n vertices and maximum degree Δ?.*

3.2 Folkman numbers

Definition 3.2.1. *For any integers t and k, we define the Folkman number $f(t, k)$ as the least number of vertices necessary to construct a graph G without $(t + 1)$-clique, but such that every k-colouring of the edges of G contains a monochromatic t-clique.*

It is not obvious that we can always construct such graph. The two-colour case was resolved in a paper by Folkman [20]. The general case was proved by Nešetřil and Rödl in [21]. We present a recent upper bound for the two-colour case by Rödl, Ruciński and Schacht in [22].

Theorem 3.2.2. [22] *For all integers $k \leq 2$ and $t \leq 3$,*

$$f(t, k) \leq t^{400t^4} \left(R(K_t, k)\right)^{40k^2} \leq 2^{c(t^4 \log t + t^3 k \log k)},$$

for some $c > 0$ independent of r and k.

In [5], Conlon, Fox and Sudakov conjectured that the Folkman number is at most exponential in t.

Conjecture 3.2.3. [5] *There exists a constant c such that*

$$f(t) \leq 2^{ct}.$$

3.3 Size Ramsey numbers

Definition 3.3.1. *For a graph H, the size Ramsey number $\hat{r}(H)$ is the smallest number m such that there exists a graph G with m edges such that every two-colouring of the edges of G contains a monochromatic copy of H.*

Observation 3.3.2. $\hat{r}(H) \leq \binom{R(H,2)}{2}$.

Proof. A complete graph on $R(H, 2)$ vertices has $\binom{R(H,2)}{2}$ edges and every two-colouring of its edges contains a monochromatic copy of H. □

In [23], Beck proved some interesting results regarding the size Ramsey numbers of paths and trees. Namely, he showed that the size Ramsey number of a path of length n is linear in n, and he proved a general upper bound for the size Ramsey number of trees. We state the results below.

Theorem 3.3.3. [23] $\hat{r}(P_n) < 900n$*, where P_n is a path of length n.*

Theorem 3.3.4. [23] *If the tree T has n edges and maximal degree at most D, then*

$$\hat{r}(T) < D \cdot n \cdot (\log n)^{12}.$$

Beck also conjectured the following:

Conjecture 3.3.5. [23] *For any tree T with n edges and maximal degree D:*

$$\hat{r}(T) < C \cdot D \cdot n,$$

where C is a universal constant.

3.4 Chromatic and degree Ramsey numbers

Similarly to the size Ramsey numbers, for a graph H we can define f-Ramsey number $R_f(H)$ for any real-valued graph parameter f in the following way: $R_f(H)$ is the minimum value of $f(G)$ among all graphs G such that every two-colouring of the edges of G contains a monochromatic copy of H.

First we observe the chromatic Ramsey number $R_\chi(H)$, where we take the chromatic number for the parameter f. We present the main results involved.

Theorem 3.4.1. [24] $R_\chi(H) \geq (t-1)^2 + 1$*, where t is the chromatic number of H.*

Burr, Erdős and Lovász also conjectured that for every t there exists a graph with chromatic number t for which this bound is sharp, and it was shown by Paul and Tardiff in [25] using a result by Zhu in [26].

Theorem 3.4.2. [5] *For every natural number t, there exists a graph H with chromatic number t such that:*

$$R_\chi(H) = (t-1)^2 + 1.$$

We also define degree Ramsey numbers: $R_\Delta(H)$ is the minimum value of $\Delta(G)$ among all graphs G such that every two-colouring of the edges of G contains a monochromatic copy of H.

There is an interesting open problem regarding degree Ramsey numbers:

Problem 3.4.3 [5] *Is it true that for every $\Delta \geq 3$ there exists a natural number Δ' such that $R_\Delta(H) \leq \Delta'$ for every graph H of maximum degree Δ?*

This is true for $\Delta = 2$, but it is suspected in [5] that the general answer is no.

3.5 Bipartite Ramsey numbers

If we set the underlying graph to be a complete bipartite graph $K_{n,n}$ instead of a complete graph K_n, we get the notion of bipartite Ramsey numbers.

Definition 3.5.1. *The k-colour bipartite Ramsey number of a bipartite graph G is the least integer n such that every k-colour edge colouring of $K_{n,n}$ contains a monochromatic copy of G. We denote this integer by $b(G, k)$.*

We state a result concerning complete bipartite graphs which was proved by Conlon in [27].

Theorem 3.5.2. [27] $b(K_{k,k}, 2) \leq (1 + o(1)) \cdot 2^{k+1} \log k$, *where the log is taken to the base* 2.

We also present three results concerning bipartite Ramsey numbers of cycles and paths that were discovered by Bucić, Letzter and Sudakov in [28].

Theorem 3.5.3. [28] *Let $k \geq 4$. The k-colour bipartite Ramsey number of a cycle or a path of order $2n$ is at most $(2k - 3 + o(1))n$.*

Theorem 3.5.4. [28] *Let $k \leq 8$. The k-colour bipartite Ramsey number of a cycle or path of order $2n$ is at most $\left(2k - 3.5 + \frac{1}{k-2} + o(1)\right)n$.*

Theorem 3.5.5. [28] *The 5-colour bipartite Ramsey number of a cycle or path of order $2(n + 1)$ is larger than $6.5n$.*

4 List Ramsey numbers

4.1 Definition

Combining ordinary Ramsey numbers and list colourings, we can define list Ramsey numbers in a natural way:

Definition 4.1.1. *The k-colour list Ramsey number of an r-uniform hypergraph G is*

$$R_l(G,k) := \min\{n : \exists L : E(K_n^{(r)}) \to \binom{\mathbb{N}}{k} \text{ s.t. } \forall L\text{-colouring of } (E(K_n^{(r)}) \exists \text{ a monochromatic copy of } G\}.$$

The idea behind list Ramsey numbers is to find an assignment of lists of fixed size k to the edges of K_n such that any edge colouring using these lists contains a monochromatic copy of a fixed graph G. The smallest integer n for which such assignment of lists exists is the list Ramsey number of G and k.

It is easy to see that the list Ramsey number is always less or equal to the ordinary Ramsey number. Namely, we can define all lists to be $\{1,\dots,k\}$ and thus force a monochromatic copy of G in any edge colouring.

Observation 4.1.2. $R_l(G,k) \leq R(G,k)$

In the next sections we are going to discuss list Ramsey numbers of stars and matchings in more detail.

4.2 Stars

4.2.1 General bounds

Definition 4.2.1.1. *An r-star is a complete bipartite graph with one part consisting of only one vertex, and the other part consisting of r vertices, i.e., $K_{1,r}$.*

In this section we are going to investigate list Ramsey numbers of stars and open questions that arise. In [29], Burr and Roberts proved the following for the ordinary Ramsey number of stars:

Theorem 4.2.1.2. [29]

$$\begin{aligned} R(K_{1,r},k) &= (r-1)k+1, \text{ if both } r \text{ and } k \text{ are even,} \\ R(K_{1,r},k) &= (r-1)k+2, \text{ otherwise.} \end{aligned}$$

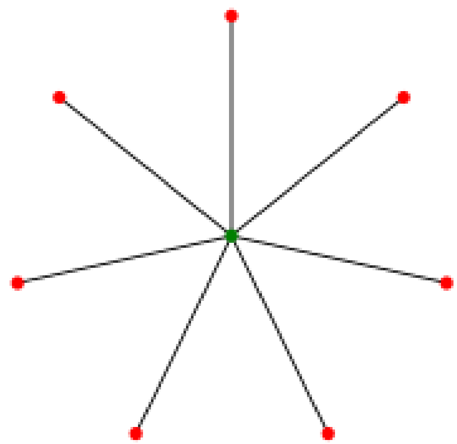

Figure 2: A star graph $K_{1,7}$

The following upper bounds are a simple corollary of Observation 4.1.2.

