Nearly-linear monotone paths in edge-ordered graphs

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ETH Zürich

Joint work with: Matthew Kwan, Alexey Pokrovskiy, Benny Sudakov, Tuan Tran and Adam Zsolt Wagner

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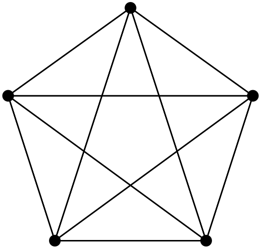

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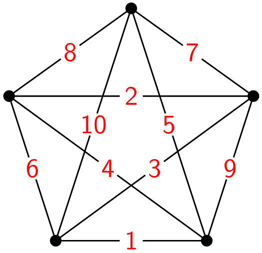

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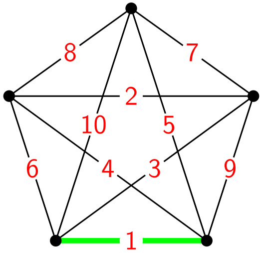

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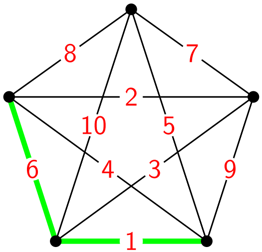

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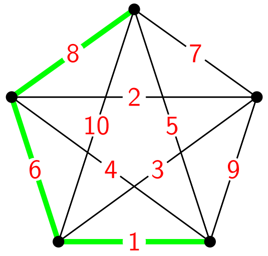

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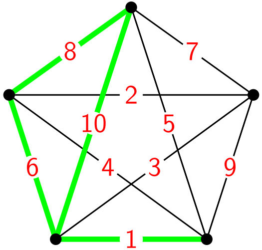

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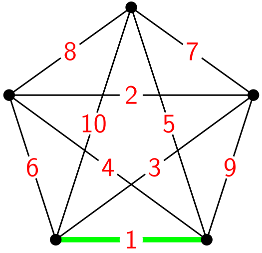

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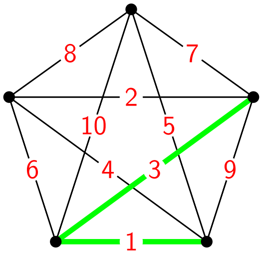

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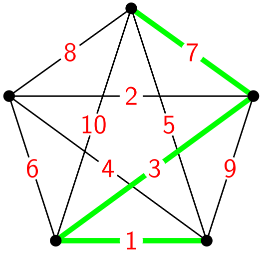

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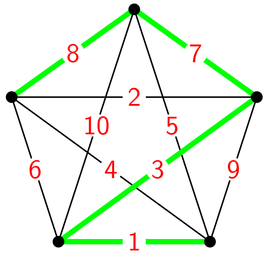

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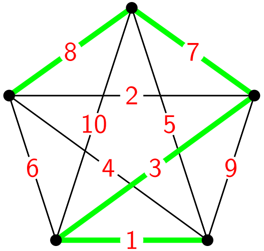

Remark: The problem for a monotone trail was solved by Graham and Kleitman, who proved that there is a trail of length $n - 1$ which is tight.

Random orderings

Question (Loh and Lavrov, 2016)

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Theorem (Angel, Ferber, S., Tassion, 2016)

Random edge ordering of K_n w.h.p. contains a monotone trail of length

$$e \quad n + o(n).$$

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Theorem 1 (Bucic, Kwan, Pokrovskiy, S., Tran, Wagner)

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Theorem 2 (Bucic, Kwan, Pokrovskiy, S., Tran, Wagner)

Let G be a graph with n vertices and average degree $d \geq 2$. Then

$$f(G) \geq \frac{d}{2^{O(\sqrt{\log d \log\log n})}}.$$

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Height tables

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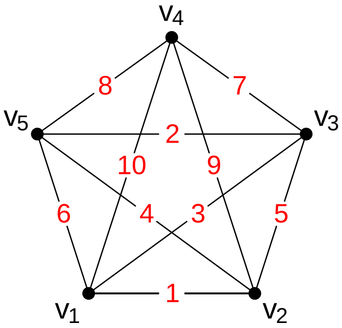

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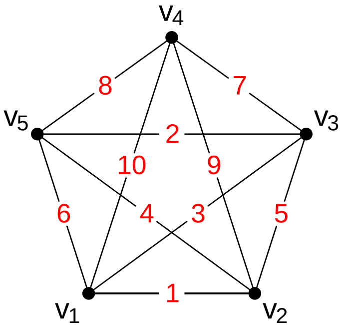

i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2					
1					

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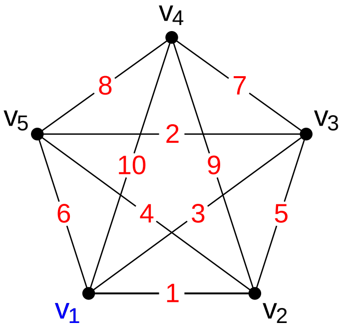

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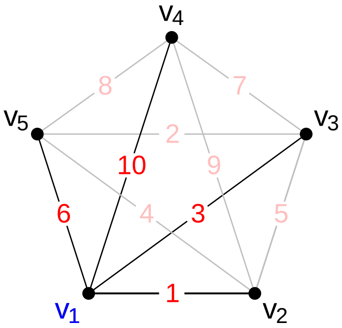

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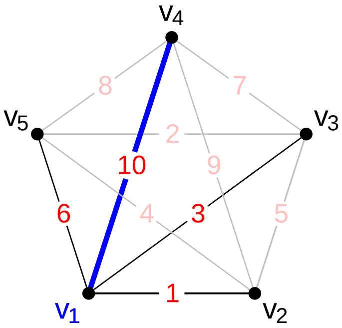

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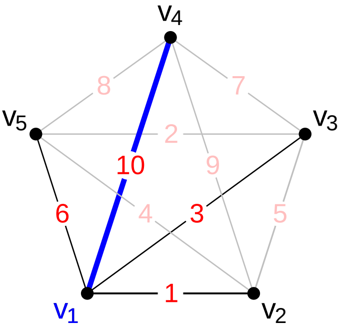

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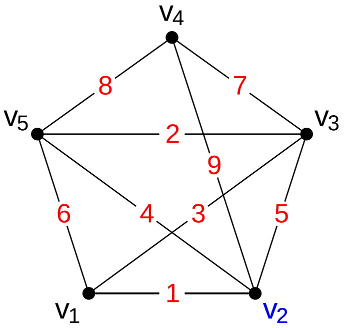

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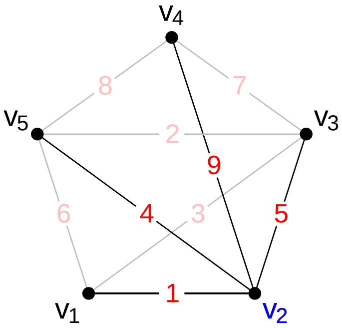

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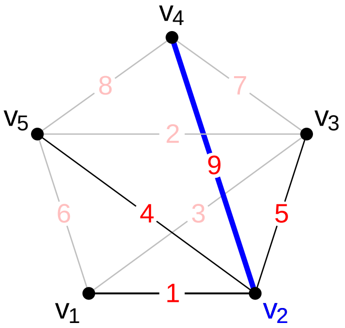

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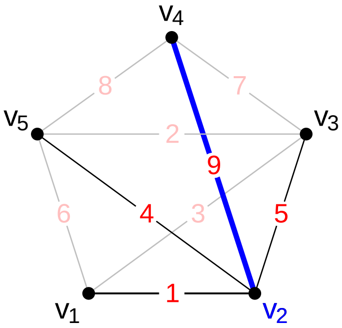

