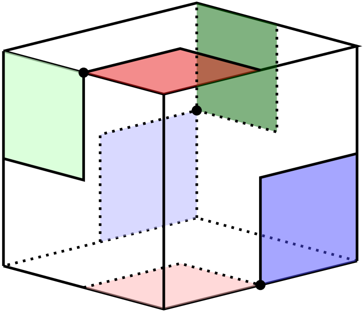
Erdős-Szekeres theorem for multidimensional arrays

Matija Bucić
joint work with Benny Sudakov and Tuan Tran



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Many different higher dimensional generalisations due to:

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Many different higher dimensional generalisations due to:
Fishburn and Graham; Kruskal; Linial and Simkin; Szabó and Tardos,...

Higher dimensional version?

What is a monotone 2D array?

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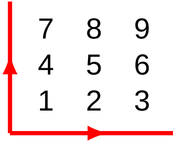

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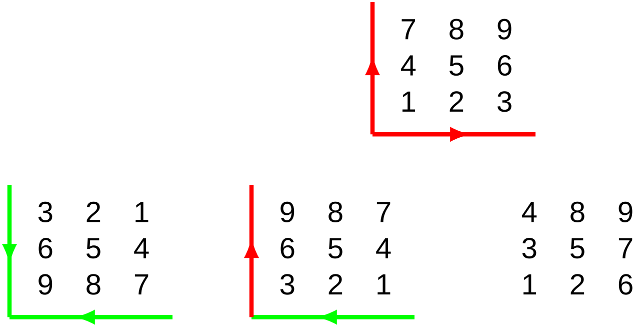

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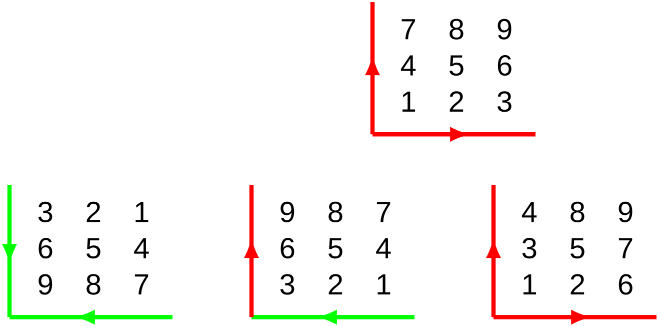

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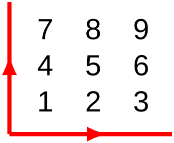

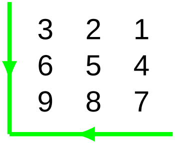

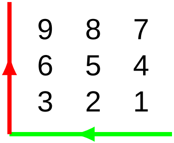

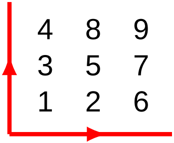

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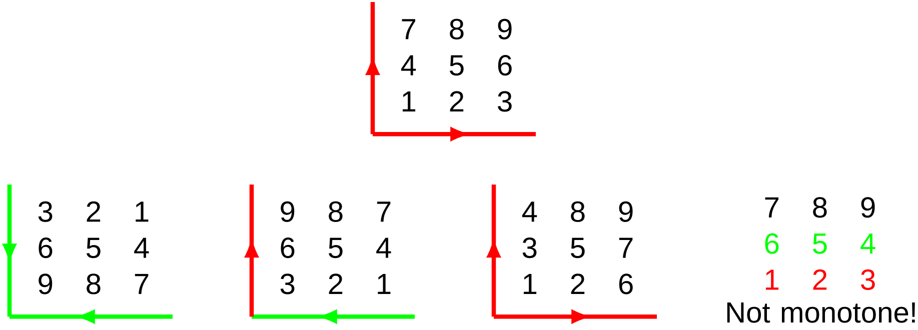

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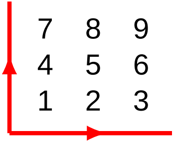

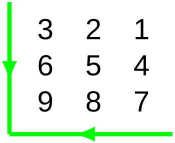

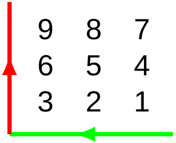

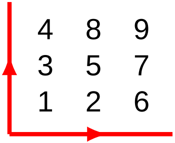

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Higher dimensional Erdős-Szekeres theorem

A d-dimensional array is an injective function from $A_1 \times \cdots \times A_d \to \mathbb{R}$ where $A_1, \dots, A_d \subseteq \mathbb{Z}$.