Corollary 4.2.1.3.

$$R_l(K_{1,r}, k) \le (r-1)k + 1, \text{ if both } r \text{ and } k \text{ are even,}$$
$$R_l(K_{1,r}, k) \le (r-1)k + 2, \text{ otherwise.}$$

We now turn our focus to lower bounds of the list Ramsey numbers of stars. We will use Galvin's Theorem [30], a result by Häggkvist and Janssen from [31], and a lemma from [32] extensively in our analysis.

Theorem 4.2.1.4. [30] *If G is a bipartite graph with maximal degree Δ, then $\chi'_l(G) = \Delta$.*

Theorem 4.2.1.5. [31] *For $n \in \mathbb{N}$, $\chi'_l(K_n) \le n$.*

Lemma 4.2.1.6. [32] *If subgraphs $H_1, \ldots, H_t$ partition the edge set of a complete graph K_n, $\chi'_l(H_i) \le k$ for $i = 1, \ldots, t$, and every vertex of K_n belongs to at most $r-1$ of these subgraphs, then $R_l(K_{1,r}, k) > n$.*

Proof.

For the sake of contradiction, let us suppose the opposite: $R_l(K_{1,r}, k) \le n$ and that the conditions of lemma are satisfied. Then there exists an assignment of lists L where lists are of size k, such that every L-colouring of K_n contains a monochromatic copy of $K_{1,r}$. Since $\chi'_l(H_i) \le k$, we have a proper edge colouring from lists L for every H_i. Since every vertex belongs to at most $r-1$ of H_i-s and all the colourings are proper, it follows that one colour can be repeated at most $r-1$ times for edges incident to a single vertex.

We constructed an L-colouring of K_n without monochromatic copy of $K_{1,r}$, which is a contradiction. □

Using these results, Alon, Bucić, Kalvari, Kuperwasser and Szabó [32] proved a general lower bound which applies for all stars.

Theorem 4.2.1.7. [32] *The following holds for the list Ramsey number of stars:*

$$R_l(K_{1,r}, k) \geq (r-1)k + 1.$$

Proof. Observe the complete graph $K_{(r-1)k}$. Partition the set of vertices into disjoint sets of k vertices each: $V_1, \ldots, V_{r-1}$. Now partition the edge set of $K_{(r-1)k}$ into $r-1$ complete graphs on $V_1, \ldots, V_{r-1}$, call them $G_1, \ldots, G_{r-1}$ and complete bipartite graphs between V_i and V_j for $i < j$, call them $H_{i,j}$. By Theorem 4.2.1.5, $\chi_l'(G_i) \leq k$ and by Theorem 4.2.1.4, $\chi_l'(H_{i,j}) = k$. Since every vertex of $K_{(r-1)k}$ belongs to exactly $r-1$ of these subgraphs, namely $H_{1,i}, \ldots, H_{i-1,i}, G_i, H_{i,i+1}, \ldots, H_{i,r-1}$, we can apply Lemma 4.2.1.6 to obtain that $R_l(K_{1,r}, k) \geq (r-1)k + 1$.

Now we can completely characterise the list Ramsey number $R_l(K_{1,r}, k)$ when both k and r are even.

Corollary 4.2.1.8. [32] $R_l(K_{1,r}, k) = R(K_{1,r}, k) = (r-1)k + 1$ *if both r and k are even.*

For the case when r or k is odd, so far we have established the following bound:

$$(r-1)k + 1 \leq R_l(K_{1,r}, k) \leq (r-1)k + 2.$$

The next result will partially resolve this case, and it was proved in [32] by Alon, Bucić, Kalvari, Kuperwasser and Szabó, using the following result by Gustavsson in [33].

Theorem 4.2.1.9. [33] *For any graph F there exist $\epsilon = \epsilon(F) > 0$ and $n_0 = n_0(F)$ such that for any graph G on $n \geq n_0$ vertices with minimum degree at least $(1-\epsilon)n$ one can partition the edge set of G into copies of F, provided:*

- $e(F) \mid e(G)$ *and*
- $\gcd(F) \mid \gcd(G)$

where $e(H)$ denotes the number of edges of a graph H and $\gcd(H)$ denotes the greatest common divisor of the degrees of vertices in H.

Theorem 4.2.1.10. [32] *For every $k \in \mathbb{N}$ there exists $\omega(k) \in \mathbb{N}$ such that the following holds:*

If $r \geq \omega(k)$ and r and k are not both even, then $R_l(K_{1,r}, k) = R(K_{1,r}, k) = (r-1)k + 2$.

Proof. Let $n = (r-1)k + 1$. We will once again apply Lemma 4.2.1.6.

First observe that $k \mid e(K_n) = \frac{((r-1)k+1)(r-1)k}{2}$. If k is odd, then $((r-1)k+1)(r-1)$ is an even number, so k divides $e(K_n)$. If k is even, r has to be odd, so $r-1$ is even, and once again we observe that $k \mid e(K_n)$.

Now define $t = \frac{e(K_n)}{k} \mod k$. Define vertex disjoint subgraphs $G_1, \ldots, G_t$ of K_n such that each G_i is isomorphic to a complete bipartite graph $K_{k+1,k+1}$ with a perfect matching removed. In order to have enough vertices, the condition $n \geq t(2k+2)$ has to be satisfied. We can achieve this by imposing $\omega(k) \geq 2k+1$. Namely, if $\omega(k) \geq 2k+1$, then

$n = (r-1)k + 1 \geq (\omega(k)-1)k + 1 \geq 2k^2 + 1 > 2k^2 - 2 = (k-1)(2k+2) \geq t(2k+2)$.

Define a graph F to be a complete bipartite graph $K_{k,k}$, and define graph G to be the graph K_n without the edges of $G_1, \ldots, G_t$. Now we show that graphs F and G satisfy the conditions of Theorem 4.2.1.9.

First observe that $e(K_n) \equiv tk \mod k^2$, since $\frac{e(K_n)}{k} \equiv t \mod k$. We have $e(G) = e(K_n) - t((k+1)(k+1) - (k+1)) \equiv tk - tk(k+1) \equiv -tk^2 \equiv 0 \mod k^2$. Since $e(F) = k^2$, it follows that $e(F) \mid e(G)$.

Degrees of vertices in G are either $(r-1)k$ or $(r-1)k - k$, so $\gcd(G)$ is a multiple of k. Degrees of vertices in F are k, so $\gcd(F) = k$, which implies that $\gcd(F) \mid \gcd(G)$.

Two more conditions have to be satisfied, i.e., $n \geq n_0(F)$ and minimum degree of G ($\delta(G)$) has to be at least $(1-\epsilon(F))n$. If we impose $\omega(k) \geq \frac{n_0(F)}{k}$, we get:

$$n = (r-1)k + 1 \geq (\omega(k)-1)k + 1 \geq (\frac{n_0(F)}{k} - 1)k + 1 = n_0(F) - k + 1 \geq n_0(F).$$

If we impose $\omega(k) \geq \frac{2}{\epsilon(F)}$, we get:

$$\begin{gathered} 1 - \epsilon(F) \leq (1-\epsilon(F)k \implies 1 \leq (1-\epsilon(F))k + \epsilon(F) \implies \\ 1 + k \leq 2k - \epsilon(F)k + \epsilon(F) = \epsilon(F)(\tfrac{2}{\epsilon(F)}k - k + 1) \leq \epsilon(F)((r-1)k+1) \\ \implies \delta(G) \geq (r-1)k - k \geq (r-1)k + 1 - \epsilon(F)((r-1)k+1) = \\ = (1-\epsilon(F))((r-1)k+1) = (1-\epsilon(F))n. \end{gathered}$$

If $\omega \geq \max(2k+1, \frac{n_0(F)}{k}, \frac{2}{\epsilon(F)})$, then F and G satisfy the conditions of Theorem 4.2.1.9, so we can partition the edge set of G into $G_{t+1}, \ldots, G_q$ which are all copies of $F = K_{k,k}$.