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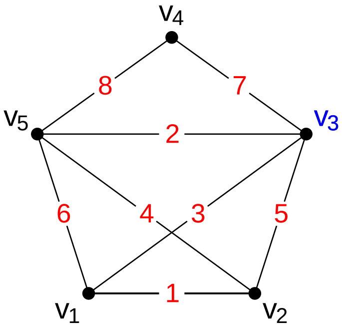

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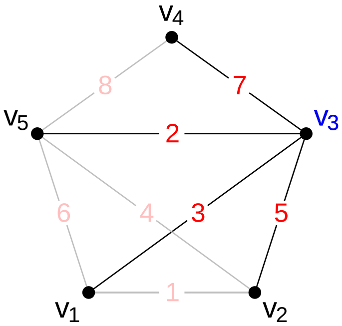

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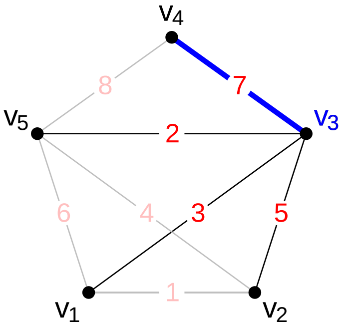

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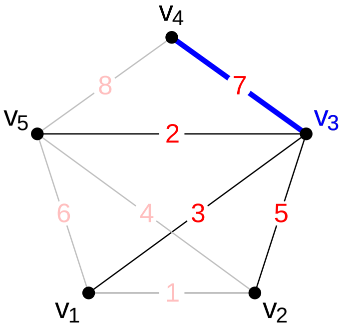

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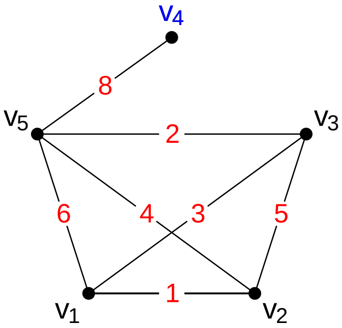

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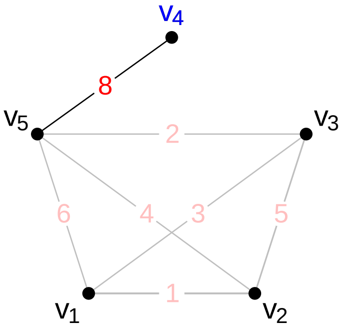

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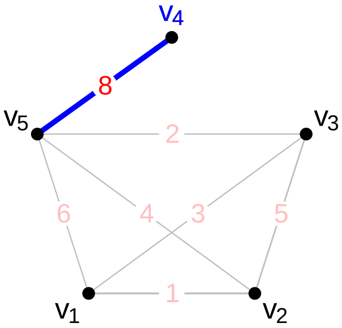

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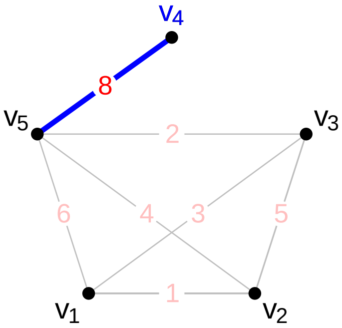

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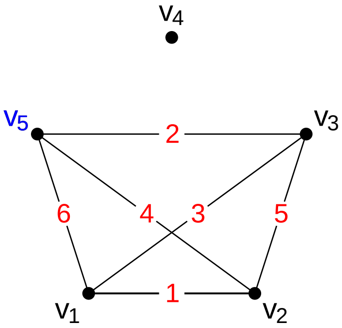

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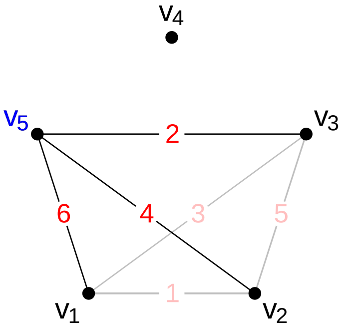

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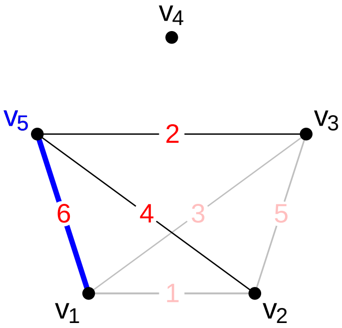

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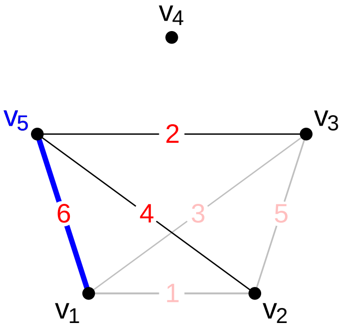

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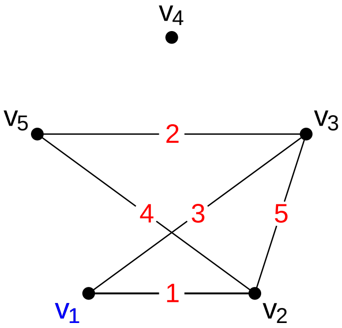

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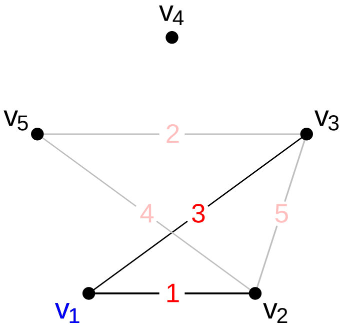
Height tables

Definition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially filled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:

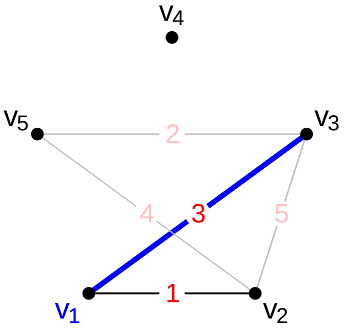


i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2					
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

De nition

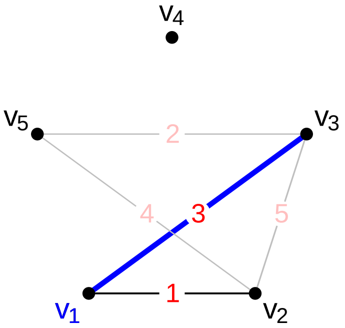
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:



⋮					
3					
2					
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:

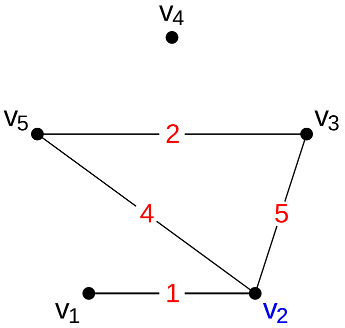


⋮					
3					
2	v_1v_3				
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Height tables

Definition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially filled array indexed by $N \quad V(G)$, constructed as follows:

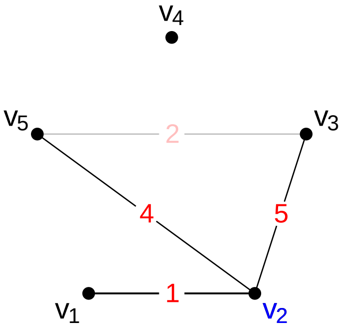


⋮					
3					
2	v_1v_3				
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3				
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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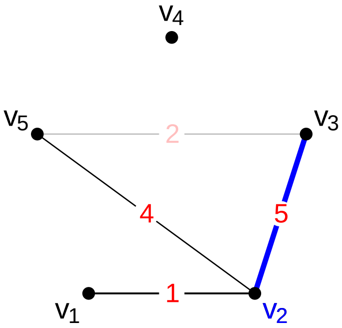
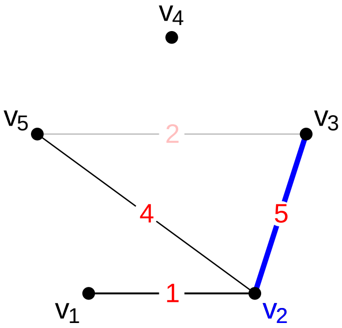
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3				
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Definition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially filled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:

