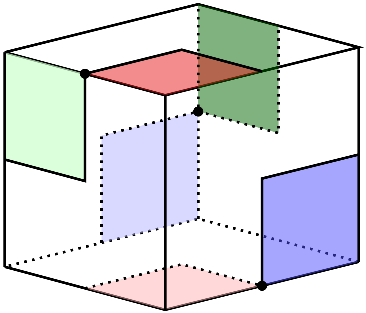
Erdős-Szekeres theorem for multidimensional arrays

Matija Bucić
joint work with Benny Sudakov and Tuan Tran



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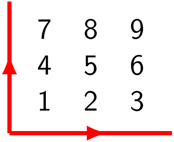

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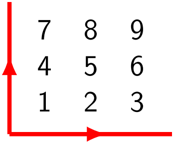

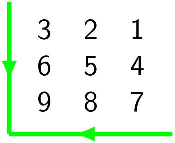

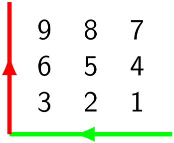

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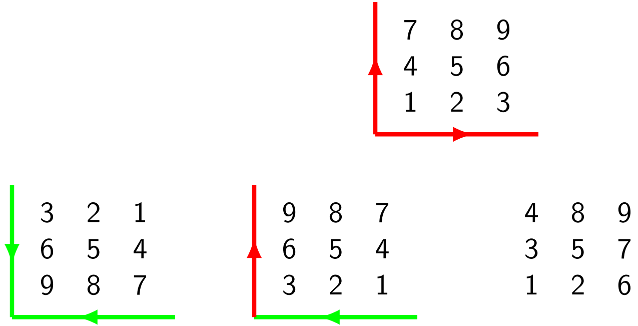

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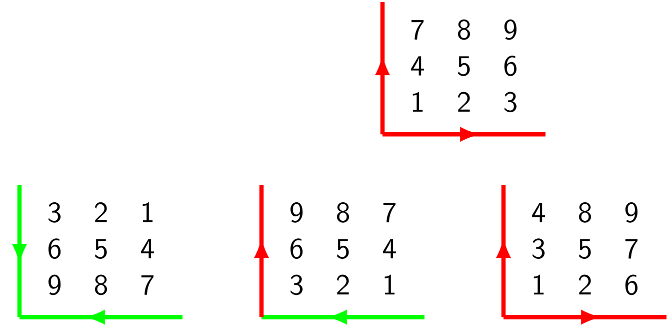

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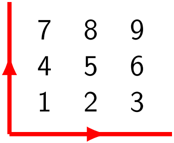

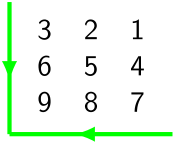

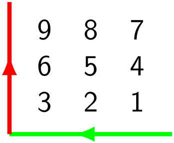

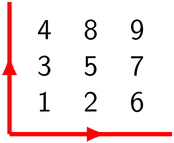

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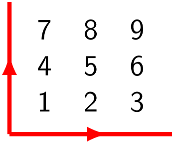

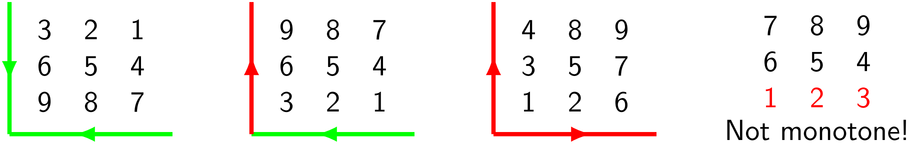

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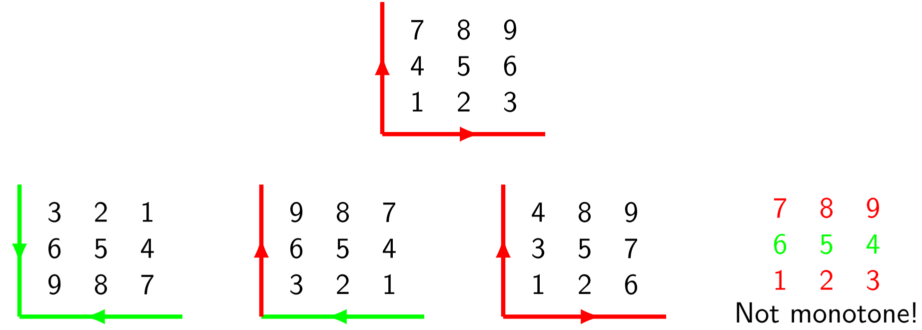

Higher dimensional Erdős-Szekeres theorem

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Proofs.

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- $M'_d(n) \leq M_d(n)$.

Theorem 2 (B., Sudakov, Tran)

For every $d \geq 2$, we have $M'_d(n) \leq 2^{2^{(1+o(1))n^{d-1}}}$.

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$$n^2 \left\{ \begin{matrix} 56 & 36 & 24 & 57 & 30 & 52 & 37 & 43 & 46 & 17 & 16 & 1 & 2 & 11 \\ 41 & 8 & 42 & 60 & 68 & 38 & 48 & 58 & 66 & 44 & 61 & 28 & 49 & 29 \\ 40 & 59 & 23 & 67 & 54 & 62 & 4 & 51 & 55 & 7 & 34 & 33 & 63 & 21 \\ 10 & 64 & 22 & 32 & 3 & 12 & 69 & 6 & 13 & 31 & 14 & 35 & 15 & 19 \\ 53 & 20 & 65 & 45 & 50 & 5 & 47 & 70 & 39 & 25 & 26 & 27 & 18 & 9 \end{matrix} \right.$$

Proof of 2D case of inconsistent monotonicity.