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(iii) $M_d(n) \quad 2^{2^{2^{n^d \quad 1}}}$; for $d \quad 4$:

Proofs.

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$M_d^0(n) \quad M_d(n)$.

Theorem 2 (B., Sudakov, Tran)

For every d 2, we have $M_d^0(n)$ $2^{2^{(1+o(1))\, n^{d} \quad 1}}$.

Proof of 2D case of inconsistent monotonicity.

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	56	36	24	57	30	52	37	43	46	17	16	1	2	11
	41	8	42	60	68	38	48	58	66	44	61	28	49	29
n^2	40	59	23	67	54	62	4	51	55	7	34	33	63	21
	10	64	22	32	3	12	69	6	13	31	14	35	15	19
	53	20	65	45	50	5	47	70	39	25	26	27	18	9

Proof of 2D case of inconsistent monotonicity.

Consider an $n^2 \times N$ array f with $n^2, N = n^{2n} \binom{n^2}{n} = 2^{2^{(1+o(1))n}}$:

Each column contains a monotone subsequence of size n

n^2 rows:

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Each column contains a monotone subsequence of size n and it can appear in $\binom{n^2}{n}$ different positions.

n^2

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40	59	23	67	54	62	4	51	55	7	34	33	63	21
10	64	22	32	3	12	69	6	13	31	14	35	15	19
53	20	65	45	50	5	47	70	39	25	26	27	18	9

Proof of 2D case of inconsistent monotonicity.

Consider an $n^2 \times N$ array f with n^2; $N = n^{2^n} \binom{n^2}{n} = 2^{2^{(1+o(1))n}}$:

Each column contains a monotone subsequence of size n and it can appear in $\binom{n^2}{n}$ different positions.

Keep the most common position to get an $M \times n$ array with all columns monotone and $M = N/\binom{n^2}{n} \geq n^{2^n}$

n^2 rows:

56	36	24	57	30	52	37	43	46	17	16	1	2	11
41	8	42	60	68	38	48	58	66	44	61	28	49	29
40	59	23	67	54	62	4	51	55	7	34	33	63	21
10	64	22	32	3	12	69	6	13	31	14	35	15	19
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	56	36	24	30	43	46	16	1	11
n	41	8	42	68	58	66	61	28	29
	40	59	23	54	51	55	34	33	21
	10	64	22	3	6	13	14	35	19
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First row contains a monotone subsequence of size $\sqrt{M}$;

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n	41	8	42	68	58	66	61	28	29
	40	59	23	54	51	55	34	33	21
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$n\left\{\right.$ 68	58	66	61	29
54	51	55	34	21
3	6	13	14	19

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Consider an $n^2 \times N$ array f with $n^2, N = n^{2^n} \binom{n^2}{n} = 2^{2^{(1+o(1))n}}$.

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First row contains a monotone subsequence of size $\sqrt{M}$; delete all columns not intersecting it.

Repeat for every row.

	68	58	66	61	29
n	54	51	55	34	21
	3	6	13	14	19

Proof of 2D case of inconsistent monotonicity.

Consider a $m^2 \times N$ array f with n^2; $N = n^{2^n} \binom{n^2}{n} = 2^{2^{(1+o(1))n}}$:

Each column contains a monotone subsequence of size n and it can appear in $\binom{n^2}{n}$ different positions.

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- Repeat for every row.

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n	54	34	21
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Proof of 2D case of inconsistent monotonicity.

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First row contains a monotone subsequence of size $\sqrt{M}$; delete all columns not intersecting it.

Repeat for every row.

We get an $\underbrace{M^{1/2^n}}_{\geq n} \times n$ array with all rows and columns monotone.

n	68	61	29
	54	34	21
	3	14	19

Proof of 2D case of inconsistent monotonicity.

Consider an n^2 N array f with $n^2; N = n^{2^n} \binom{n^2}{n} = 2^{2^{(1+o(1))n}}$:

Each column contains a monotone subsequence of size n and it can appear in $\binom{n^2}{n}$ different positions.

Keep the most common position to get an M n array with all columns monotone and $M = N / \binom{n^2}{n}$ n^{2^n}

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First row contains a monotone subsequence of size $\sqrt{M}$; delete all columns not intersecting it.

