Clique minors in graphs with a forbidden subgraph

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joint work with Jacob Fox and Benny Sudakov

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- Robertson-Seymour-Thomas, 1993: true for $t = 6$.

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- Balogh, Kostochka 2011: currently best bound $0.513n/\alpha(G)$.

Large clique minors due to forbidden subgraphs

Theorem (Kuhn and Osthus; Krivelevich and Sudakov)

If G does not have a bipartite graph F as a subgraph then it has a clique minor of size $(n/\alpha(G))^{1+c}$ for some $c = c(F) > 0$.

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Question (Dvořák and Yepremyan)

Do we get a similar improvement over what Hadwiger's conjecture implies for F-free graphs for any forbidden graph F?

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- Simpler proof for $F = K_3$ with $c = \frac{1}{4}$.
- Simpler proof for F bipartite as well.

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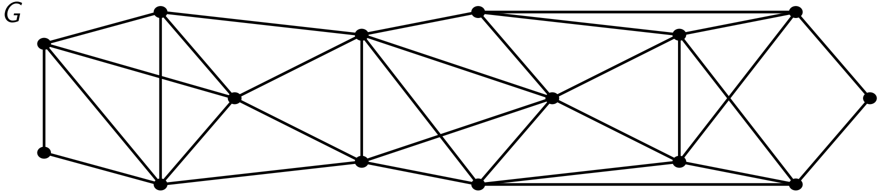

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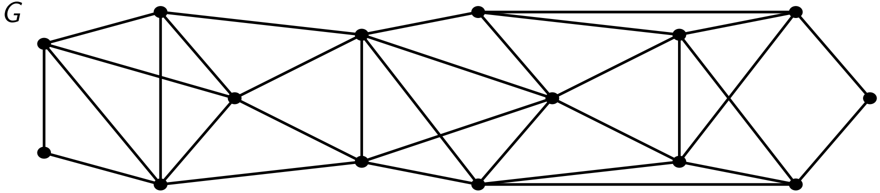

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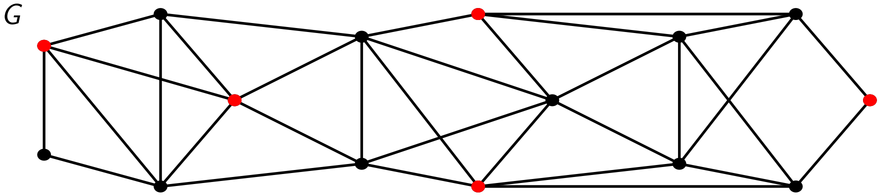

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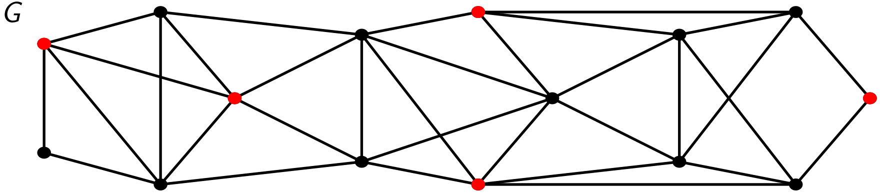

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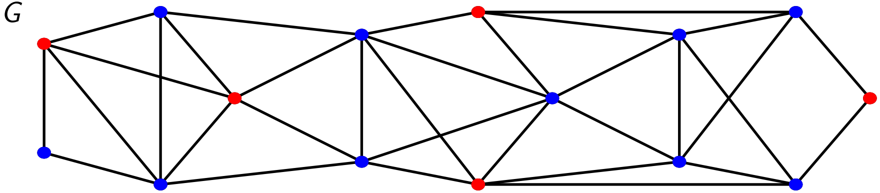

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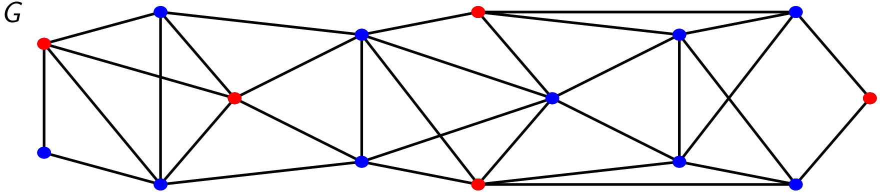