Now we have a partition of the edge set of K_n into $G_1, \ldots, G_q$, where $\chi'_l(G_1) = \cdots = \chi'_l(G_t) = k$ by Galvin's Theorem and $\chi'_l(G_{t+1}) = \cdots = \chi'_l(G_q) \leq k$ by Theorem 4.2.1.5. Since graphs $G_1, \ldots, G_q$ are k-regular, every vertex of $K_{(r-1)k+1}$ belongs to exactly $r-1$ of these graphs. We can apply Lemma 4.2.1.6 to obtain $R_l(K_{1,r}, k) > (r-1)k+1$, and together with Corollary 4.2.1.3 we get $R_l(K_{1,r}, k) = (r-1)k+2$ for $r \geq \omega(k)$. □

These results lead us to a reasonable conjecture, that ordinary Ramsey numbers and list Ramsey numbers for stars coincide.

Conjecture 4.2.1.11. [32] *$R_l(K_{1,r}, k) = R(K_{1,r}, k)$ for every $k, r \in \mathbb{N}$.*

For the moment, this remains an open question. It is worth noting that the List Colouring Conjecture actually implies the previous conjecture, and that the case of $r = 2$ is equivalent to the List Colouring Conjecture for cliques.

4.2.2 Special cases

In [32], Alon, Bucić, Kalvari, Kuperwasser and Szabó proved the exact bounds for two-colour list Ramsey numbers of stars, using results by Alspach and Galvas in [34] and Šajna in [35]. First, we present these preliminary results.

Theorem 4.2.2.1. [34] *For positive even integers m and n with $4 \leq m \leq n$, the complete graph K_n can be decomposed into a perfect matching and cycles of length m if and only if $|E(K_n)| - \frac{n}{2}$ is a multiple of m.*

Corollary 4.2.2.2. *K_n can be decomposed into a perfect matching and Hamilton cycles when n is even.*

Proof. Since $|E(K_n)| - \frac{n}{2} = \frac{n(n-1)}{2} - \frac{n}{2} = \frac{n^2}{2} - n$ is a multiple of n (n is even), it follows by Theorem 4.2.2.1 that, for $n \geq 4$, K_n can be decomposed into a perfect matching and cycles of length n, which are exactly Hamilton cycles. The case of $n = 2$ is trivial. □

Theorem 4.2.2.3. [35] *Let n be an odd integer and m be an even integer with $3 \leq m \leq n$. The complete graph K_n can be decomposed into cycles of length m whenever m divides the number of edges in K_n.*

Using these results, we can completely characterise the two-colour list Ramsey numbers of stars.

Theorem 4.2.2.4. [32] *The following holds for the two-colour list Ramsey numbers of stars:*

$$R_l(K_{1,r}, 2) = R(K_{1,r}, 2) = \begin{cases} 2r - 1 & \text{if } r \text{ is even,} \\ 2r & \text{if } r \text{ is odd.} \end{cases}$$

Proof.

We will apply Lemma 4.2.1.6 in both even and odd case.[-2ex]

Even case

By Corollary 4.2.1.3, we already have the upper bound: $R(K_{1,r}, 2) \leq 2r - 1$. We prove the lower bound $R(K_{1,r}, 2) > 2r - 2$ using Lemma 4.2.1.6.

By Corollary 4.2.2.2, we can partition the edge set of K_{2r-2} into a perfect matching H_1 and $r - 2$ Hamilton cycles $H_2, \ldots, H_{r-1}$. Since $\chi'_l(H_i) \leq 2$ for $i = 1, \ldots, t$, by Galvin's Theorem, and every vertex belongs to exactly $r - 1$ H_i-s (all of them), we can apply Lemma 4.2.1.6 to obtain $R(K_{1,r}, 2) > 2r - 2$.

Odd case

By Corollary 4.2.1.3, we already have the upper bound: $R(K_{1,r}, 2) \leq 2r$. We prove the lower bound $R(K_{1,r}, 2) > 2r - 1$ using Lemma 4.2.1.6.

For $r \geq 5$, we observe the edge set of K_{2r-1}. The number of edges is $\frac{(2r-1)(2r-2)}{2} = (2r-1)(r-1)$, and since $r-1$ is even and divides the number of edges, we can partition the edge set of K_{2r-1} into even cycles of length $r-1$ by Theorem 4.2.2.3. Every vertex belongs to exactly $\frac{2r-2}{2} = r-1$ of these cycles so we can again apply Lemma 4.2.1.6 to obtain $R(K_{1,r}, 2) > 2r-1$.

For $r = 3$, we observe the graph K_5. Partition edges of K_5 into two 5-cycles: C_1 and C_2. Suppose the opposite, that $R_l(K_{1,3}, 2) \leq 5$. Then there exists an assignment of lists of size 2, call it L, such that every L-colouring of the edges contains a monochromatic copy of $K_{1,3}$.

Case 1 The lists assigned to the edges of C_1 are all equal.
In this case we colour all edges of C_1 with the same colour, and then colour the edges of C_2 with a different colour. Now every vertex of K_5 has at most two incident edges of the same colour, hence no monochromatic $K_{1,3}$.

Case 2 The lists assigned to the edges of C_2 are all equal.
Analogous to Case 1.

Case 3 Neither are the lists assigned to the edges of C_1 all equal, nor are the lists assigned to the edges of C_2 all equal.

First observe the following: if we have lists of size 2 which are not all equal assigned to the edges of an odd cycle, we can always find a proper edge colouring from these lists. The process is as follows:

- First find a pair of neighbouring edges with different lists.

- Choose a colour for one of these edges which does not appear in the list of the other edge, leave this other edge for the end.

- Choose colours iteratively, always choosing a colour different than the colour of the neighbouring edge.

- For the last edge, we have at least one colour in the list which was not used for the neighbouring edges.

Now we can choose proper colourings for C_1 and C_2. It follows that every vertex of K_5 has at most two incident edges of the same colour, hence no monochromatic $K_{1,3}$.

After observing all cases, we know that we cannot find an assignment of lists such that every colouring from this assignment contains a monochromatic $K_{1,3}$. We conclude that $R_l(K_{1,3}, 2) > 5$. □

In the previous section, it was shown that ordinary and list Ramsey numbers always coincide for large enough stars, and whenever both k and r are even. Therefore, we turn our attention to smaller stars, namely $K_{1,3}$. We will use a result by Schauz from [36].

Theorem 4.2.2.5. [36] $\chi'_l(K_{p+1}) = p$ *for every odd prime* p.

Corollary 4.2.2.6. $R_l(K_{1,3}, k) = 2k + 2$ *whenever* k *is a prime number.*

Proof. The case $k = 2$ was proved in Theorem 4.2.2.4.