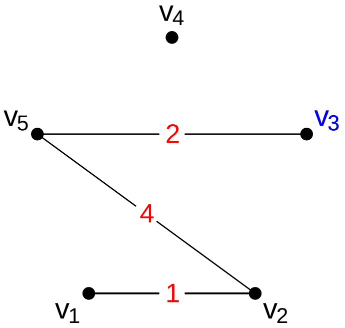


⋮					
3					
2	v_1v_3	v_2v_3			
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Height tables

De nition

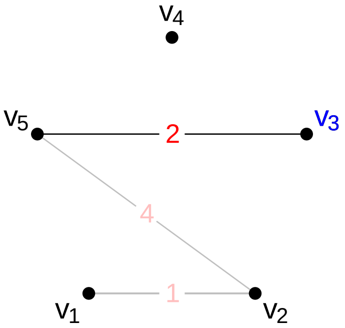
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:



⋮					
3					
2	v_1v_3	v_2v_3			
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i / v	v_1	v_2	v_3	v_4	v_5

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:

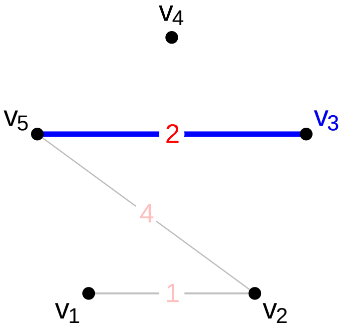


⋮					
3					
2	v_1v_3	v_2v_3			
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3			
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

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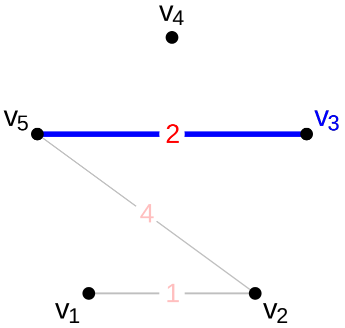
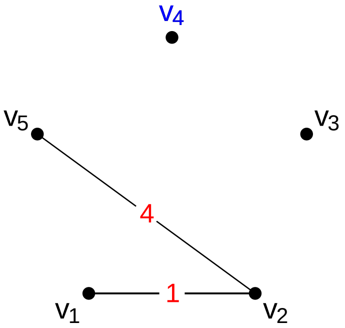
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

De nition

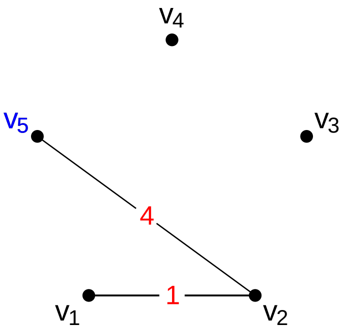
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $\mathbb{N}$ $V(G)$, constructed as follows:



⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Definition

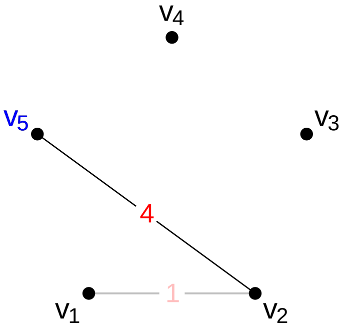
A height table of an edge ordered graph G with vertex set $[n]$ is a partially filled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

De nition

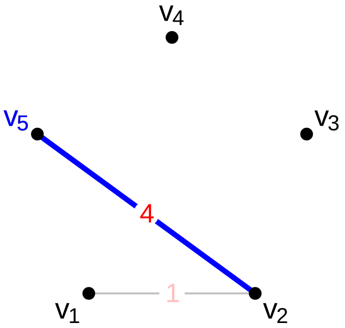
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $\mathbb{N}$ $V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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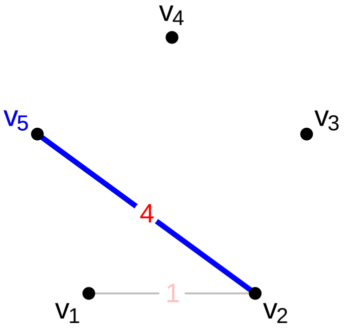
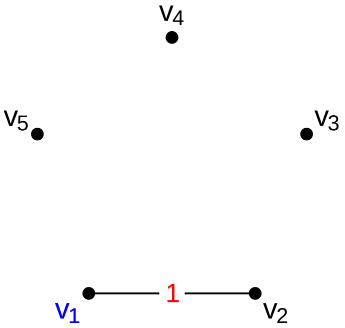
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:



$i \diagdown v$	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

De nition

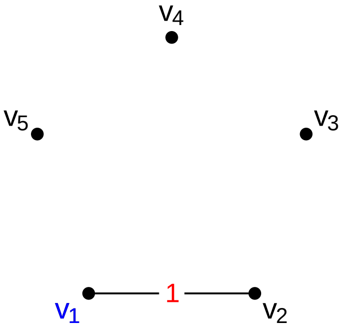
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Definition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially filled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:

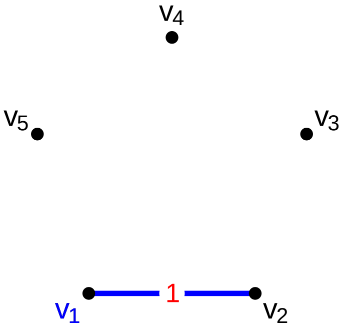


i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Definition

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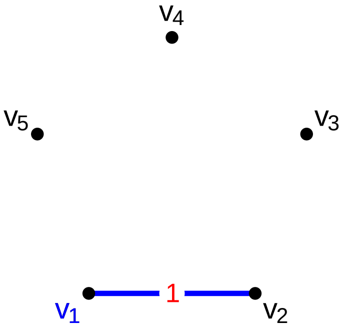
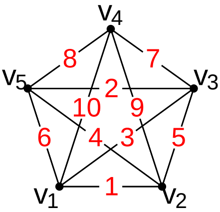


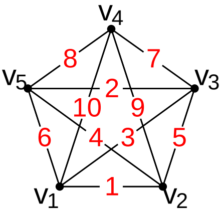
i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

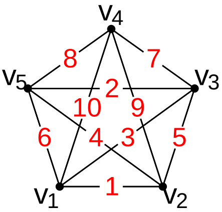
Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

There are $|E(G)|$ non-empty positions.

Basic properties of height tables

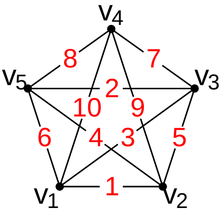


$i \backslash v$	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Basic properties of height tables

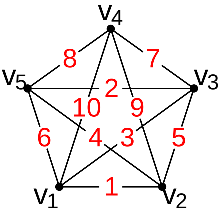


3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
$i \diagup v$	v_1	v_2	v_3	v_4	v_5

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge $v_i v_j$ is entered into column v_i or column v_j .



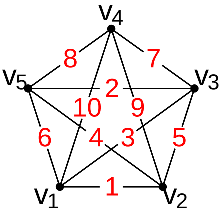
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge $v_i v_j$ is entered into column v_i or column v_j - column vertex.

Basic properties of height tables



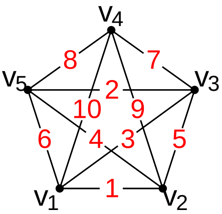
i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

Basic properties of height tables



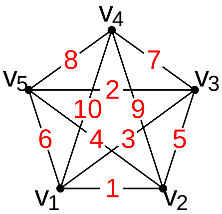
3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
1	$v_1 v_4$	$v_2 v_4$	$v_3 v_4$	$v_4 v_5$	$v_5 v_1$
i \ v	v_1	v_2	v_3	v_4	v_5

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge $v_i v_j$ is entered into column v_i or column v_j - column vertex.

Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

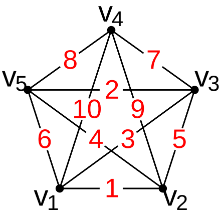
There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty

Basic properties of height tables



i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

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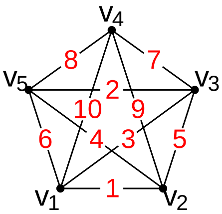
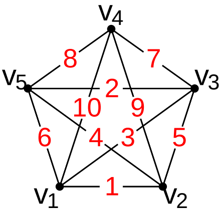
Basic properties of height tables



i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

- There are $|E(G)|$ non-empty positions.
- The height of e, denoted by $h_G(e)$, is the row index of its position
- Any edge v_iv_j is entered into column v_i or column v_j - column vertex.
- If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty

Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

- There are $|E(G)|$ non-empty positions.
- The height of e, denoted by $h_G(e)$, is the row index of its position
- Any edge v_iv_j is entered into column v_i or column v_j - column vertex.
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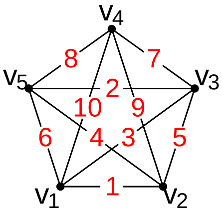
Basic properties of height tables



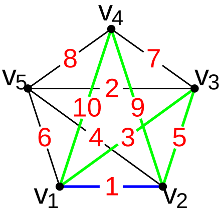
i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty and contain edges larger than e.



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

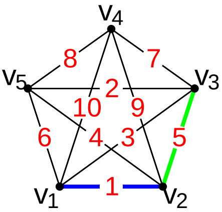
There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge $v_i v_j$ is entered into column v_i or column v_j - column vertex.

If edge $e = v_i v_j$ is entered at position (h, v_i) all positions $(a, v_i), (a, v_j)$ for $a < h$ are non-empty and contain edges larger than e.

Basic properties of height tables



i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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The height of e, denoted by $h_G(e)$, is the row index of its position

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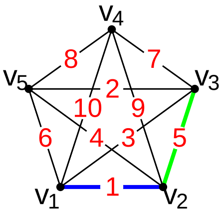
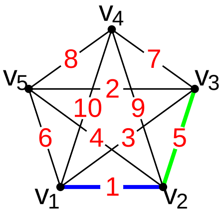


3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

- There are $|E(G)|$ non-empty positions.
- The height of e, denoted by $h_G(e)$, is the row index of its position
- Any edge v_iv_j is entered into column v_i or column v_j - column vertex.
- If edge $e = v_iv_j$ is entered at position (h, v_i) all positions $(a, v_i), (a, v_j)$ for $a < h$ are non-empty and contain edges larger than e.
  - Any such position was considered before (h, v_i).

Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

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- The height of e, denoted by $h_G(e)$, is the row index of its position
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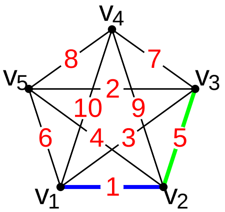
- If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty and contain edges larger than e.
  - Any such position was considered before $(h; v_i)$.
  - At that point edge v_iv_j was unused.

Basic properties of height tables



i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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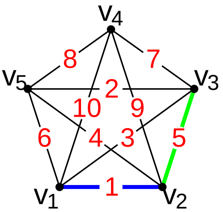
  - Any such position was considered before $(h; v_i)$.
  - At that point edge v_iv_j was unused.

Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

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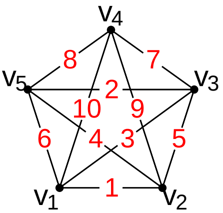
- The height of e, denoted by $h_G(e)$, is the row index of its position
- Any edge v_iv_j is entered into column v_i or column v_j - column vertex.
- If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty and contain edges larger than e:
  - Any such position was considered before $(h; v_i)$.
  - At that point edge v_iv_j was unused.
  - Since v_iv_j was not entered, there had to be a larger edge available.

Basic properties of height tables



i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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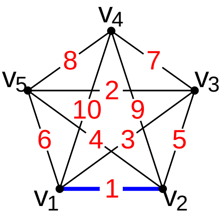
Definition

A vertex w is called an *extender* of an edge vu, entered at position $(h; v)$, if uw is an edge entered at position $(a; u)$ for some $a < h$.

Basic properties of height tables



$i \backslash v$	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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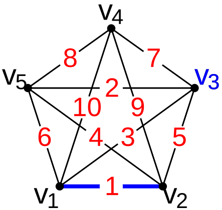

$i \diagdown v$	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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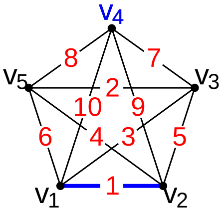
Basic properties of height tables



i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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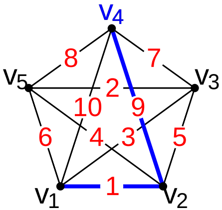
Basic properties of height tables



i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

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Definition

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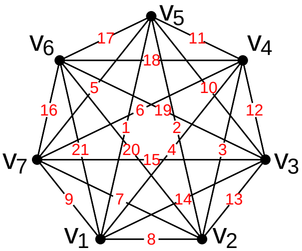
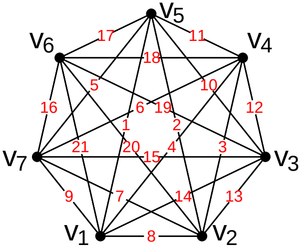
Application of height tables



5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

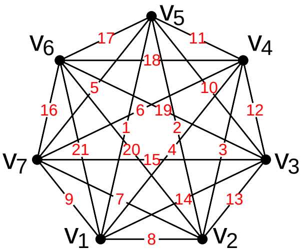
Application of height tables



$i \backslash v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3

Theorem (Rödl)

In any edge ordered graph there is a monotone path of length $\sqrt{\overline{d}(G)}$:



5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

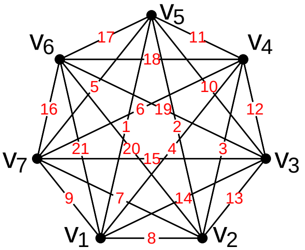
Theorem (Rödl)

In any edge ordered graph there is a monotone path of length $\sqrt{d(G)}$:

Proof.

□

Application of height tables



5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

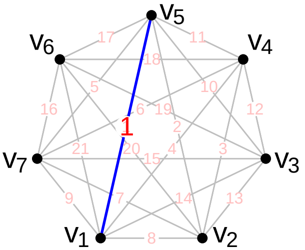
Theorem (Rödl)

In any edge ordered graph there is a monotone path of length $\sqrt{d(G)}$.

Proof.

There is an edge u_1u_2 of height at least $|E(G)|/n = d(G)/2$.

□



5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
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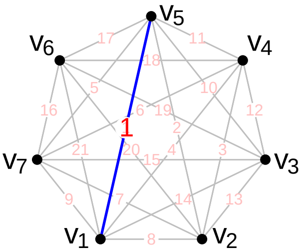


$i \backslash v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3

Theorem (Rödl)

In any edge ordered graph there is a monotone path of length $\sqrt{d(G)}$.

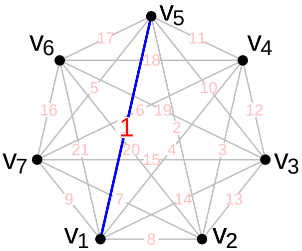
Proof.

There is an edge u_1u_2 of height at least $|E(G)|/n = d(G)/2$.

Let u_3 be its highest extender.

□

Application of height tables



5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Theorem (Rödl)

In any edge ordered graph there is a monotone path of length $\sqrt{d(G)}$.