Repeat for every row.

We get an $\underbrace{M^{1/2^n}}_{n} \times n$ array with all rows and columns monotone.

Theorem (B., Sudakov, Tran)

$$M_2^0(n) \leq 2^{2^{(1+o(1))n}}.$$

Monotone 2D case

Notice that $M_2(n) \le M_2'(2n - 1)$.

Monotone 2D case

Notice that $M_2(n) \quad M_2^0(2n \quad 1)$:

- Delete the less common of increasing/decreasing rows.

Monotone 2D case

Notice that $M_2(n) \leq M_2'(2n-1)$:

- Delete the less common of increasing/decreasing rows.
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Notice that $M_2(n) \le M_2^0(2n - 1)$:

- Delete the less common of increasing/decreasing rows.
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Theorem (B., Sudakov, Tran)

$$M_2(n) \le 2^{2^{(2+o(1))n}}:$$

Monotone 2D case

Notice that $M_2(n) \geq M_2^0(2n - 1)$:

- Delete the less common of increasing/decreasing rows.
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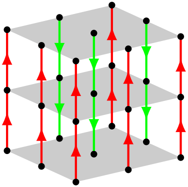
Theorem (B., Sudakov, Tran)

$$M_2(n) = 2^{2^{(2+o(1))n}}.$$



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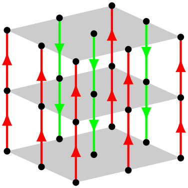

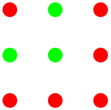

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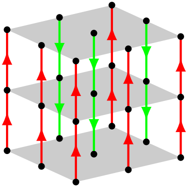
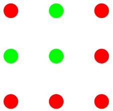
Theorem (B., Sudakov, Tran)

$$M_2(n) \quad 2^{2^{(2+\,o(1))\,n}}:$$





Lemma (Grid Ramsey)

9C = C(d; k) : for $N \quad 2^{Cn^{d} \quad 1}$, any k-colouring of thed-dimensional $N \quad ::: \quad N$ grid contains a monochromatic subgrid of size $::: \quad n$.

From inconsistency to consistency

Theorem (B., Sudakov, Tran)

For $d \geq 3$ we have $M_d(n) \leq M'_d(2^{C_d n^{d-1}})$.

Theorem (B., Sudakov, Tran)

For d 3 we have $M_d(n)$ $M^0_d(2^{C_d n^d \; 1})$:

Proof.

□

Theorem (B., Sudakov, Tran)

For $d \geq 3$ we have $M_d(n) \leq M_d^0(2^{C_d n^{d-1}})$:

Proof.

Consider an inconsistently monotone array of size $2^{Cn^{d-1}} \times \cdots \times 2^{Cn^{d-1}}$.

□

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For d 3 we have $M_d(n)$ $M_d^0(2^{C_d n^{d\;\;1}})$:

Proof.

Consider an inconsistently monotone array of size $2^{Cn^{d\;\;1}}$ $2^{Cn^{d\;\;1}}$.
Colour each entry into one of 2^d many colours corresponding to the monotonicity pattern of 1D subarrays passing through it.

□

From inconsistency to consistency

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For d 3 we have $M_d(n)$ $M_d^0(2^{C_d n^d \; 1})$:

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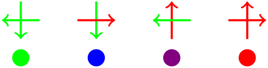


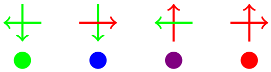
Theorem (B., Sudakov, Tran)

For $d \geq 3$ we have $M_d(n) \leq M'_d(2^{C_d n^{d-1}})$.

Proof.

Consider an inconsistently monotone array of size $2^{Cn^d} 2^{1} \times \cdots \times 2^{Cn^{d-1}}$.
Colour each entry into one of 2^d many colours corresponding to the monotonicity pattern of 1D subarrays passing through it.

□



8	7	3
6	5	4
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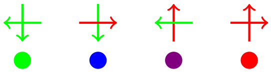

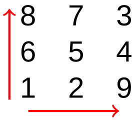

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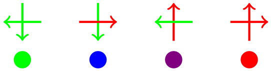

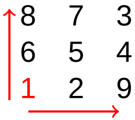

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For $d \geq 3$ we have $M_d(n) \leq M_d^0(2^{C_d n^{d-1}})$.