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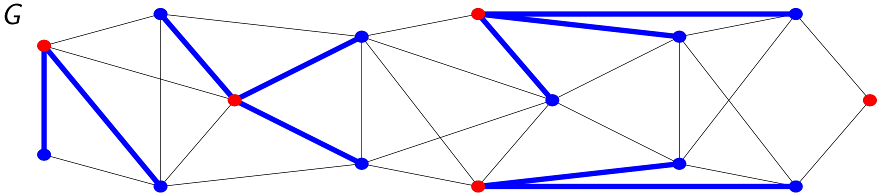

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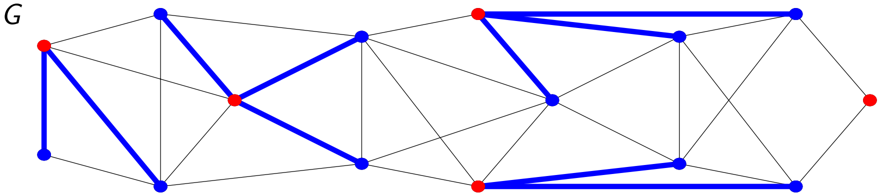

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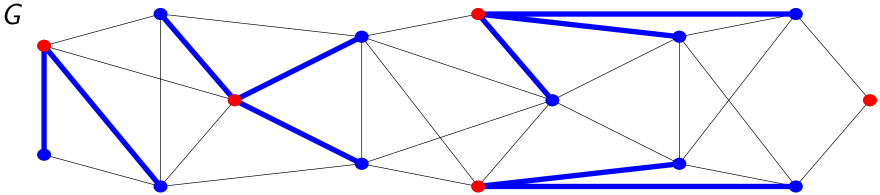

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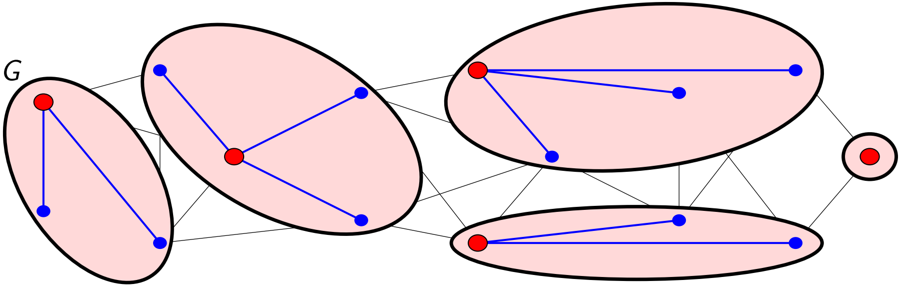

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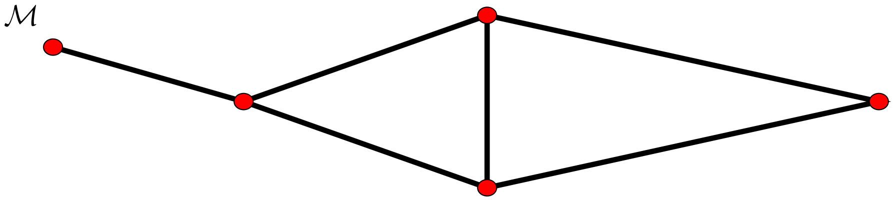

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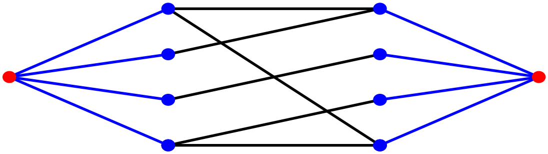

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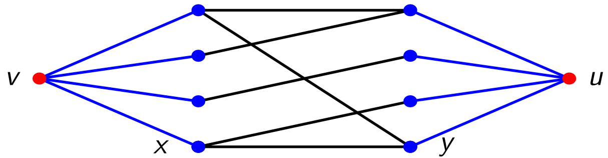