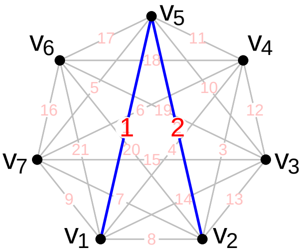
Proof.

There is an edge u_1u_2 of height at least $|E(G)|/n = d(G)/2$.

Let u_3 be its highest extender.

□

Application of height tables



5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Theorem (Rödl)

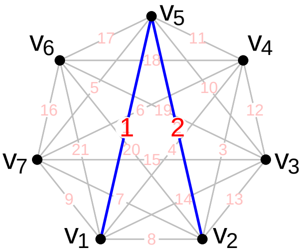
In any edge ordered graph there is a monotone path of length $\sqrt{d(G)}$.

Proof.

There is an edge u_1u_2 of height at least $|E(G)|/n = d(G)/2$.

Let u_3 be its highest extender.

□



$i \backslash v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3

Theorem (Rödl)

In any edge ordered graph there is a monotone path of length $\sqrt{d(G)}$.

Proof.

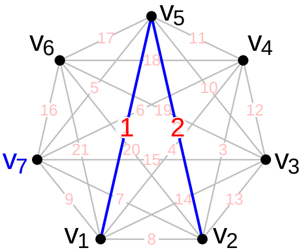
There is an edge u_1u_2 of height at least $|E(G)|/n = d(G)/2$.

Let u_3 be its highest extender.

Repeat, let u_{i+1} be the highest extender of $u_{i-1}u_i$

□

Application of height tables



5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Theorem (Rödl)

In any edge ordered graph there is a monotone path of length $\sqrt{d(G)}$.

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Repeat, let u_{i+1} be the highest extender of $u_{i-1}u_i$

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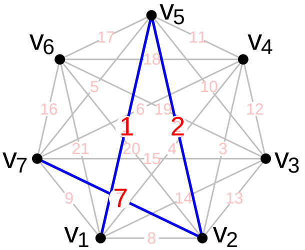
Application of height tables



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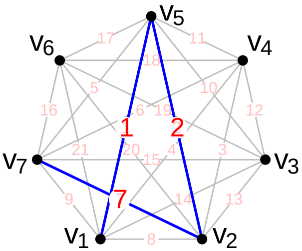

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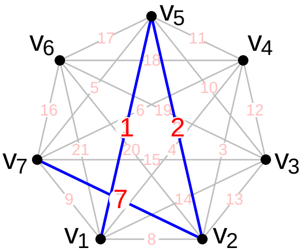

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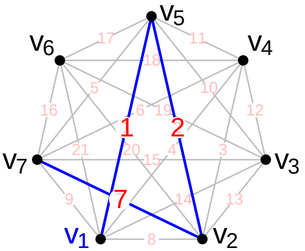

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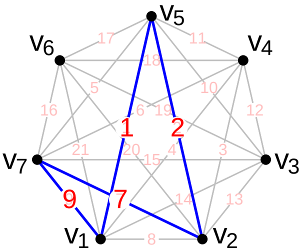

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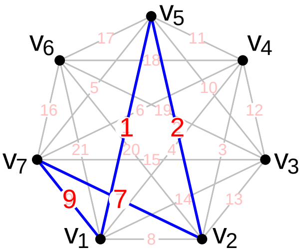

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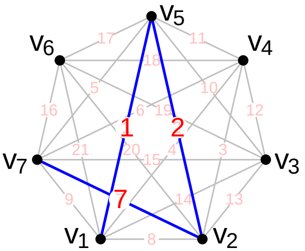

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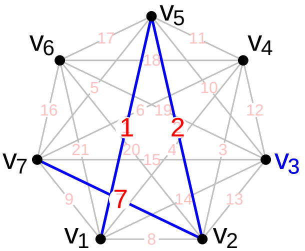

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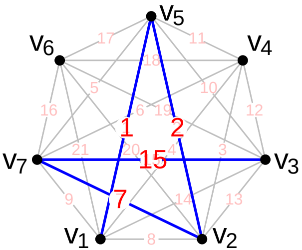

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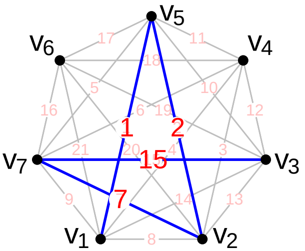

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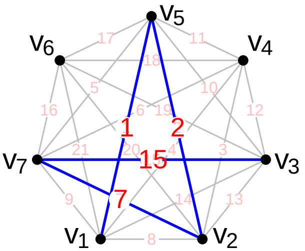

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Repeat as long as $d/2 - 1 - \dots - i = d/2 - \binom{i}{2} > 0$, $\Leftarrow$ $\sqrt{d} > i$. □

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Remark: Let $\varepsilon > 0$; then there exists an n vertex graph G with average degree $d(G) = n^{\varepsilon}$ for which this result is tight up to a constant factor.

Proof strategy

Theorem

Let G be an ordered graph, $e \in E(G)$ an edge with $h_G(e) > a$. Then there is an increasing path P starting with e, having length at least

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such that $h_G(f) \geq h_G(e) - a$ for every $f \in E(P)$.

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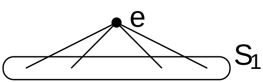

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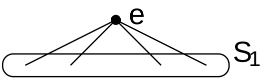

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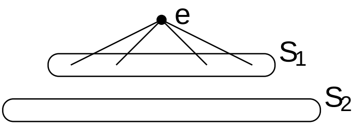

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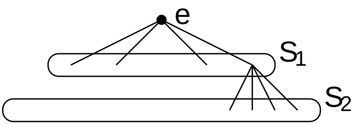

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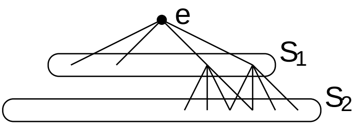

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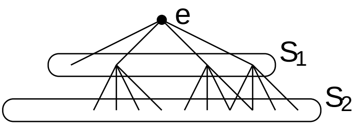

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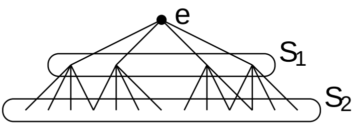

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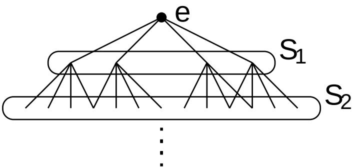

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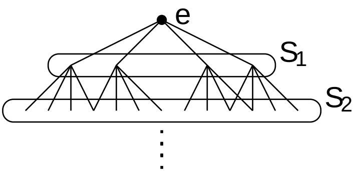

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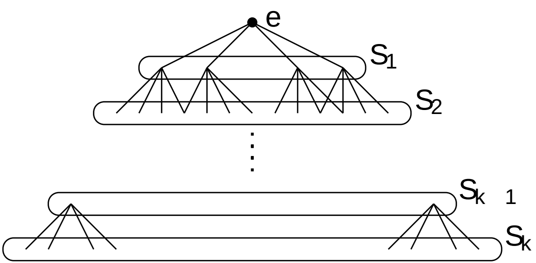

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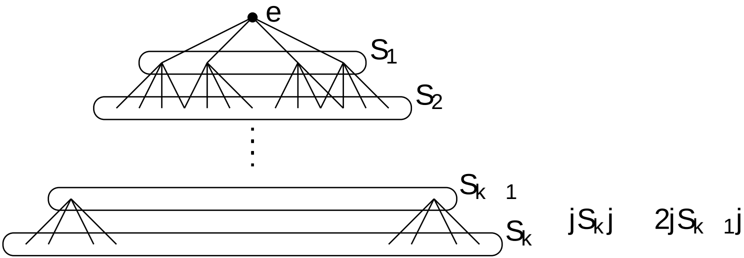