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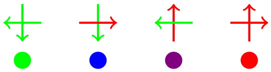

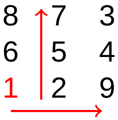

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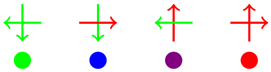

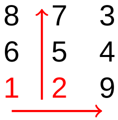

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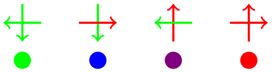

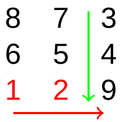

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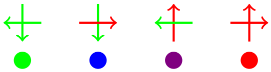

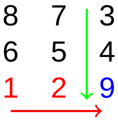

From inconsistency to consistency

Theorem (B., Sudakov, Tran)

For $d \geq 3$ we have $M_d(n) \leq M_d^0(2^{C_d n^{d-1}})$.

Proof.

Consider an inconsistently monotone array of size $2^{Cn^{d-1}} \times \cdots \times 2^{Cn^{d-1}}$.

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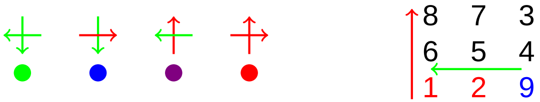

From inconsistency to consistency

Theorem (B., Sudakov, Tran)

For $d \geq 3$ we have $M_d(n) \leq M'_d(2^{C_d n^{d-1}})$.

Proof.

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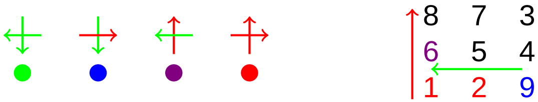

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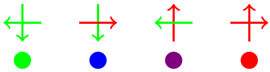

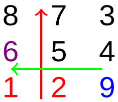

From inconsistency to consistency

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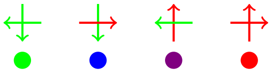

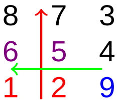

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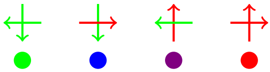

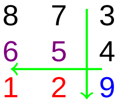

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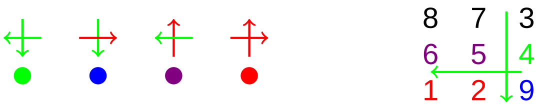

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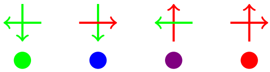

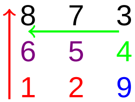

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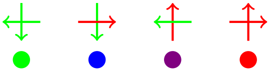

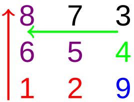

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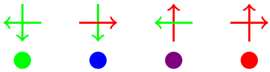

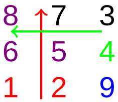

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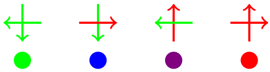

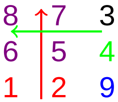

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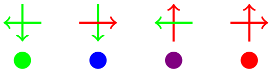

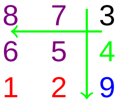

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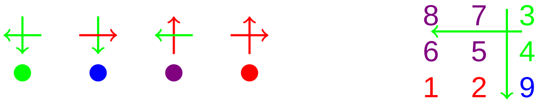

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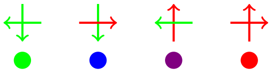

8	7	3
6	5	4
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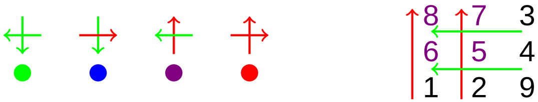

Part 2: Lex-monotone arrays

Lex-monotone arrays.

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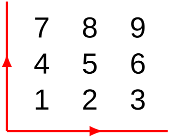

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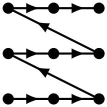

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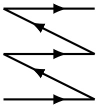

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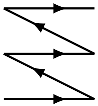

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An array is lex-monotone if we can permute the coordinates and re ect th array along some dimensions to obtain the lexicographic array.