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n^2 rows:

56	36	24	57	30	52	37	43	46	17	16	1	2	11
41	8	42	60	68	38	48	58	66	44	61	28	49	29
40	59	23	67	54	62	4	51	55	7	34	33	63	21
10	64	22	32	3	12	69	6	13	31	14	35	15	19
53	20	65	45	50	5	47	70	39	25	26	27	18	9

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n^2

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$$n \begin{cases} 41 \quad 8 \quad 42 \qquad 68 \qquad\qquad 58 \quad 66 \qquad 61 \quad 28 \qquad 29 \\ 40 \quad 59 \quad 23 \qquad 54 \qquad\qquad 51 \quad 55 \qquad 34 \quad 33 \qquad 21 \\ 10 \quad 64 \quad 22 \qquad 3 \qquad\qquad 6 \quad 13 \qquad 14 \quad 35 \qquad 19 \end{cases}$$

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Theorem (B., Sudakov, Tran)

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- Notice that $M_2(n) \leq M_2'(2n-1)$.

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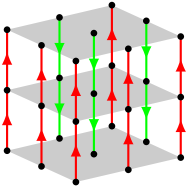

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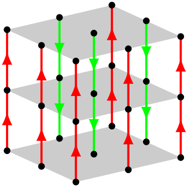

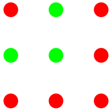

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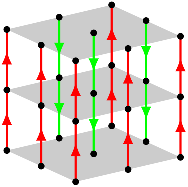

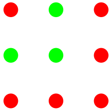

Lemma (Grid Ramsey)

$\exists C = C(d,k)$: for $N \geq 2^{Cn^{d-1}}$, any k-colouring of the d-dimensional $N \times \ldots \times N$ grid contains a monochromatic subgrid of size $n \times \ldots \times n$.

From inconsistency to consistency

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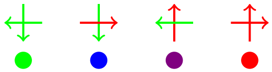

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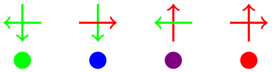

8	7	3
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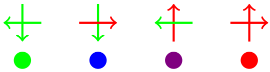

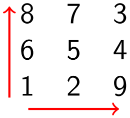

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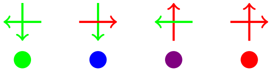

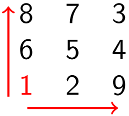

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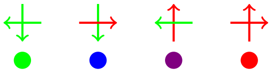

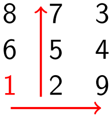

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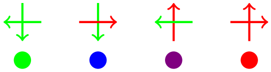

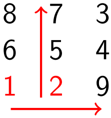

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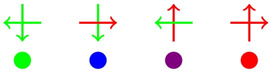

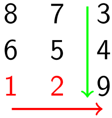

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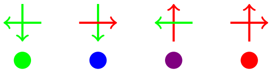

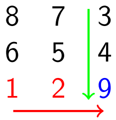

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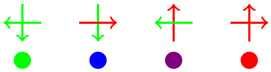

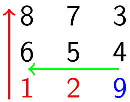

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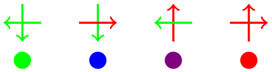

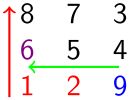

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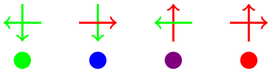

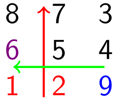

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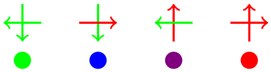

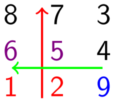

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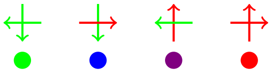

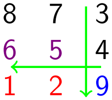

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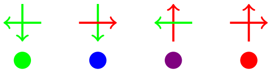

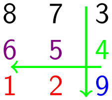

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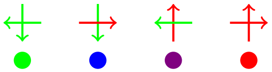

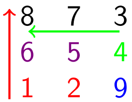

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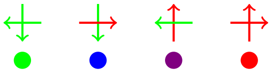

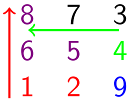

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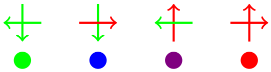

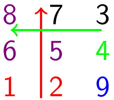

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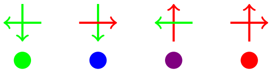

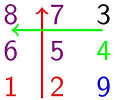

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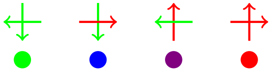

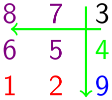

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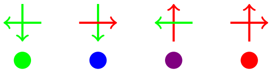

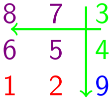

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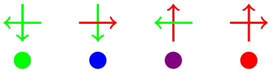

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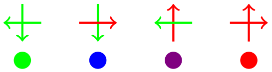

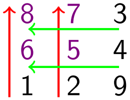

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Is $M_d(n)$ always at most double exponential in n?


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
dziękuję
bedankt
спасибо
Баярлалаа
рахмат
sukriya
kop khun krap
go raibh maith agat
mochchakkeram
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dakujem
grazie
terima kasih
мерси
ευχαριστώ
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ありがとう
תודה רבה
dank je
tapadh leat
hvala
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