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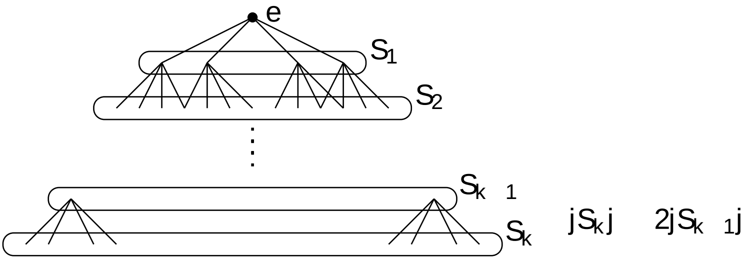

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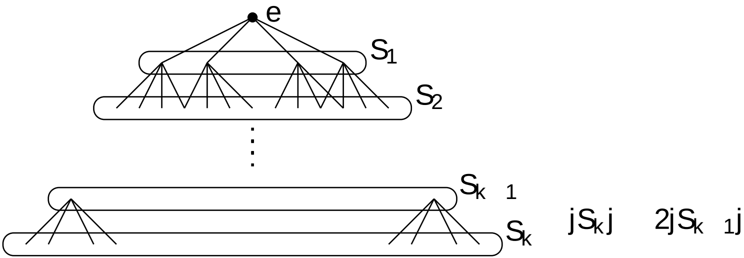

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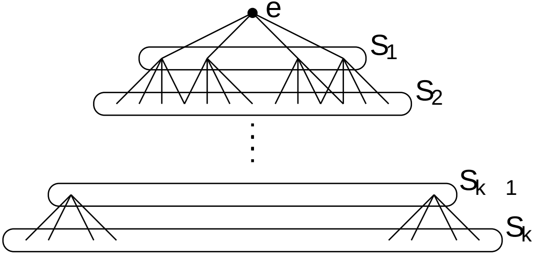
Consider the subgraph G' of G induced by $S_{k-1} \cup S_k$.

Finding a dense almost regular subgraph of extenders



$|S_k| \geq 2|S_{k-1}|$

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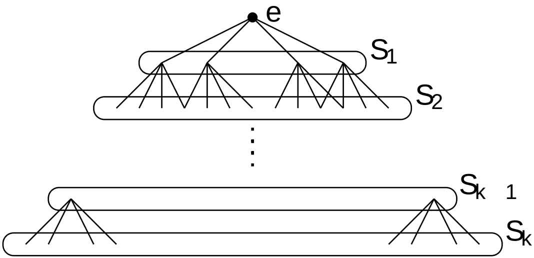
Consider the subgraph G' of G induced by $S_{k-1} \cup S_k$. By construction every vertex in S_{k-1} has degree at least $a/\log n$ in G'.



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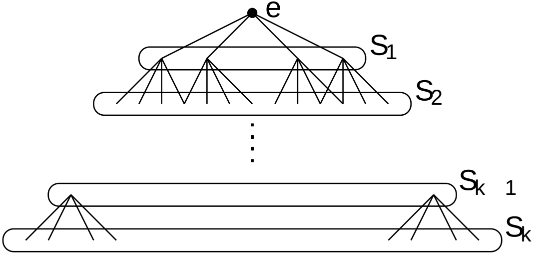
Consider the subgraph G' of G induced by $S_{k-1} \cup S_k$. By construction every vertex in S_{k-1} has degree at least $a/\log n$ in G'. Therefore,

$$d(G') \ge a/(6 \log n).$$



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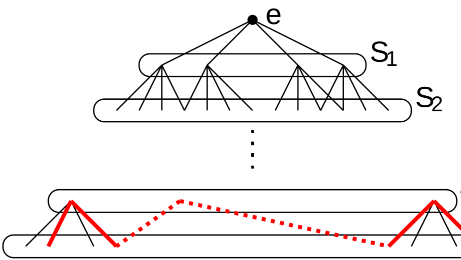
Apply regularisation lemma to get an almost regular subgraph H of G':

$$a/(6 \log n)^2 \quad d(H) \quad (\ H) \quad 2d(H)$$



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Consider the subgraph G^0 of G induced by $S_{k \; 1} [\; S_k$. By construction every vertex in $S_{k \; 1}$ has degree at least $a = \log n$ in G^0. Therefore,

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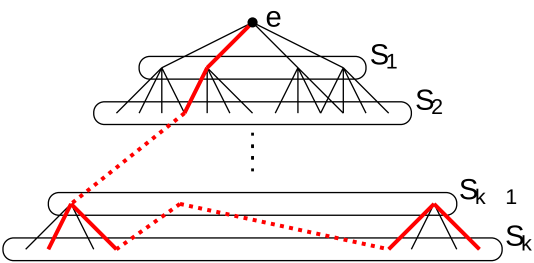

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Apply induction within H using only top $a^{3/4}$ rows.

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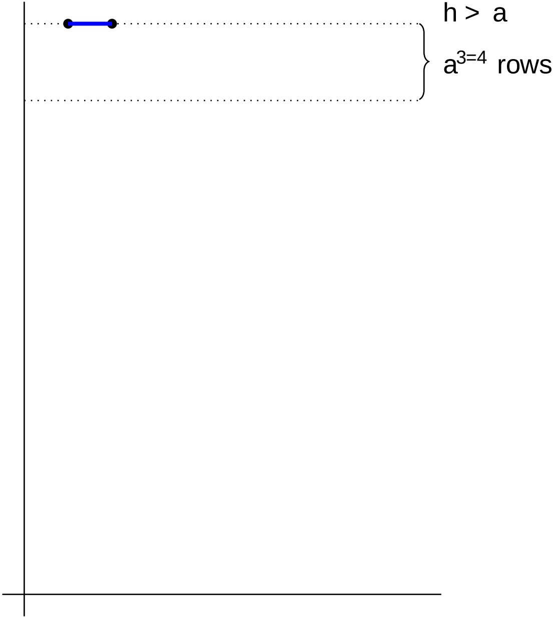

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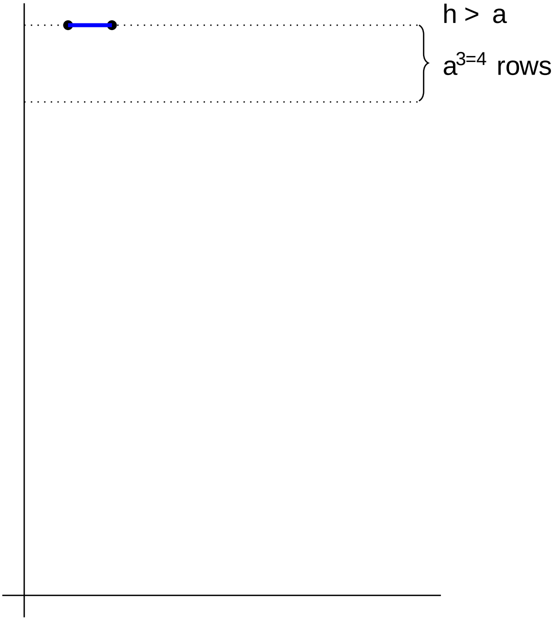

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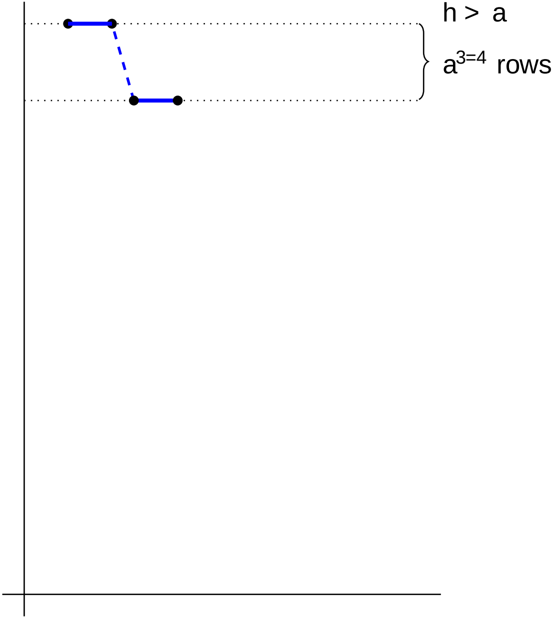