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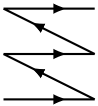

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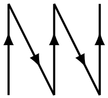

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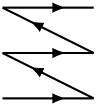

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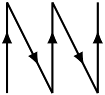

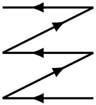

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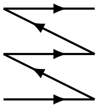

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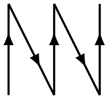

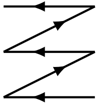

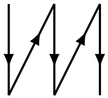

Lex version of Higher dimensional Erdős-Szekeres

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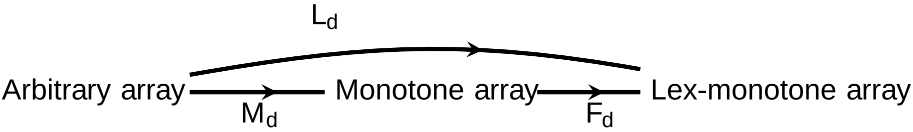

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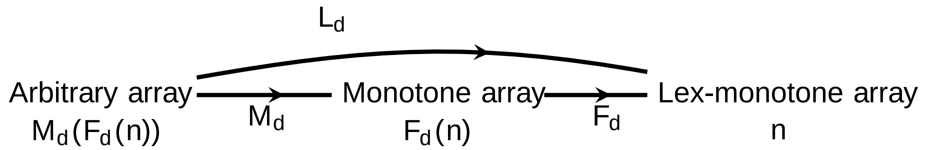

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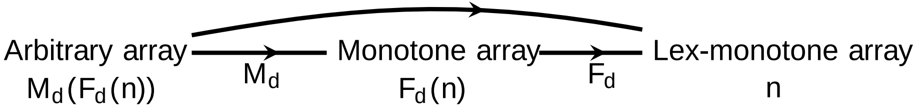

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Fishburn and Graham show:

(i) $n^2 \quad F_2(n) \quad 2n^2 \implies L_2(n) \quad \mathrm{towr}_5(O(n^2))$ and

(ii) $n^d \quad F_d(n) \quad \mathrm{towr}_d(n) \implies L_d(n)$ is still Ackermann type.

Theorem 3 (B., Sudakov, Tran, 2019+)

(i) $F_2(n) = 2n^2 \quad 5n + 4$,

(ii) $F_d(n) \quad 2^{n^{d} \quad 2}$ for $d \quad 3$.

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Enough to look for an increasing lex-monotone array in a general increasing array.

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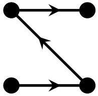

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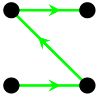

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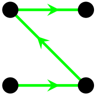

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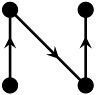

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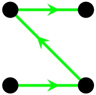

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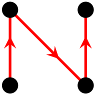

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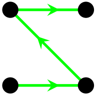

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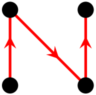

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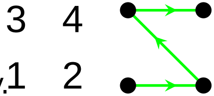

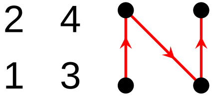

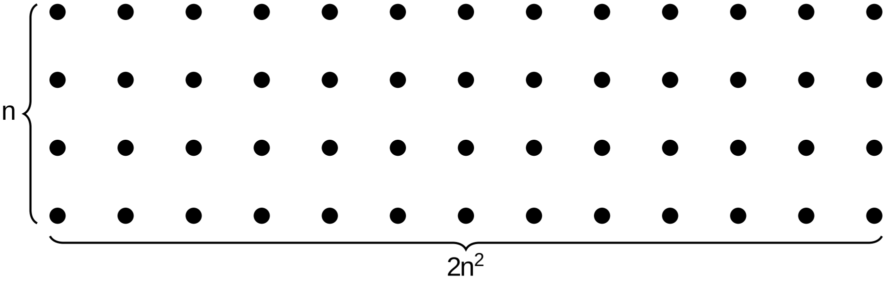

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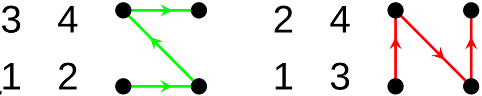

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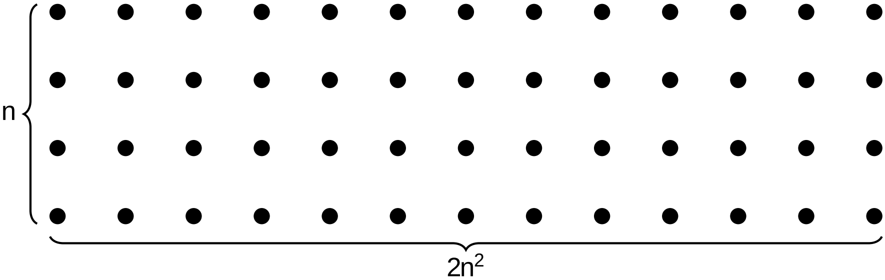

