Erdős-Szekeres theorem for multidimensional arrays

Matija Bucić

joint work with Benny Sudakov and Tuan Tran
Theorem (Erdős-Szekeres, 1935)

Any sequence of \((n - 1)^2 + 1\) distinct real numbers contains a monotone subsequence of length \(n\).
Erdős-Szekeres theorem

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3 2 1 6 5 4 9 8 7 10

Many different beautiful proofs.
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Many different beautiful proofs.

Many very natural generalisations and extensions.
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Many different higher dimensional generalisations due to: 

\[
3 \quad 2 \quad 1 \quad 6 \quad 5 \quad 4 \quad 9 \quad 8 \quad 7 \quad 10
\]
Theorem (Erdős-Szekeres, 1935)

Any sequence of \((n - 1)^2 + 1\) distinct real numbers contains a monotone subsequence of length \(n\).

Many different beautiful proofs.

Many very natural generalisations and extensions.

Many different higher dimensional generalisations due to: Fishburn and Graham; Kruskal; Linial and Simkin; Szabó and Tardos,...
Higher dimensional version?

- What is a monotone 2D array?
Higher dimensional version?

- What is a monotone 2D array?
- Fishburn and Graham:
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- What is a monotone 2D array?
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  - all rows are monotone in the same way
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\begin{array}{ccc}
7 & 8 & 9 \\
4 & 5 & 6 \\
1 & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
3 & 2 & 1 \\
6 & 5 & 4 \\
9 & 8 & 7 \\
\end{array}
\]

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- **What is a monotone 2D array?**
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Erdős-Szekeres theorem for multidimensional arrays
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4 5 6
1 2 3
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```
3 2 1
6 5 4
9 8 7
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6 5 4
3 2 1
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Not monotone!
A $d$-dimensional array is an injective function from $A_1 \times \cdots \times A_d \rightarrow \mathbb{R}$ where $A_1, \ldots, A_d \subseteq \mathbb{Z}$. 

Definition (Fishburn and Graham, 1993) An array is monotone if for each dimension all the 1-dimensional subarrays along this dimension are monotone in the same way.

Question (Fishburn and Graham, 1993) What is the smallest $N$ such that any $d$-dimensional array of size $N \times \cdots \times N$ contains a monotone subarray of size $n \times \cdots \times n$? We denote the answer to this question by $M_d(n)$.
A \( d \)-dimensional array is an injective function from \( A_1 \times \cdots \times A_d \to \mathbb{R} \) where \( A_1, \ldots, A_d \subseteq \mathbb{Z} \), we say it has size \( |A_1| \times \cdots \times |A_d| \).
A $d$-dimensional array is an injective function from $A_1 \times \cdots \times A_d \rightarrow \mathbb{R}$ where $A_1, \ldots, A_d \subseteq \mathbb{Z}$, we say it has size $|A_1| \times \cdots \times |A_d|$.

**Definition (Fishburn and Graham, 1993)**

An array is monotone if for each dimension all the 1-dimensional subarrays along this dimension are monotone in the same way.
Higher dimensional Erdős-Szekeres theorem

- A $d$-dimensional array is an injective function from $A_1 \times \cdots \times A_d \to \mathbb{R}$ where $A_1, \ldots, A_d \subseteq \mathbb{Z}$, we say it has size $|A_1| \times \cdots \times |A_d|$.

**Definition (Fishburn and Graham, 1993)**

An array is monotone if for each dimension all the 1-dimensional subarrays along this dimension are monotone in the same way.

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What is the smallest $N$ such that any $d$-dimensional array of size $N \times \cdots \times N$ contains a monotone subarray of size $n \times \cdots \times n$?
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4/12
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5 & 4 & 3 \\
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\begin{array}{ccc}
5 & 4 & 3 \\
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\end{array}
\begin{array}{ccc}
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```
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```

shows $M_2(2) \geq 4$
Results

- Erdős-Szekeres is precisely $M_1(n) = (n - 1)^2 + 1$. 
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- A random ordering gives $M_d(n) \geq n^{d-1}$.
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Fishburn and Graham show:

(i) $M_2(n) \leq \text{towr}_5(O(n))$ and
Erdős-Szekeres is precisely $M_1(n) = (n - 1)^2 + 1$.