For odd prime numbers we use Theorem 4.2.2.5 and Lemma 4.2.1.6. We already know by Corollary 4.2.1.3 that $R_l(K_{1,3}, k) \leq 2k + 2$. Observe the complete graph K_{2k+1}. Partition its vertex set into a single vertex v and two sets of size k, call them V_1 and V_2. Now partition the edge set into a complete graph on $\{v\} \cup V_1$, call it G_1, a complete graph on $\{v\} \cup V_2$, call it G_2, and a complete bipartite graph between V_1 and V_2, call it H. By Theorem 4.2.2.5, $\chi'_l(G_1) = \chi'_l(G_2) = k$, and by Galvin's Theorem, $\chi'_l(H) = k$. Vertex v belongs to G_1 and G_2, vertices of V_1 belong to G_1 and H, and vertices of V_2 belong to G_2 and H, so we can apply Lemma 4.2.1.6 to obtain $R_l(K_{1,3}, k) > 2k + 1$. Together with Corollary 4.2.1.3, we conclude that $R_l(K_{1,3}, k) = 2k + 2$ whenever k is a prime number. □

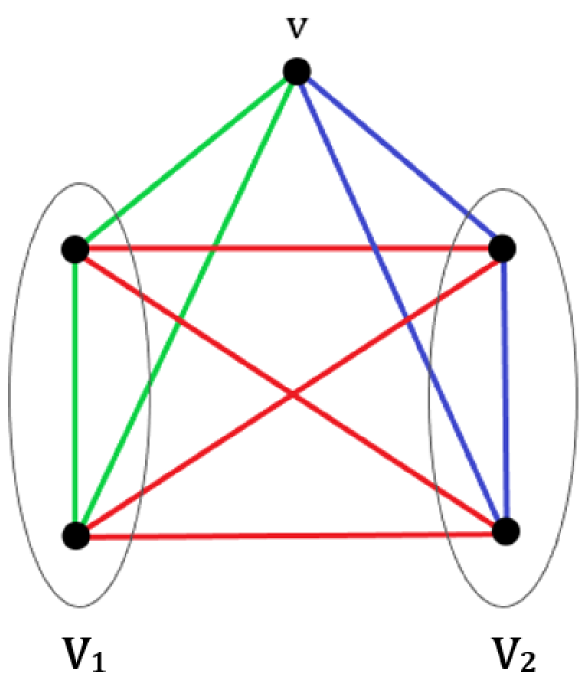

Figure 3: Proof of Corollary 4.2.2.6

4.3 Matchings

Definition 4.3.1. *A matching rK_2 is a graph on $2r$ vertices with r independent edges.*

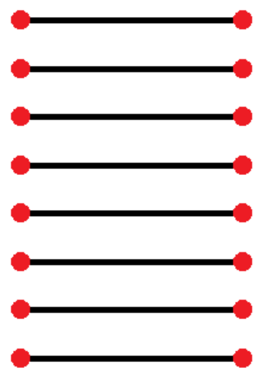

Figure 4: A matching $8K_2$

In this section we are going to investigate list Ramsey numbers of matchings and open questions that arise. In [37], Cockayne and Lorimer proved the following for the ordinary Ramsey number of matchings:

Theorem 4.3.2. [37] *For $r_1, \dots, r_k \in \mathbb{N}$ and $r := max(r_1, \dots, r_k)$, the following holds:*

$$R(r_1K_2, \dots, r_kK_2) = r + 1 + \sum_{i=1}^{k}(r_i - 1).$$

In particular, $R(rK_2, k) = r + 1 + rk - k = (r-1)k + r + 1$.

The following two lemmas show certain lower and upper bounds for list Ramsey numbers of matchings, and they were stated and proved in [32] by Alon, Bucić, Kalvari, Kuperwasser and Szabó. We present their proofs in detail.

Lemma 4.3.3. [32] *Let $r, k \in \mathbb{N}$, and assume $k \geq 2$. Then:*

$$R_l(rK_2, k) \ \geq \ \max\left(2r,\ (r-1) \cdot \lfloor \tfrac{k}{2\log(rk)} \rfloor + r + 1\right).$$

Proof:

First suppose that $2r \geq (r-1) \cdot \lfloor \frac{k}{2\log(rk)} \rfloor + r + 1$. We want to prove that $R_l(rK_2, k) > 2r - 1$, so we observe the complete graph K_{2r-1}. We prove that by showing that for every assignment of lists L of size k to the edges of K_{2r-1} we can find an L-colouring of edges without a monochromatic rK_2. But that is trivial because K_{2r-1} cannot contain a monochromatic rK_2.

Now suppose that $2r < (r-1) \cdot \lfloor \frac{k}{2\log(rk)} \rfloor + r + 1$ and define $N := (r-1) \cdot \lfloor \frac{k}{2\log(rk)} \rfloor + r$. We want to prove that $R_l(rK_2, k) > N$ so we observe K_N. We prove that by showing that for every assignment of lists L of size k to the edges of K_N we can find an L-colouring of edges without a monochromatic rK_2.

Observe that $\lfloor \frac{k}{2\log(rk)} \rfloor \geq 2$ since $(r-1) \cdot \lfloor \frac{k}{2\log(rk)} \rfloor > r - 1$, and define $t := \lfloor \frac{k}{2\log(rk)} \rfloor$. Observe that $N = (r-1)r + r$. Since $R(rK_2, t) = (r-1)t + r + 1$ by Theorem 4.3.2, there exists a t-colouring of the edges of K_N without a monochromatic rK_2. Fix one such t-colouring and call it $c : E(K_N) \to [t]$.

Choose any assignment of lists L of size k to the edges of K_N and denote by L_e the list assigned to the edge e. Now, to each colour in $\bigcup_{e \in E(K_N)} L_e$ assign a colour from $[t]$, randomly and uniformly (assign each colour with probability $\frac{1}{t}$). Denote by B_e the event that no colour from L_e was assigned the colour $c(e)$, and denote by B the event that at least one list L_e has no colour with $c(e)$ as an assigned colour (B is the event that at least one B_e happens).

$$\begin{aligned}
\mathbb{P}(B) &\underset{\text{Boole's inequality}}{\leq} \sum_{e \in E(K_N)} \mathbb{P}(B_e) = \sum_{e \in E(K_N)} \left(1 - \frac{1}{t}\right)^k = \sum_{e \in E(K_N)} \left(1 - \frac{1}{\lfloor \frac{k}{2\log(rk)} \rfloor}\right)^k \leq \\
&\leq \sum_{e \in E(K_N)} \left(1 - \frac{2\log(rk)}{k}\right)^k \leq \sum_{e \in E(K_N)} e^{2\log(rk)} = \sum_{e \in E(K_N)} \frac{1}{r^2k^2} = \frac{N(N-1)}{2r^2k^2} < \\
&< \frac{N^2}{2r^2k^2} \leq \frac{\left((r-1) \cdot \frac{k}{2\log(rk)} + r\right)^2}{2r^2k^2} \leq \frac{r^2 \cdot \left(\frac{k}{2\log(rk)}\right)^2 + r^2 \cdot \frac{k}{\log(rk)} + r^2}{2r^2k^2} = \\
&= \frac{1}{8(\log(rk))^2} + \frac{1}{2k\log(rk)} + \frac{1}{2k^2} \leq \frac{1}{8(\log(2))^2} + \frac{1}{2 \cdot 2\log(2)} + \frac{1}{2 \cdot 2^2} < \\
&< 0.2602 + 0.3607 + 0.125 = 0.7459 < 1
\end{aligned}$$

Since $\mathbb{P}(B) < 1$, there exists an assignment of colours from $[t]$ to colours in $\bigcup_{e \in E(K_N)} L_e$ such that B does not happen, i.e., such that every list L_e has at least one colour with $c(e)$ as an assigned colour. Using one such assignment, we define an L-colouring of the edges of K_N: from list L_e choose any colour which was assigned the colour $c(e)$. Notice that if two edges are coloured differently with colouring c, they are again coloured differently with this L-colouring. Since there is no monochromatic rK_2 with colouring c, there is no monochromatic rK_2 with our L-colouring.

We proved that for any assignment of lists L of size k to the edges of K_N we can find an L-colouring of edges without a monochromatic rK_2. It follows that $R_l(rK_2, k) > N = (r-1) \cdot \lfloor \frac{k}{2\log(rk)} \rfloor + r$. □

Lemma 4.3.4. [32] *Let $r, k \in \mathbb{N}$. If $2(k+1) \leq \log(r)$, then:*

$$R_l(rK_2, k) \;\leq\; 2r + 2 \cdot \left\lceil 20r^{\frac{k}{k+1}} \right\rceil.$$

If $2(k+1) > \log(r)$, then:

$$R_l(rK_2, k) \;\leq\; 2 \left\lceil \frac{16rk}{\log(rk)} \right\rceil.$$

Proof:

If $r = 1$, then $2(k+1) > \log(r)$, and $R_l(rK_2, k) = 2 \leq 2 \left\lceil \dfrac{16k}{\log(k)} \right\rceil$.