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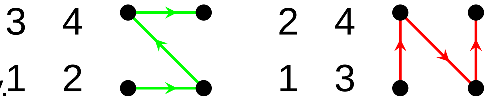

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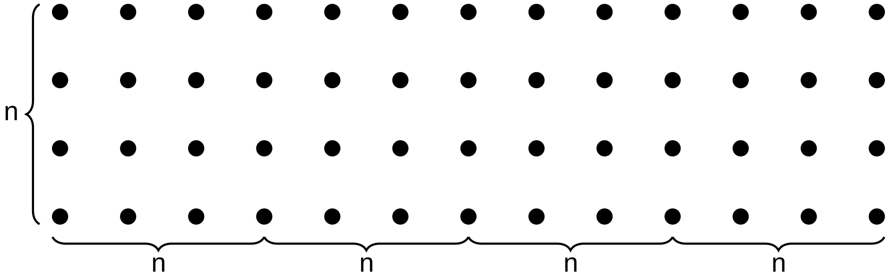

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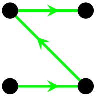

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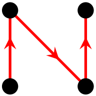

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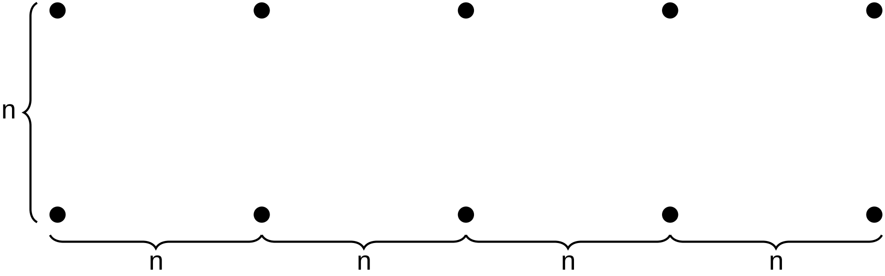

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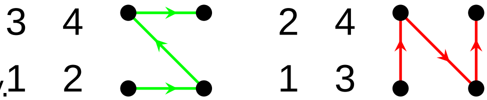

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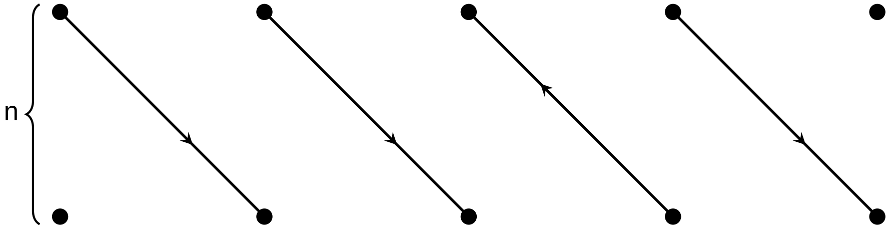

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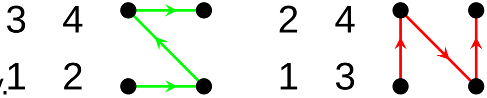

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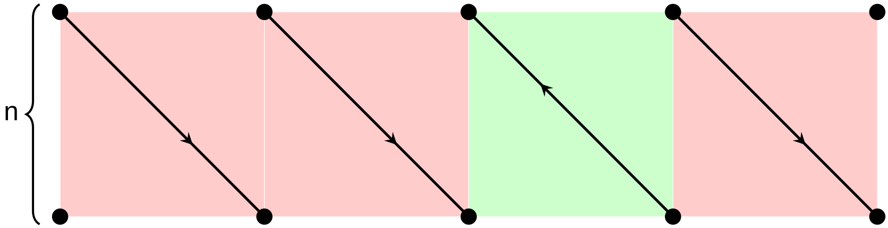

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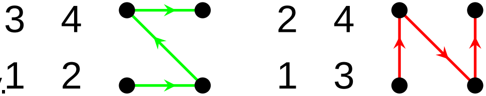

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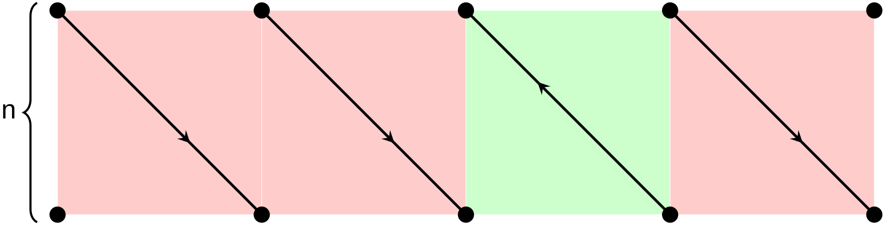
Keep only entries with both coordinates divisible by n.