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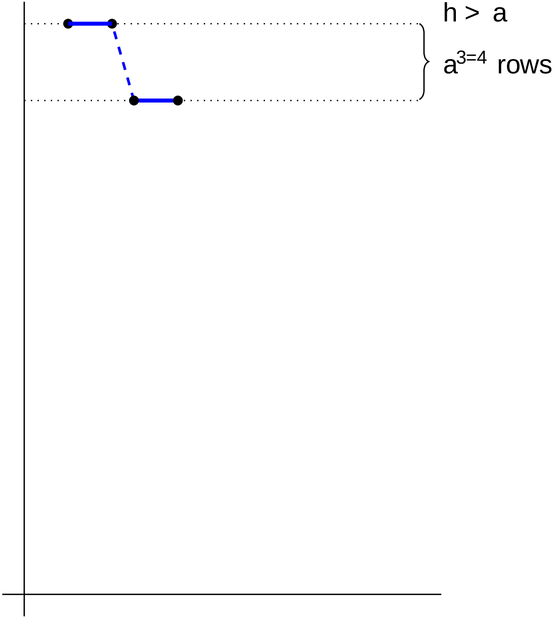
Remove all but its last two vertices.



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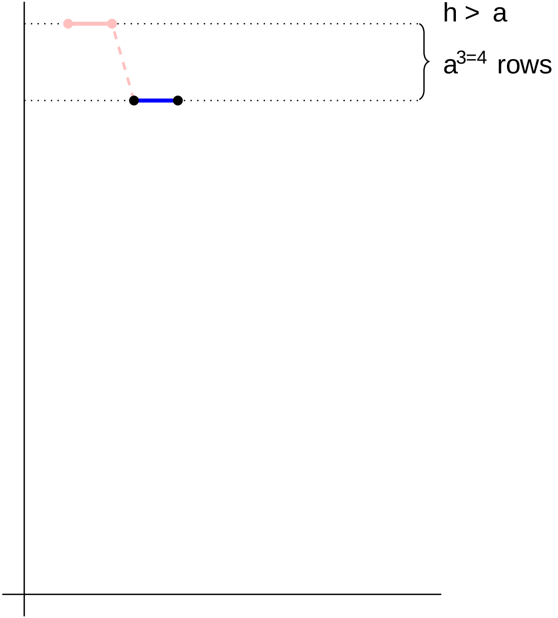

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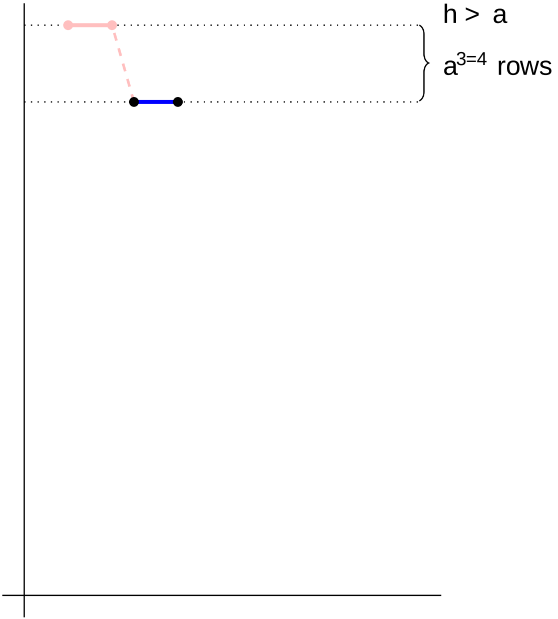

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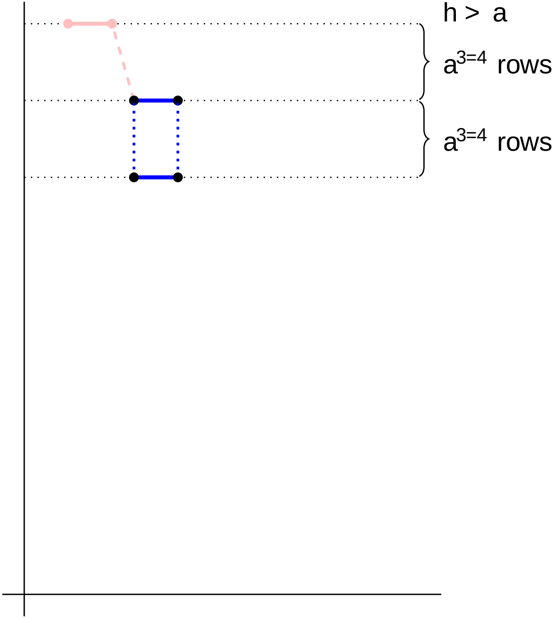

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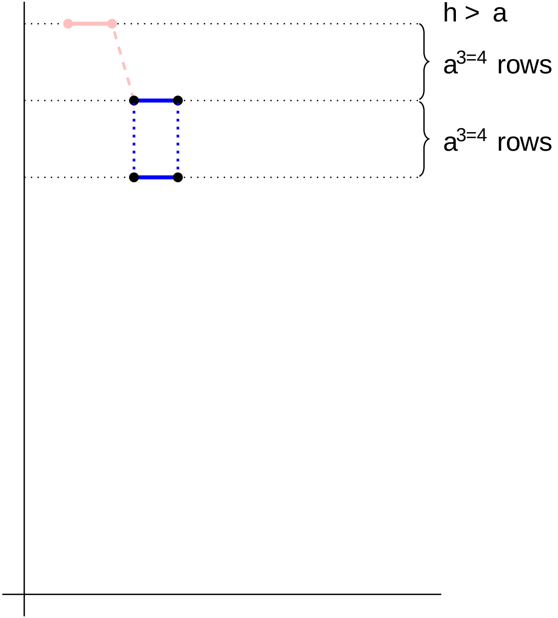

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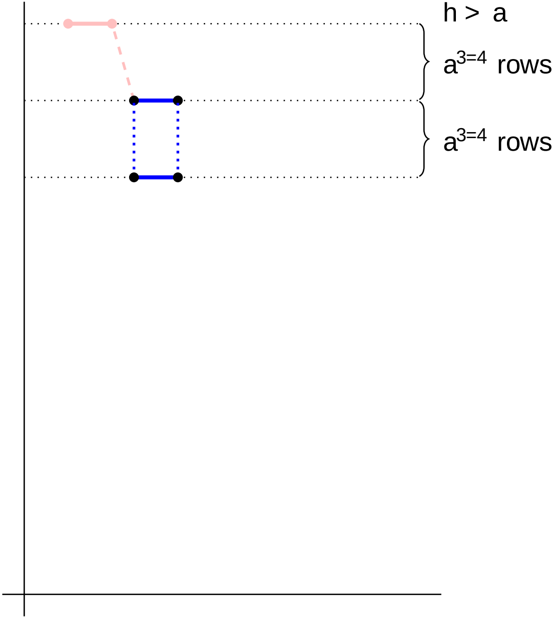