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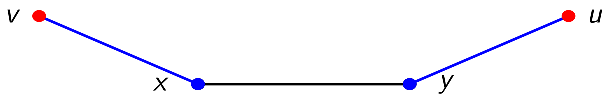

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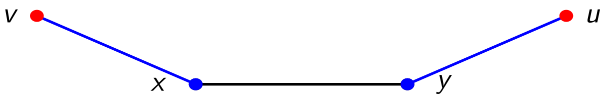

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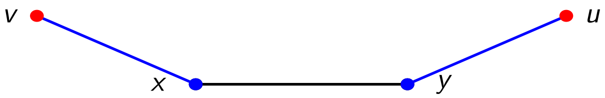

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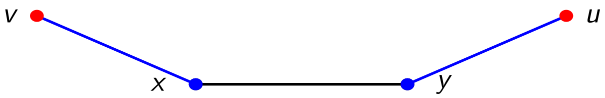

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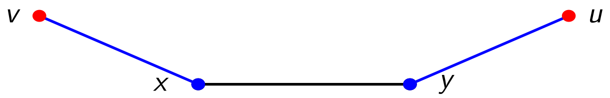

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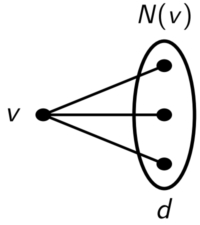

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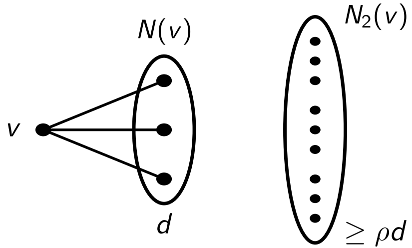

Example: triangle-free case

- Let G be a triangle-free, d-regular graph with $\alpha = \alpha(G)$ and $p = 1/\sqrt{d}$.
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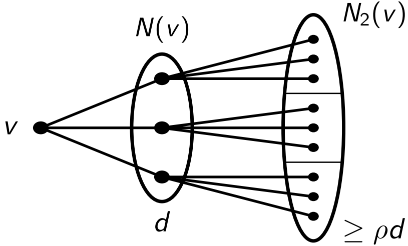

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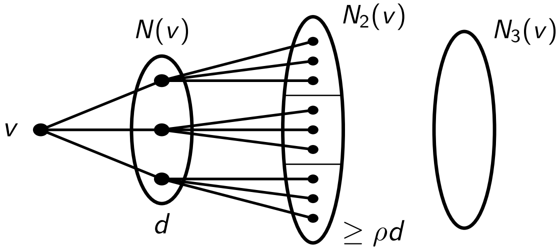

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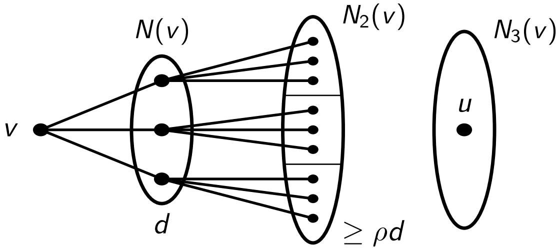

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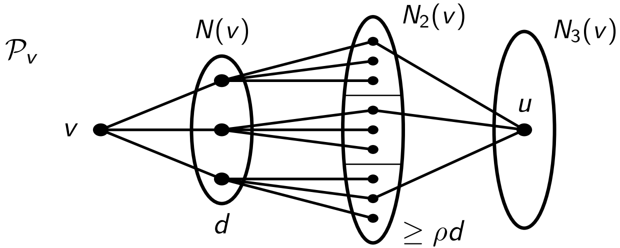

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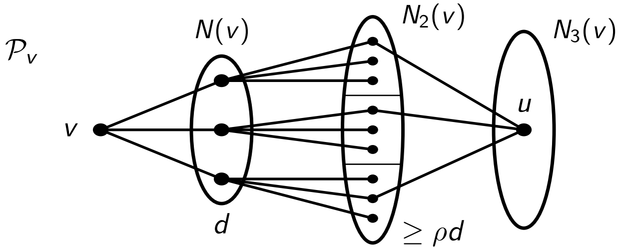