If $k = 1$, then $R_l(rK_2, k) = 2r \leq 2r + 2 \cdot \left\lceil 20r^{\frac{k}{k+1}} \right\rceil$, and if $\log(r) < 4$, then $R_l(rK_2, k) = 2r \leq 2 \left\lceil \dfrac{16r}{4} \right\rceil \leq 2 \left\lceil \dfrac{16r}{\log(r)} \right\rceil$.

From now on, suppose that $r \geq 2$ and $k \geq 2$.

First suppose that $2(k+1) \leq \log(r)$, and define $N := 2r + 2 \cdot \left\lceil 20r^{\frac{k}{k+1}} \right\rceil$. We want to find an assignment of lists L of size k to the edges of K_N such that every L-colouring contains a monochromatic rK_2. We will choose colours for L independently, randomly and uniformly from $I := \{1, \ldots, t\}$, where $t := (k-1) \cdot \left(\left\lceil \dfrac{N}{20\, r^{\frac{k}{k+1}}} \right\rceil + 1 \right)$.

Denote by B the event that the lists L induce a colouring c without a monochromatic rK_2. We want to prove that $\mathbb{P}(B) < 1$. This implies that there exists an assignment of lists L such that no L-colouring is without a monochromatic rK_2.

Let us isolate a single complete bipartite graph $K_{\frac{N}{2}, \frac{N}{2}}$ from K_N and call it H. If B happens, it implies that there exists an L-colouring c such that H does not have a monochromatic matching of size r. By bipartite Kőnig's theorem, a minimum vertex cover of a single colour class of H has at most $r-1$ vertices. This implies that every colour class of colouring c is contained in one of the following graphs:

$$C_1,\ C_2,\ \ldots,\ C_{\binom{N}{r-1}},$$

where for each $(r-1)$-vertex subset of H we define one C_i as a subgraph of H containing all edges incident to some vertex in this $(r-1)$-vertex subset.

Denote by c_i the subgraph of H induced by the colour class $i \in I$ under colouring c, denote by L_e the list assigned to the edge e, and define $m := \binom{N}{r-1}$. Now we have the following inequalities:

$$
\begin{aligned}
\mathbb{P}(B) &\le \mathbb{P}(\exists \text{ an } L\text{-colouring } c \text{ s.t. } H \text{ is without monochromatic } rK_2) \\
&\le \mathbb{P}\left(\exists \text{ an } L\text{-colouring } c \text{ s.t. } \forall\, i \in I\ \exists\, j \in [m] \text{ s.t. } c_i \subseteq C_j\right) \\
&\le \sum_{j_1=1}^{m} \cdots \sum_{j_t=1}^{m} \mathbb{P}\left(\exists \text{ an } L\text{-colouring } c \text{ s.t. } \forall\, i \in [t] : c_i \subseteq C_{j_i}\right) \\
&\le m^t \max_{j_1,\ldots,j_t \in [m]} \mathbb{P}\left(\exists \text{ an } L\text{-colouring } c \text{ s.t. } \forall\, i \in [t] : c_i \subseteq C_{j_i}\right) \\
&= m^t \max_{j_1,\ldots,j_t \in [m]} \mathbb{P}\left(\forall e \in E(H)\ \exists\, i \in L_e \text{ s.t. } e \in C_{j_i}\right)
\end{aligned}
$$

Now we analyse the last term in this inequality. Fix $j_1, \ldots, j_t$ (not necessarily distinct). Since we assigned lists to the edges independently, we have the following equality:

$$
\mathbb{P}\left(\forall e \in E(H)\ \exists\, i \in L_e \text{ s.t. } e \in C_{j_i}\right) = \prod_{e \in E(H)} \mathbb{P}\left(\exists\, i \in L_e \text{ s.t. } e \in C_{j_i}\right).
$$

Remember that $j_1, \ldots, j_t$ are fixed. Denote by d_e the number of distinct graphs C_{j_i} which contain the edge e. Fix one edge e. The list L_e contains k colours from the universe of t colours. Since we assigned colours to the lists randomly and uniformly, the probability that we chose k colours for L_e such that $e \notin C_{j_i}$ for every colour $i \in L_e$ is $\frac{\binom{t-d_e}{k}}{\binom{t}{k}}$. This implies that:

$$
\mathbb{P}\left(\exists\, i \in L_e \text{ s.t. } e \in H_{j_i}\right) = 1 - \frac{\binom{t-d_e}{k}}{\binom{t}{k}}.
$$

Now we have:

$$
\begin{aligned}
\prod_{e \in E(H)} \mathbb{P}\left(\exists\, i \in L_e \text{ s.t. } e \in H_{j_i}\right) &= \prod_{e \in E(H)} \left(1 - \frac{\binom{t-d_e}{k}}{\binom{t}{k}}\right) \\
&= \prod_{e \in E(H)} \left(1 - \frac{\frac{(t-d_e)\cdot \cdots \cdot (t-d_e-k+1)}{k \cdot \cdots \cdot 1}}{\frac{t \cdot \cdots \cdot (t-k+1)}{k \cdot \cdots \cdot 1}}\right) \\
&= \prod_{e \in E(H)} \left(1 - \frac{(t-d_e)\cdot \cdots \cdot (t-d_e-k+1)}{t \cdot \cdots \cdot (t-k+1)}\right) \\
&= \prod_{e \in E(H)} \left(1 - \left(1 - \frac{d_e}{t}\right) \cdot \cdots \cdot \left(1 - \frac{d_e}{t-k+1}\right)\right) \\
&\le \prod_{e \in E(H)} \left(1 - \left(1 - \frac{d_e}{t-k+1}\right)^k\right)
\end{aligned}
$$

We apply logarithm to this expression so we can easily use mean inequalities.

$$\log\left(\prod_{e\,\in\,E(H)}\left(1-\left(1-\frac{d_e}{t-k+1}\right)^k\right)\right) = \sum_{e\,\in\,E(H)} \log\left(1-\left(1-\frac{d_e}{t-k+1}\right)^k\right)$$

$$\underset{\text{logarithm of AM-GM inequality}}{\leq} \frac{N^2}{4}\cdot\log\left(\frac{\sum_{e\,\in\,E(H)}\left(1-\left(1-\frac{d_e}{t-k+1}\right)^k\right)}{\frac{N^2}{4}}\right)$$

$$= \frac{N^2}{4}\cdot\log\left(1-\frac{\sum_{e\,\in\,E(H)}\left(1-\frac{d_e}{t-k+1}\right)^k}{\frac{N^2}{4}}\right)$$

$$\underset{\text{generalised mean inequality}}{\leq} \frac{N^2}{4}\cdot\log\left(1-\left(\frac{\sum_{e\,\in\,E(H)}\left(1-\frac{d_e}{t-k+1}\right)}{\frac{N^2}{4}}\right)^k\right)$$

$$= \log\left(1-\left(1-\frac{\widetilde{d_e}}{t-k+1}\right)^k\right)^{\frac{N^2}{4}},$$

where $\widetilde{d_e} = \frac{\sum_{e\in E(H)} d_e}{N^2/4}$. Now we have the following inequality:

$$\prod_{e\,\in\,E(H)} \mathbb{P}\left(\exists\, i \in L_e \text{ s.t. } e \in H_{j_i}\right) \leq \left(1-\left(1-\frac{\widetilde{d_e}}{t-k+1}\right)^k\right)^{\frac{N^2}{4}}.$$

Since the number of edges of H_{j_i} is at most $(r-1)\cdot\frac{N}{2}$, we observe the following:

$$\widetilde{d_e} = \frac{\sum_{e\in E(H)} d_e}{N^2/4} = \frac{\sum_{i=1}^{t} E(H_{j_i})}{N^2/4} \leq \frac{t\cdot(r-1)\cdot\frac{N}{2}}{N^2/4} = \frac{2t(r-1)}{N}.$$

Combining inequalities, we obtain:

$$\mathbb{P}(B) \leq \binom{N}{r-1}^t \left(1-\left(1-\frac{2t(r-1)}{N(t-k+1)}\right)^k\right)^{\frac{N^2}{4}}$$

$$\leq \left(\frac{eN}{r-1}\right)^{t(r-1)} \left(1 - \left(1 - \frac{2t(r-1)}{N(t-k+1)}\right)^k\right)^{\frac{N^2}{4}}.$$

Remember that our goal is to prove that $\mathbb{P}(B) < 1$, or equivalently, $\log(\mathbb{P}(B)) < 0$. Remember that $N = 2r + 2 \cdot \left\lceil 20r^{\frac{k}{k+1}} \right\rceil$, and $t = (k-1) \cdot \left(\left\lceil \frac{N}{20\, r^{\frac{k}{k+1}}} \right\rceil + 1 \right)$.