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Fishburn and Graham show:

(i) $M_2(n) \leq \text{towr}_5(O(n))$ and

(ii) $M_d(n)$ is upper bounded by an Ackermann type function.
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  1. $M_2(n) \leq \text{towr}_5(O(n))$ and
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Theorem 1 (B., Sudakov, Tran, 2019+)

(1) $M_2(n) \leq \text{towr}_5(O(n))$

(ii) $M_d(n)$ is upper bounded by an Ackermann type function.
- Erdős-Szekeres is precisely $M_1(n) = (n - 1)^2 + 1$.

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**Theorem 1 (B., Sudakov, Tran, 2019+)**

(i) $M_2(n) \leq 2^{2^n}$,
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Fishburn and Graham show:

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Theorem 1 (B., Sudakov, Tran, 2019+)

(i) $M_2(n) \leq 2^{2^{2n}}$,

(ii) $M_3(n) \leq 2^{2^{2n^2}}$ and

(iii) $M_d(n) \leq 2^{2^{2^{n^{d-1}}}}$, for $d \geq 4$. 
Proofs.

YOU WANT PROOF?
I'LL GIVE YOU PROOF!
Inconsistently monotone arrays.

**Definition**

An array is inconsistently monotone if all its 1D subarrays are monotone.
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\begin{array}{ccc}
8 & 7 & 3 \\
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\end{array}
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Theorem 2 (B., Sudakov, Tran)

For every \( d \geq 2 \), we have

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M'_d(n) \leq 2^{2^{(1+o(1))(d-1) n}}.
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Let \( M'_d(n) \) be the smallest \( N \) such that any \( d \)-dimensional \( N \times \cdots \times N \) array contains an inconsistently monotone subarray of size \( n \times \cdots \times n \).
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An array is inconsistently monotone if all its 1D subarrays are monotone.

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- \( M'_d(n) \leq M_d(n) \).
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Theorem 2 (B., Sudakov, Tran)

For every \( d \geq 2 \), we have \( M'_d(n) \leq 2^{2(1+o(1))n^{d-1}} \).
Proof of 2D case of inconsistent monotonicity.

- Consider an $n^2 \times N$ array $f$
Proof of 2D case of inconsistent monotonicity.

- Consider an $n^2 \times N$ array $f$ with $n^2, N = n^{2^n} \cdot \binom{n^2}{n}$
Proof of 2D case of inconsistent monotonicity.

- Consider an $n^2 \times N$ array $f$ with $n^2, N = n^{2n} \cdot \binom{n^2}{n} = 2^{(1+o(1))n}$. 

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Proof of 2D case of inconsistent monotonicity.

- Consider an $n^2 \times N$ array $f$ with $n^2, N = n^{2n} \cdot \binom{n^2}{n} = 2^{2(1+o(1))n}$.

\begin{align*}
\begin{bmatrix}
56 & 36 & 24 & 57 & 30 & 52 & 37 & 43 & 46 & 17 & 16 & 1 & 2 & 11 \\
41 & 8 & 42 & 60 & 68 & 38 & 48 & 58 & 66 & 44 & 61 & 28 & 49 & 29 \\
40 & 59 & 23 & 67 & 54 & 62 & 4 & 51 & 55 & 7 & 34 & 33 & 63 & 21 \\
10 & 64 & 22 & 32 & 3 & 12 & 69 & 6 & 13 & 31 & 14 & 35 & 15 & 19 \\
53 & 20 & 65 & 45 & 50 & 5 & 47 & 70 & 39 & 25 & 26 & 27 & 18 & 9
\end{bmatrix}
\end{align*}
Proof of 2D case of inconsistent monotonicity.

- Consider an $n^2 \times N$ array $f$ with $n^2, N = n^{2n} \cdot \left(\frac{n^2}{n}\right) = 2^{(1+o(1))n}$.
- Each column contains a monotone subsequence of size $n$.

$$
\begin{pmatrix}
56 & 36 & 24 & 57 & 30 & 52 & 37 & 43 & 46 & 17 & 16 & 1 & 2 & 11 \\
41 & 8 & 42 & 60 & 68 & 38 & 48 & 58 & 66 & 44 & 61 & 28 & 49 & 29 \\
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Proof of 2D case of inconsistent monotonicity.