$|\mathcal{P}_v| \geq \rho^2 d$

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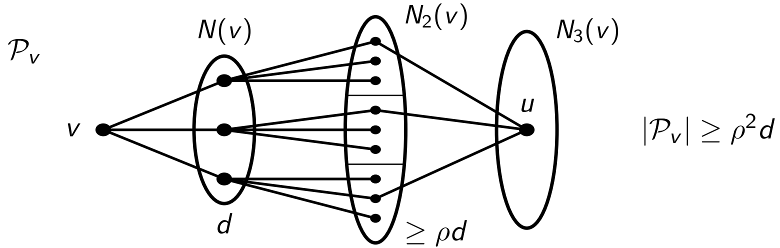

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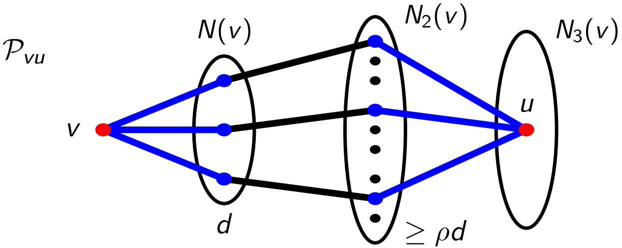

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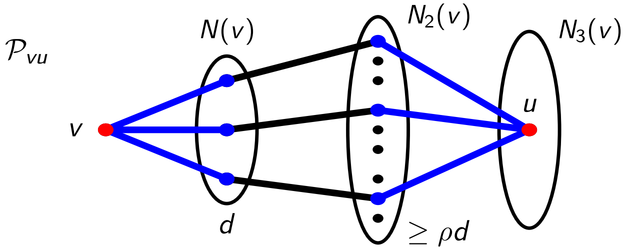

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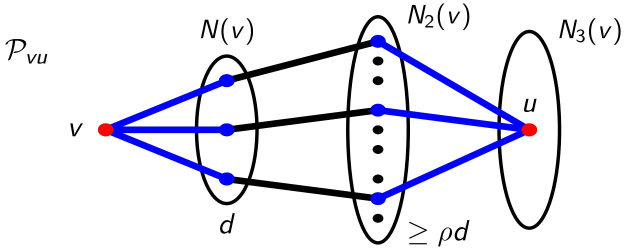

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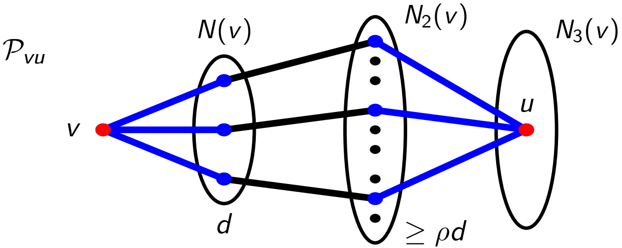

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Open problems

Question

What is the max. $c(F)$ *s.t. any* F*-free graph has a clique minor of size* $(n/\alpha)^{1+c(F)-o(1)}$*?*

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Question

For which graphs F does G being F-free implies G has a clique minor of size $(\chi(G))^{1+c}$ for some $c = c(F) > 0$?

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- True if F is bipartite

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- True if F is bipartite
- Not true if F contains a triangle.


рахмат
danke
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
faafetai lava
mersi
kia ora
barka
welalin
tack
spas
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
mahalo
tapadh leat
kiitos
dankie
thank you
gracias
хвала
nanni
nandri
dhanyavad
asante
manana
obrigada
bedankt
bayarlalaa
gràcie
hvala
maururu
köszönöm
akun dankon
aciú
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tenki
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arigatô
takk
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trugarez
najis tuke
terima kasih
rahmet
diolch
dhanyavadagalu
shukriya
mercê
мерси
kam sah hamnida
rahmat
তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci
תודה רבה