Observe the following:

$$\frac{2r}{\left\lceil 20r^{\frac{k}{k+1}} \right\rceil} \leq \frac{2r}{20r^{\frac{k}{k+1}}} \leq \left\lceil \frac{2r}{20r^{\frac{k}{k+1}}} \right\rceil \leq \left\lceil \frac{N}{20r^{\frac{k}{k+1}}} \right\rceil$$

$$\frac{2r(k-1)\left(\left\lceil \frac{N}{20\, r^{\frac{k}{k+1}}} \right\rceil + 1\right))}{(k-1) \cdot \left\lceil \frac{N}{20\, r^{\frac{k}{k+1}}} \right\rceil} = 2r + \frac{2r}{\left\lceil \frac{N}{20\, r^{\frac{k}{k+1}}} \right\rceil} \leq 2r + \left\lceil 20r^{\frac{k}{k+1}} \right\rceil$$

$$\frac{2tr}{t-(k-1)} < 2r + 2 \cdot \left\lceil 20r^{\frac{k}{k+1}} \right\rceil = N$$

$$0 \leq \frac{2t(r-1)}{N(t-k+1)} < \frac{2tr}{N(t-k+1)} < 1$$

In our analysis we are going to use the following inequalities:

(a) $\log\left(1 - \left(1 - \frac{2t(r-1)}{N(t-k+1)}\right)^k\right) < -\left(1 - \frac{2t(r-1)}{N(t-k+1)}\right)^k$, since $0 < \left(1 - \frac{2t(r-1)}{N(t-k+1)}\right)^k < 1$

(b) $N > 2r$

(c) $\frac{N-2r}{N} - \frac{k-1}{t-k+1} \leq \frac{N-2r}{N} - \frac{2r(k-1)}{N(t-k+1)} = 1 - \frac{2tr}{N(t-k+1)} \leq 1 - \frac{2t(r-1)}{N(t-k+1)}$

(d) $\log\left(\frac{N}{r-1}\right) = \log\left(\frac{2r + 2 \cdot \left\lceil 20r^{\frac{k}{k+1}} \right\rceil}{r-1}\right) \leq \log\left(\frac{2r + 2 \cdot 21r}{\frac{r}{2}}\right) = \log(88) < 5$

(e) $1 + \lceil x \rceil < 2x$, for $x > 2$

(f) $r \geq e^{2(k+1)} \geq e^2$, and $r^{\frac{1}{k+1}} \geq e^2$, since $\log(r) \geq 2(k+1)$.

We are going to prove that $\log(\mathbb{P}(B)) < 0$:

$$\log(\mathbb{P}(B)) \leq \log\left(\left(\frac{eN}{r-1}\right)^{t(r-1)} \left(1 - \left(1 - \frac{2t(r-1)}{N(t-k+1)}\right)^k\right)^{\frac{N^2}{4}}\right)$$

$$
\begin{aligned}
&= t(r-1)\cdot\log\left(\frac{eN}{r-1}\right) \;+\; \frac{N^2}{4}\cdot\log\left(1-\left(1-\frac{2t(r-1)}{N(t-k+1)}\right)^k\right)\\
&\underset{(a)}{\leq}\; t(r-1)\cdot\left(1+\log\left(\frac{N}{r-1}\right)\right) \;+\; \frac{N^2}{4}\cdot\left(-\left(1-\frac{2t(r-1)}{N(t-k+1)}\right)^k\right)\\
&\underset{(d)}{\leq}\; t(r-1)\cdot 6 \;-\; \frac{N^2}{4}\cdot\left(1-\frac{2t(r-1)}{N(t-k+1)}\right)^k\\
&\underset{(c)}{\leq}\; (k-1)\cdot\left(\left\lceil\frac{N}{20\,r^{\frac{k}{k+1}}}\right\rceil+1\right)(r-1)\cdot 6 \;-\; \frac{N^2}{4}\cdot\left(\frac{N-2r}{N}-\frac{k-1}{t-k+1}\right)^k\\
&\underset{(e)}{\leq}\; 12(r-1)(k-1)\cdot\frac{N}{20\,r^{\frac{k}{k+1}}} \;-\; \frac{N^2}{4}\cdot\left(\frac{2\cdot\left\lceil 20r^{\frac{k}{k+1}}\right\rceil}{N}-\frac{1}{\left\lceil\frac{N}{20\,r^{\frac{k}{k+1}}}\right\rceil}\right)^k\\
&\leq 12r(k-1)\cdot\frac{N}{20\,r^{\frac{k}{k+1}}} \;-\; \frac{N^2}{4}\cdot\left(\frac{2\cdot\left\lceil 20r^{\frac{k}{k+1}}\right\rceil}{N}-\frac{20\,r^{\frac{k}{k+1}}}{N}\right)^k\\
&\underset{(b)}{\leq}\; 12r(k-1)\cdot\frac{N}{20\,r^{\frac{k}{k+1}}} \;-\; r^2\cdot\left(\frac{20r^{\frac{k}{k+1}}}{N}\right)^k\\
&= \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot 12(k-1) \;-\; r\cdot rN\cdot\frac{20^k\cdot r^{\frac{k^2}{k+1}}}{N^{k+1}}\\
&= \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot 12(k-1) \;-\; \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot\frac{20^{k+1}\cdot r^{\frac{k^2}{k+1}+1+\frac{k}{k+1}}}{N^{k+1}}\\
&= \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot\left(12(k-1) \;-\; \left(\frac{20r}{N}\right)^{k+1}\right)\\
&= \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot\left(12(k-1) \;-\; \left(\frac{20r}{2r+2\cdot\left\lceil 20r^{\frac{k}{k+1}}\right\rceil}\right)^{k+1}\right)\\
&\leq \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot\left(12(k-1) \;-\; \left(\frac{20r}{2r+2\cdot 20r^{\frac{k}{k+1}}+2}\right)^{k+1}\right)\\
&= \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot\left(12(k-1) \;-\; \left(\frac{10}{1+20r^{\frac{-1}{k+1}}+r^{-1}}\right)^{k+1}\right)\\
&\underset{(f)}{\leq}\; \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot\left(12(k-1) \;-\; \left(\frac{10}{1+\frac{20}{e^2}+\frac{1}{e^2}}\right)^{k+1}\right)\\
&\leq \frac{rN}{20\,r^{\frac{k}{k+1}}}\cdot\left(12(k-1) \;-\; \left(\frac{10}{4}\right)^{k+1}\right)\\
&\leq 0.
\end{aligned}
$$

We proved that $\log(\mathbb{P}(B)) < 0$, i.e., $\mathbb{P}(B) < 1$. This implies that there exists an assignment of lists L of size k to the edges of K_N such that lists L do not induce a colouring c without a monochromatic rK_2, i.e., no L-colouring is without a monochromatic rK_2. This implies that $R_l(rK_2, k) \leq N = 2r + 2 \cdot \left\lceil 20r^{\frac{k}{k+1}} \right\rceil$, when $2(k+1) \leq \log(r)$.