- Consider an \( n^2 \times N \) array \( f \) with \( n^2, N = n^{2^n} \cdot \binom{n^2}{n} = 2^{(1+o(1))n} \).
- Each column contains a monotone subsequence of size \( n \).
Proof of 2D case of inconsistent monotonicity.

- Consider an $n^2 \times N$ array $f$ with $n^2$, $N = n^{2n} \cdot \binom{n^2}{n} = 2^{(1+o(1))n}$.
- Each column contains a monotone subsequence of size $n$ and it can appear in $\binom{n^2}{n}$ different positions.
Proof of 2D case of inconsistent monotonicity.

- Consider an $n^2 \times N$ array $f$ with $N = n^{2n} \cdot \left(\frac{n^2}{n}\right) = 2^{(1+o(1))n}$.
- Each column contains a monotone subsequence of size $n$ and it can appear in $\left(\frac{n^2}{n}\right)$ different positions.
- Keep the most common position to get an $M \times n$ array with all columns monotone and $M = N/\left(\frac{n^2}{n}\right) \geq n^{2n}$.
Proof of 2D case of inconsistent monotonicity.

- Consider an \( n^2 \times N \) array \( f \) with \( n^2, N = n^{2n} \cdot \binom{n^2}{n} = 2^{(1+o(1))n} \).
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- Keep the most common position to get an \( M \times n \) array with all columns monotone and \( M = N/\binom{n^2}{n} \geq n^{2n} \).
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- Keep the most common position to get an \( M \times n \) array with all columns monotone and \( M = N/\binom{n^2}{n} \geq n^{2n} \).
Proof of 2D case of inconsistent monotonicity.

- Consider an \( n^2 \times N \) array \( f \) with \( n^2, N = n^{2n} \cdot \binom{n^2}{n} = 2^{2(1+o(1))n} \).
- Each column contains a monotone subsequence of size \( n \) and it can appear in \( \binom{n^2}{n} \) different positions.
- Keep the most common position to get an \( M \times n \) array with all columns monotone and \( M = N/\binom{n^2}{n} \geq n^{2n} \).
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Consider an $n^2 \times N$ array $f$ with $n^2, N = n^{2n} \cdot \binom{n^2}{n} = 2^{2(1+o(1))n}$.

Each column contains a monotone subsequence of size $n$ and it can appear in $\binom{n^2}{n}$ different positions.

Keep the most common position to get an $M \times n$ array with all columns monotone and $M = N/\binom{n^2}{n} \geq n^{2n}$

First row contains a monotone subsequence of size $\sqrt{M}$,
Proof of 2D case of inconsistent monotonicity.

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\[
\begin{pmatrix}
41 & 8 & 42 & 68 & 58 & 66 & 61 & 28 & 29 \\
40 & 59 & 23 & 54 & 51 & 55 & 34 & 33 & 21 \\
10 & 64 & 22 & 3 & 6 & 13 & 14 & 35 & 19
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**Theorem (B., Sudakov, Tran)**

\[ M'_2(n) \leq 2^{2(1+o(1))n}. \]
Monotone 2D case

- Notice that \( M_2(n) \leq M'_2(2n - 1) \).
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**Lemma (Grid Ramsey)**

\( \exists C = C(d, k) : \text{for } N \geq 2^{Cn^{d-1}} , \text{any } k\text{-colouring of the } d\text{-dimensional } N \times \ldots \times N \text{ grid contains a monochromatic subgrid of size } n \times \ldots \times n. \)
From inconsistency to consistency

Theorem (B., Sudakov, Tran)

For $d \geq 3$ we have $M_d(n) \leq M'_d(2^n C_d n^{d-1})$. 

Proof. Consider an inconsistently monotone array of size $2^C n^{d-1} \times \cdots \times 2^C n^{d-1}$. Colour each entry into one of $2^d$ many colours corresponding to the monotonicity pattern of 1D subarrays passing through it. Grid Ramsey gives us a monochromatic $n \times \cdots \times n$ subarray, which is monotone by definition of our colouring.
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\[
\begin{array}{cccc}
\downarrow & \downarrow & \uparrow & \uparrow \\
\bullet & \bullet & \bullet & \bullet \\
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