If we can nd n red squares we nd n n red lex-monotone array



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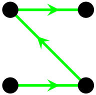

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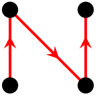

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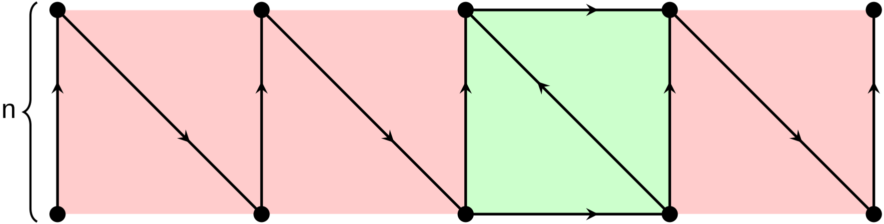

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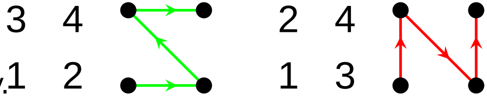
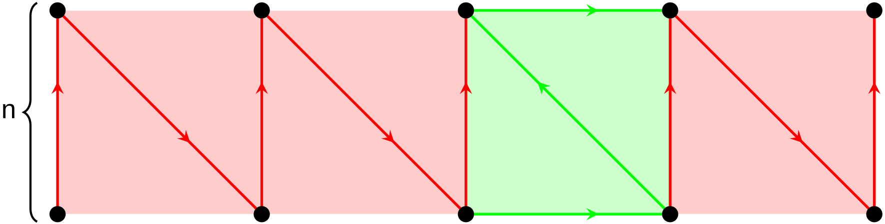


Take an $n \times 2n^2$ increasing array.

Keep only entries with both coordinates divisible by n.

If we can find n red squares we find $n \times n$ red lex-monotone array



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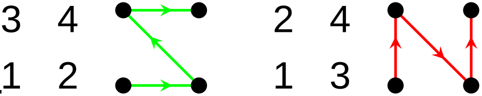

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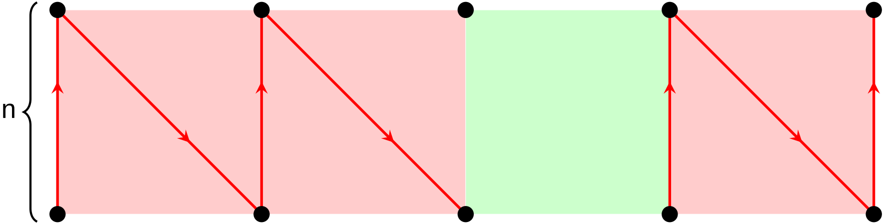

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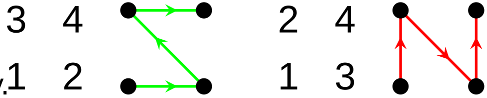

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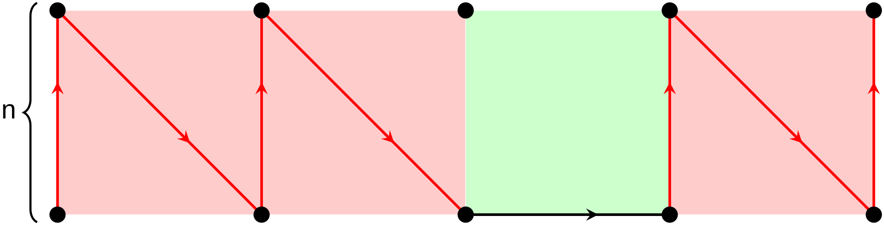

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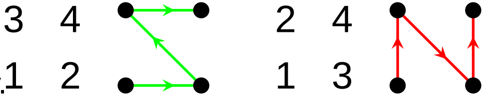