Now we turn to the other case: $2(k+1) > \log(r)$. Define $N := 2\left\lceil \frac{16rk}{\log(rk)} \right\rceil$, and $t := 2k$. By performing the same analysis, we again obtain:

$$\mathbb{P}(B) \leq \left(\frac{eN}{r-1}\right)^{t(r-1)} \left(1 - \left(1 - \frac{2t(r-1)}{N(t-k+1)}\right)^k\right)^{\frac{N^2}{4}}.$$

In the following analysis we are going to use these inequalities:

(a) $\log\left(\frac{eN}{r-1}\right) \leq \log\left(\frac{e \cdot 2 \cdot 17rk}{r/2}\right) \leq 1 + \log(68k) \leq 8 + \log(k)$

(b) $(1-x) \geq e^{-2x}$ for $0 \leq x \leq \frac{1}{2}$, $x = \frac{\log(rk)}{8k} \leq \frac{2k+2+\log(k)}{8k} \leq \frac{1}{2}$, $x \geq \frac{\log(4)}{8k} \geq 0$

(c) $8 + \log(k) \leq 8k^{\frac{1}{4}}$

(d) $\log\left(1 - \frac{1}{(rk)^{\frac{1}{4}}}\right) < -\frac{1}{(rk)^{\frac{1}{4}}}$, since $0 < \frac{1}{(rk)^{\frac{1}{4}}} < 1$

(e) $\log(rk) \leq 4(rk)^{\frac{1}{4}}$

We are going to prove that $\log(\mathbb{P}(B)) < 0$:

$$\log(\mathbb{P}(B)) \leq \log\left(\left(\frac{eN}{r-1}\right)^{t(r-1)} \left(1 - \left(1 - \frac{2t(r-1)}{N(t-k+1)}\right)^k\right)^{\frac{N^2}{4}}\right)$$

$$= t(r-1) \cdot \log\left(\frac{eN}{r-1}\right) + \frac{N^2}{4} \cdot \log\left(1 - \left(1 - \frac{2 \cdot 2k(r-1)}{2\left\lceil \frac{16rk}{\log(rk)} \right\rceil (2k-k+1)}\right)^k\right)$$

$$\underset{(a)}{\leq} 2k(r-1)(8+\log(k)) + \frac{N^2}{4} \cdot \log\left(1 - \left(1 - \frac{4kr}{2 \cdot \frac{16rk}{\log(rk)} \cdot k}\right)^k\right)$$

$$\leq 2kr(8+\log(k)) + \frac{N^2}{4} \cdot \log\left(1 - \left(1 - \frac{\log(rk)}{8k}\right)^k\right)$$

$$
\begin{aligned}
&\underset{(b),\,(c)}{\leq} 2kr \cdot 8k^{\frac{1}{4}} + \frac{N^2}{4} \cdot \log\left(1 - \left(e^{-2 \cdot \frac{\log(rk)}{8k}}\right)^k\right) \\
&= 16kr \cdot k^{\frac{1}{4}} + \frac{N^2}{4} \cdot \log\left(1 - \frac{1}{(rk)^{\frac{1}{4}}}\right) \\
&\underset{(d)}{\leq} 16rk \cdot k^{\frac{1}{4}} - \frac{N^2}{4} \cdot \frac{1}{(rk)^{\frac{1}{4}}} \\
&\leq 16rk \cdot k^{\frac{1}{4}} - \frac{4 \cdot 16^2 (rk)^2}{4 \log^2(rk)} \cdot \frac{1}{(rk)^{\frac{1}{4}}} \\
&= 16rk \cdot \left(k^{\frac{1}{4}} - \frac{16(rk)^{\frac{3}{4}}}{\log^2(rk)}\right) \\
&\underset{(e)}{\leq} 16rk \cdot \left(k^{\frac{1}{4}} - (rk)^{\frac{1}{4}}\right) \\
&< 0
\end{aligned}
$$

We proved that $\log(\mathbb{P}(B)) < 0$, i.e., $\mathbb{P}(B) < 1$. This implies that there exists an assignment of lists L of size k to the edges of K_N such that lists L do not induce a colouring c without a monochromatic rK_2, i.e., no L-colouring is without a monochromatic rK_2. This implies that $R_l(rK_2, k) \leq N = 2\left\lceil \frac{16rk}{\log(rk)} \right\rceil$, when $2(k+1) > \log(r)$. □

Combining two previous lemmas, we obtain the following result:

Theorem 4.3.5. [32] *Let $r, k \in \mathbb{N}$. If $2(k+1) \leq \log(r)$, then:*

$$2r \leq R_l(rK_2, k) \leq 2r + 42r^{\frac{k}{k+1}}.$$

If $2(k+1) > \log(r) > 0$, then:

$$\frac{rk}{4\log(rk)} \leq R_l(rK_2, k) \leq \frac{34rk}{\log(rk)}.$$

4.4 Cliques and hypergraphs

In this section, we are going to discuss bounds for list Ramsey numbers of cliques and hypergraphs that were discovered by Alon, Bucić, Kalvari, Kuperwasser and Szabó in [32]. Their proofs rely heavily on the container method introduced by Saxton and Thomason [38] and Balogh, Morris and Somatij [39]. This method is not going to be discussed here in more detail, so we present the results without proofs.

4.4.1 Upper bounds

The first result gives us an upper bound for l-uniform cliques. In particular, it gives us a bound exponential in clique size, for a fixed uniformity and a fixed number of colours.

Theorem 4.4.1.1. [32] *For arbitrary positive integers $r \geq l$ and $k \in \mathbb{N}$ we have*

$$R_l(K_r^{(l)}, k) \leq 2^{4r^{3l-1}+4kr^{l-1}\log_2 r}.$$

In order to present our next result, we first introduce the following definitions:

Definition 4.4.1.2. *The maximum number of edges among all graphs on n vertices that do not contain graph H as a subgraph is denoted by $ex(H, n)$.*

Definition 4.4.1.3. *For an l-uniform hypergraph H we define:*

$$\pi(H) := \lim_{n\to\infty} \frac{ex(H, n)}{\binom{n}{l}}.$$

Definition 4.4.1.4. *Given a graph H with at least 2 edges, we define:*

$$m(H) := \max_{H'\subset H,\, e(H')>1} \frac{e(H') - 1}{v(H') - l},$$

where $e(G)$ denotes the number of edges of G, and $v(G)$ denotes the number of vertices of G.

The following result applies to general uniform hypergraphs, for a large number of colours.

Theorem 4.4.1.5. [32] *Let H be an l-uniform hypergraph. Then, as k tends to infinity, we have*

$$R_l(H, k) \leq (1 - \pi(H) + o(1))^{-km(H)}.$$

4.4.2 Lower bounds

First we state a lower bound which uses ordinary Ramsey numbers.

Lemma 4.4.2.1. [32] *If* $R\left(H, \left\lfloor \frac{k}{l \log n} \right\rfloor\right) > n$ *then:*

$$R_l(H, k) > n.$$

Using this lemma, Alon, Bucić, Kalvari, Kuperwasser and Szabó [32] proved three results that apply to special cases.

Theorem 4.4.2.2. [32] *If* H *is an* l*-uniform hypergraph with* $\chi(H) > r$*, then we have*

$$R_l(H, k) \geq e^{\sqrt{k \log r/(4l)}},$$

In particular $R_l(K_3, k) > e^{\sqrt{k}/4}$.

