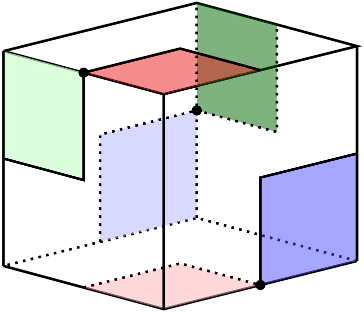
Erdős-Szekeres theorem for multidimensional arrays

Matija Bucić
joint work with Benny Sudakov and Tuan Tran



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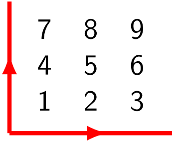

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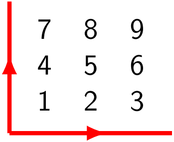

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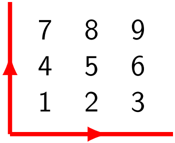

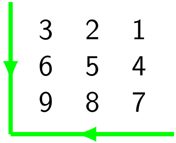

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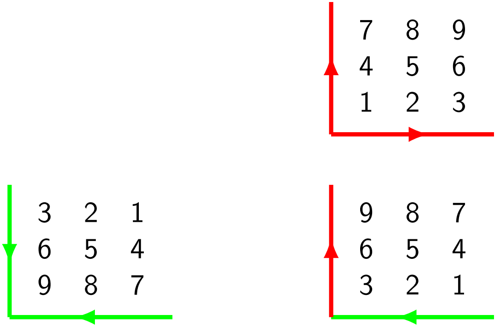

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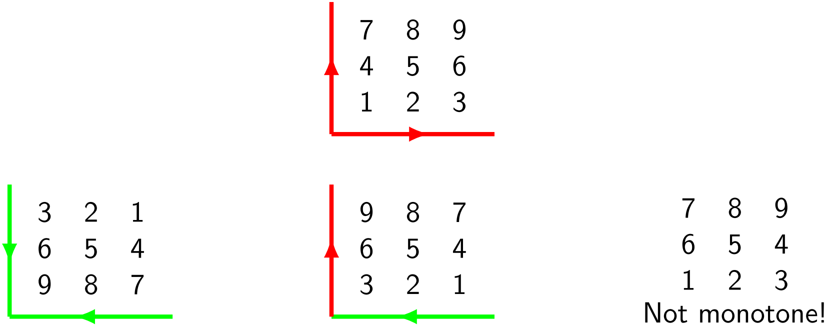

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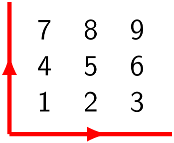

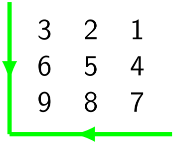

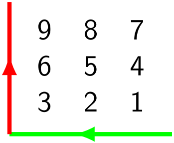

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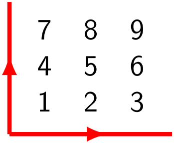

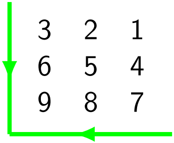

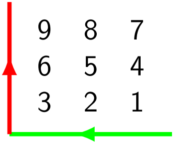

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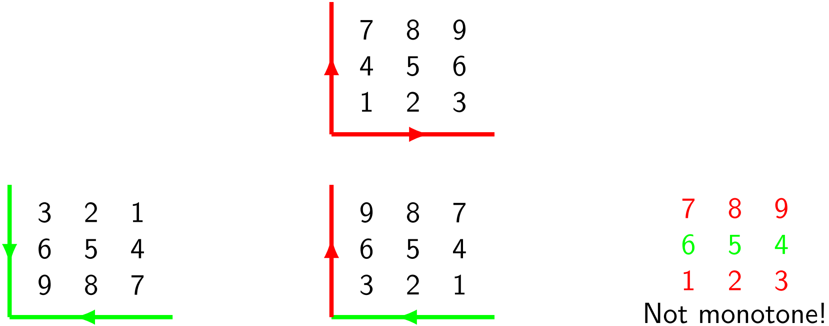

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- We denote the answer to this question by $M_d(n)$.

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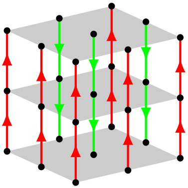

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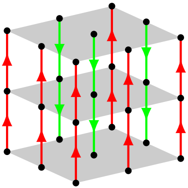

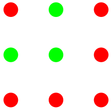

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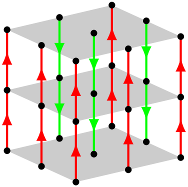

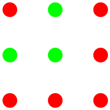

Lemma (Grid Ramsey)

$\exists C = C(d,k)$: for $N \geq 2^{Cn^{d-1}}$, any k-colouring of the d-dimensional $N \times \ldots \times N$ grid contains a monochromatic subgrid of size $n \times \ldots \times n$.