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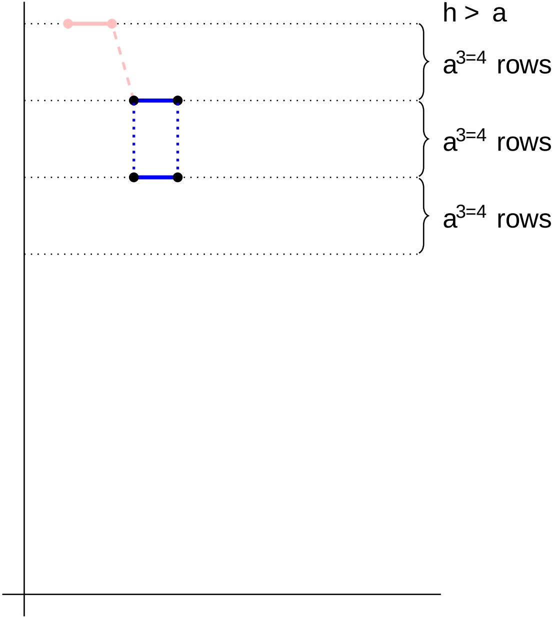

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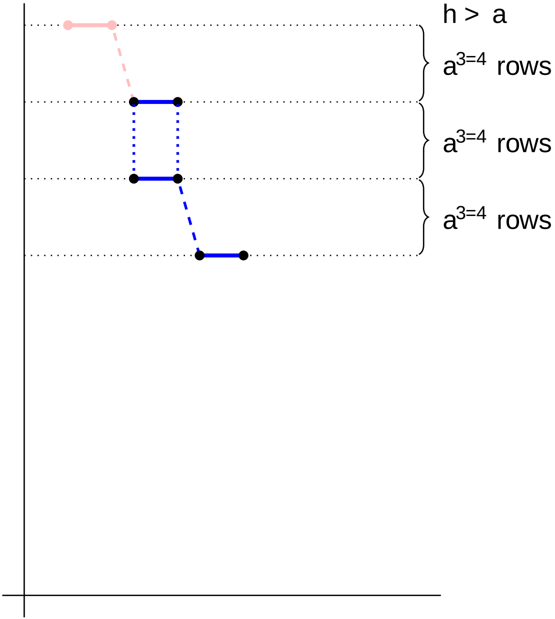

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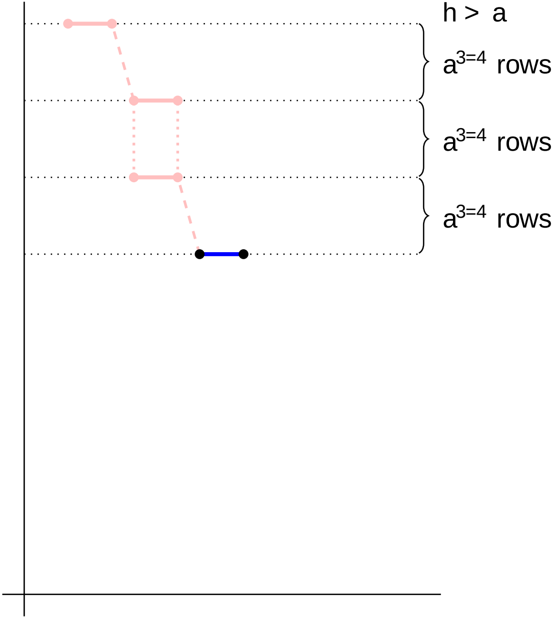

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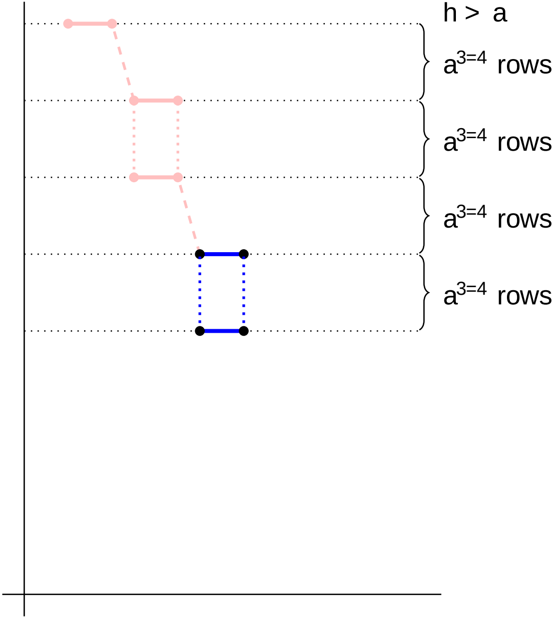

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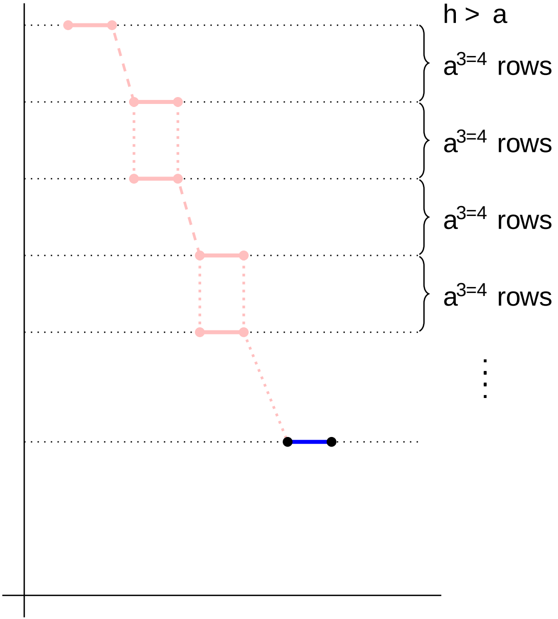

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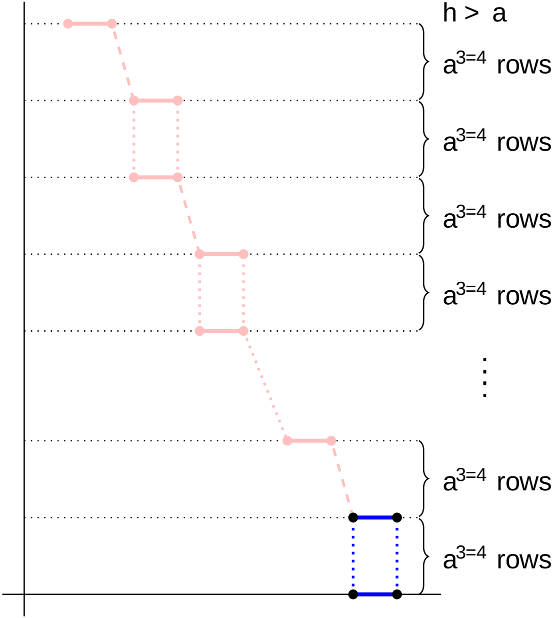

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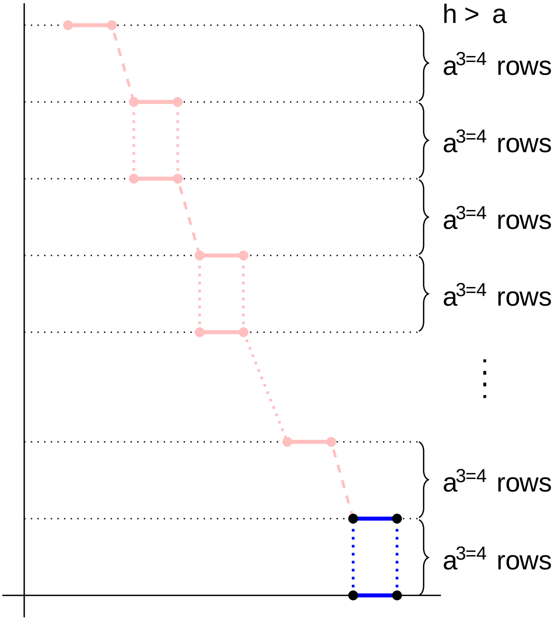
Repeat $a=a^{3=4} = a^{1=4}$ times to obtain a path of length $a^{1=4}\ \ a^{1=2} = a^{3=4}$:



Concluding remarks

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Proposition

Let G be an edge-ordered graph with average degree d, such that every set of at most εd vertices induces at most $(1/2 - \varepsilon)d$ edges. Then G has a monotone path of length εd.