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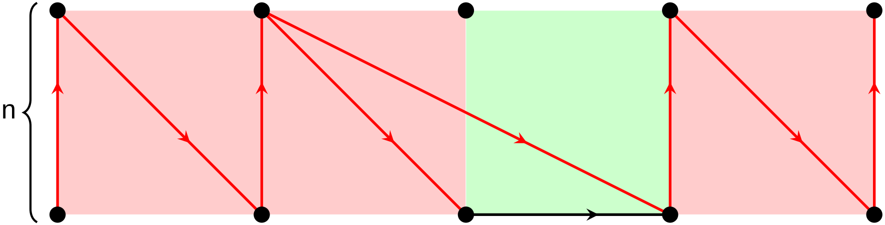

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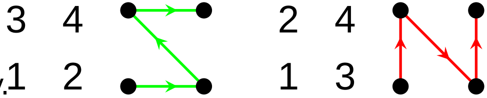

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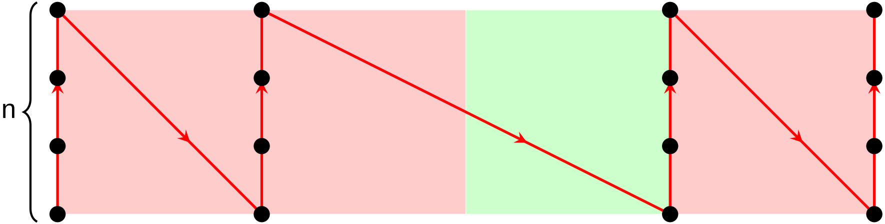

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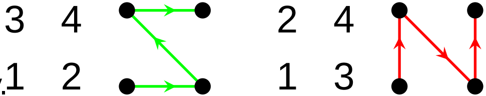

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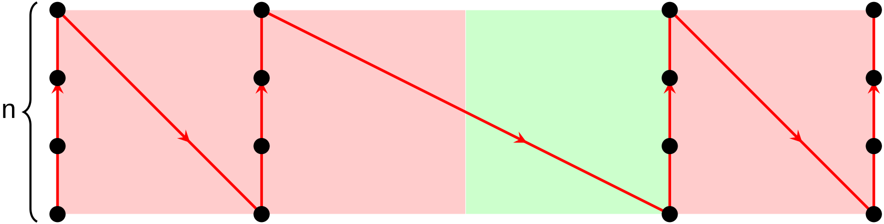

Same works if we take $2n^2 \times n$ and find n green squares.

Theorem

Any $2n^2 \times 2n^2$ increasing array contains an $n \times n$ lex-monotone array.

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$\square$

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Take an increasing $2n^2 \times 2n^2$ array.

□

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Any $2n^2 \quad 2n^2$ increasing array contains an $n \quad n$ lex-monotone array.

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Take an increasing $2n^2 \quad 2n^2$ array.

Keep only points with both coordinates divisible by n.

□

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Any $2n^2 \quad 2n^2$ *increasing array contains an* $n \quad n$ *lex-monotone array.*

Proof.

Take an increasing $2n^2 \quad 2n^2$ array.

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Colour each $2 \quad 2$ subsquare red or green depending on diagonal edge.

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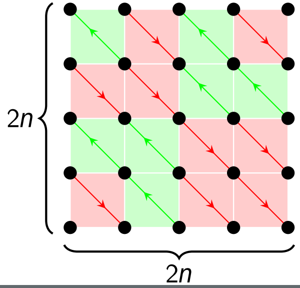

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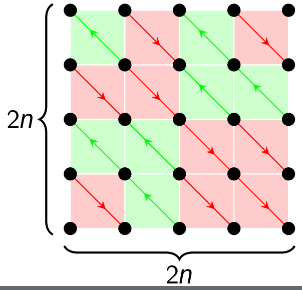

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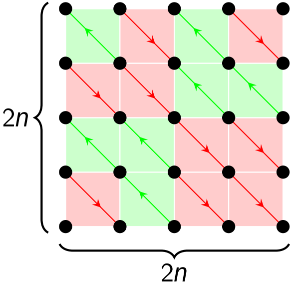
This occurs for more frequent colour since we have a $2n$ $2n$ grid. □



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Is $M_2(n)$ exponential or double exponential?

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Is $M_d(n)$ always at most double exponential in n?

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Question

Is $F_3(n)$ polynomial or exponential in n?