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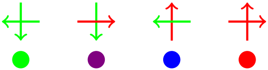

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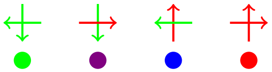

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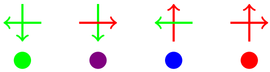

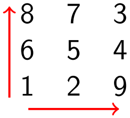

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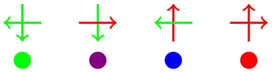

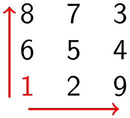

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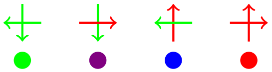

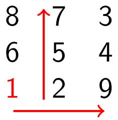

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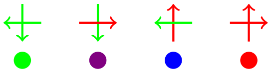

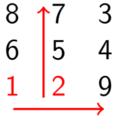

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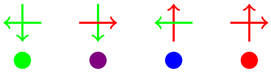

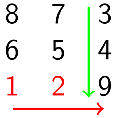

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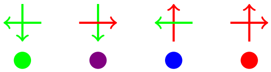

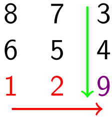

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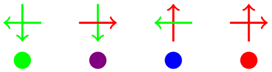

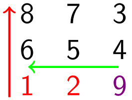

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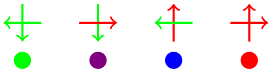

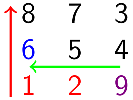

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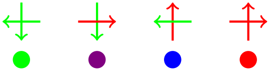

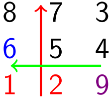

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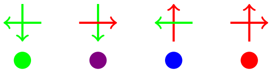

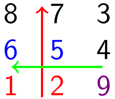

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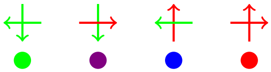

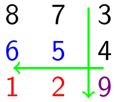

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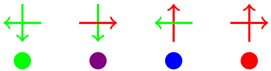

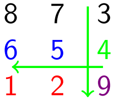

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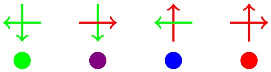

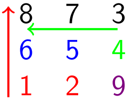

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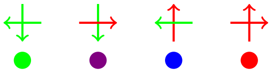

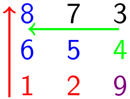

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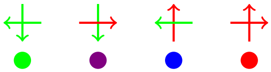

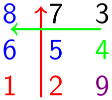

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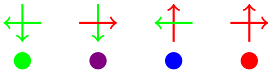

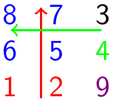

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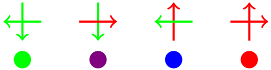

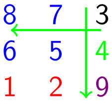

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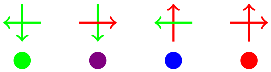

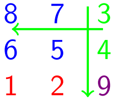

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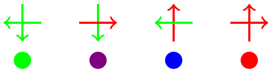

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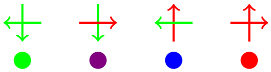

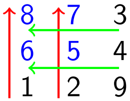

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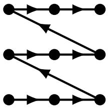

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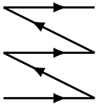

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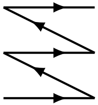

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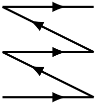

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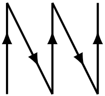

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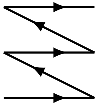

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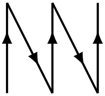

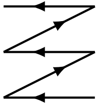

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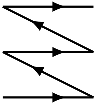

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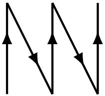

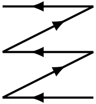

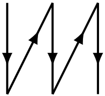

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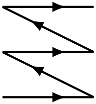

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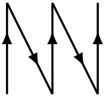

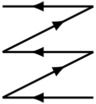

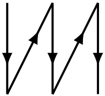

- There are $d! \cdot 2^d$ different lex-monotone arrays of fixed size.

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- First find a big monotone array f and then lex-monotone one within f.

Concluding remarks and open problems

- Finding (lex-)monotone subarrays is related to:
  - Finding long common monotone subsequences of permutations.
  - Ramsey type problems for vertex ordered graphs.
  - Canonical orderings of discrete structures.

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Is $M_2(n)$ exponential or double exponential?

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Is $M_2(n)$ exponential or double exponential?

Question

Is $M_d(n)$ always at most double exponential in n?


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
рахмат
Баярлалаа
спасибо
gracias
tapadh leat
merci
obrigado
dziękuję
bedankt
hvala
sukriya
kop khun krap
go raibh maith agat
mochchakkeram
arigatô
dakujem
grazie
terima kasih
감사합니다
ありがとう
תודה רבה
мерси
ευχαριστώ
dank je
mersi
barka
welalin
tack
spas
kia ora
blagodaram
misaotra
matondo
paldies
grazzi
mahalo
faafetai lava
vinaka
спасибі
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takk
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mercé
trugarez