Theorem 4.4.2.3. [32] *Let* H *be an* l*-uniform hypergraph which is not* l*-partite. We have*

$$R_l(H, k) \geq e^{c_l \sqrt{k}},$$

where $\frac{1}{c_l} = 2le^{l/2}$.

Theorem 4.4.2.4. [32] *Let* H *be an* l*-partite* l*-uniform hypergraph with parts of size at most* r*. There is a constant* $c = c(r, l)$ *such that*

$$R_l(H, k) \geq R\left(H, \left\lfloor \frac{ck}{\log k} \right\rfloor\right).$$

5 Conclusion

This thesis is primarily a glimpse into the vast area of Ramsey theory. The main goals were to introduce the reader to the field, to cover some recent upper and lower bounds for ordinary Ramsey numbers and to list and analyse some of the most interesting variants of Ramsey numbers. Specifically, the variants that are mentioned are induced Ramsey numbers, Folkman numbers, size, chromatic, degree, bipartite and list Ramsey numbers. The list Ramsey numbers of stars and matchings are analysed extensively.

Important open questions that were stated in the thesis are the List colouring conjecture, Problem 2.2.3, Conjecture 2.2.10, Problem 3.1.9, Problem 3.1.10, Conjecture 3.2.3, Conjecture 3.3.5, Problem 3.4.3, whether $R_{ind}(H) \leq 2^{cn}$ for some constant c, and whether ordinary and list Ramsey numbers of stars coincide.

For a more extensive analysis, we encourage the reader to take a look at the survey by Conlon, Fox and Sudakov [5].

References

[1] Benny Sudakov. Graph theory lecture notes, July 2019.

[2] Frank P. Ramsey. On a problem of formal logic. In *Classic Papers in Combinatorics*, pages 1–24. Springer, 2009.

[3] Paul Erdös. Some remarks on the theory of graphs. *Bulletin of the American Mathematical Society*, 53(4):292–294, 1947.

[4] Joel Spencer. Ramsey's theorem—a new lower bound. *Journal of Combinatorial Theory, Series A*, 18(1):108–115, 1975.

[5] David Conlon, Jacob Fox, and Benny Sudakov. Recent developments in graph Ramsey theory. *Surveys in combinatorics*, 424:49–118, 2015.

[6] Ronald L. Graham and Vojtěch Rödl. Numbers in Ramsey theory. *Surveys in combinatorics*, 123:111–153, 1987.

[7] Andrew Thomason. An upper bound for some Ramsey numbers. *Journal of graph theory*, 12(4):509–517, 1988.

[8] David Conlon. A new upper bound for diagonal Ramsey numbers. *Annals of Mathematics*, pages 941–960, 2009.

[9] Ashwin Sah. Diagonal Ramsey via effective quasirandomness. *arXiv preprint arXiv:2005.09251*, 2020.

[10] Paul Erdös and Richard Rado. Combinatorial theorems on classifications of subsets of a given set. *Proceedings of the London Mathematical Society*, s3-2(1):417–439, 1952.

[11] Paul Erdös, András Hajnal, and Richard Rado. Partition relations for cardinal numbers. *Acta Mathematica Academiae Scientiarum Hungarica*, 16(1):93–196, Mar 1965.

[12] David Conlon, Jacob Fox, and Benny Sudakov. Hypergraph Ramsey numbers. *Journal of the American Mathematical Society*, 23(1):247–266, 2010.

[13] Václav Chvátal. Tree-complete graph Ramsey numbers. *Journal of Graph Theory*, 1(1):93–93, 1977.

[14] Paul Erdös. On some problems in graph theory, combinatorial analysis and combinatorial number theory. *Graph Theory and Combinatorics (Cambridge, 1983), Academic Press, London*, pages 1–17, 1984.

[15] David Conlon, Jacob Fox, and Benny Sudakov. On two problems in graph Ramsey theory. *Combinatorica*, 32(5):513–535, 2012.

[16] Yoshiharu Kohayakawa, Hans J. Prömel, and Vojtěch Rödl. Induced Ramsey numbers. *Combinatorica*, 18(3):373–404, 1998.

[17] Jacob Fox and Benny Sudakov. Induced Ramsey-type theorems. *Advances in Mathematics*, 219(6):1771–1800, 2008.

[18] Izolda Gorgol. A note on lower bounds for induced Ramsey numbers. *Discussiones Mathematicae Graph Theory*, 39(3):647–654, 2019.

[19] David Conlon, Domingos Dellamonica, Steven La Fleur, Vojtěch Rödl, and Mathias Schacht. A note on induced Ramsey numbers. In *A Journey Through Discrete Mathematics*, pages 357–366. Springer, 2017.

[20] Jon Folkman. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM Journal on Applied Mathematics*, 18(1):19–24, 1970.

[21] Jaroslav Nešetřil and Vojtěch Rödl. The Ramsey property for graphs with forbidden complete subgraphs. *Journal of Combinatorial Theory, Series B*, 20(3):243–249, 1976.

[22] Vojtěch Rödl, Andrzej Ruciński, and Mathias Schacht. An exponential-type upper bound for Folkman numbers. *Combinatorica*, 37(4):767–784, 2017.

[23] József Beck. On size Ramsey number of paths, trees, and circuits. I. *Journal of Graph Theory*, 7(1):115–129, 1983.

[24] Stefan A. Burr, Paul Erdös, and Laszlo Lovász. On graphs of Ramsey type. *Ars Combinatoria*, 1(1):167–190, 1976.

[25] Nathalie Paul and Claude Tardif. The chromatic Ramsey number of odd wheels. *Journal of Graph Theory*, 69(2):198–205, 2012.

[26] Xuding Zhu. The fractional version of Hedetniemi's conjecture is true. *European Journal of Combinatorics*, 32(7):1168–1175, 2011.

[27] David Conlon. A new upper bound for the bipartite Ramsey problem. *Journal of Graph Theory*, 58(4):351–356, 2008.

[28] Matija Bucić, Shoham Letzter, and Benny Sudakov. Multicolour bipartite Ramsey number of paths. *arXiv preprint arXiv:1901.05834*, 2019.

[29] Stefan A. Burr and John A. Roberts. On Ramsey numbers for stars. *Utilitas Math*, 4:217–220, 1973.

[30] Fred Galvin. The list chromatic index of a bipartite multigraph. *Journal of Combinatorial Theory, Series B*, 63(1):153–158, 1995.

[31] Roland Häggkvist and Jeannette Janssen. New bounds on the list-chromatic index of the complete graph and other simple graphs. *Combinatorics Probability and Computing*, 6(3):295–313, 1997.

[32] Noga Alon, Matija Bucić, Tom Kalvari, Eden Kuperwasser, and Tibor Szabó. List Ramsey numbers. *arXiv preprint arXiv:1902.07018*, 2019.

[33] Torbjörn Gustavsson. *Decompositions of large graphs and digraphs with high minimum degree.* PhD thesis, Univ., 1991.

[34] Brian Alspach and Heather Gavlas. Cycle decompositions of K_n and $K_n - I$. *Journal of Combinatorial Theory, Series B*, 81(1):77–99, 2001.

[35] Mateja Šajna. Cycle decompositions III: Complete graphs and fixed length cycles. *Journal of Combinatorial Designs*, 10(1):27–78, 2002.

[36] Uwe Schauz. Proof of the list edge coloring conjecture for complete graphs of prime degree. *The Electronic Journal of Combinatorics*, pages P3–43, 2014.

[37] Ernest J. Cockayne and Peter J. Lorimer. The Ramsey number for stripes. *Journal of the Australian Mathematical Society*, 19(2):252–256, 1975.

[38] David Saxton and Andrew Thomason. Hypergraph containers. *Inventiones mathematicae*, 201(3):925–992, Sep 2015.

[39] József Balogh, Robert Morris, and Wojciech Samotij. Independent sets in hypergraphs. *Journal of the American Mathematical Society*, 28(3):669–709, 2015.