Towards the Erdős-Gallai Cycle Decomposition Conjecture

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Institute for Advanced Study and Princeton University

joint work with Richard Montgomery

Graph decomposition problems

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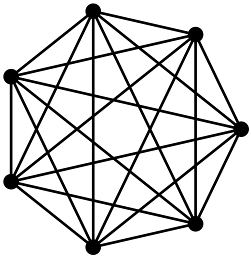

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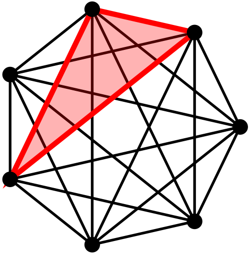

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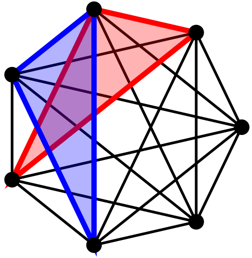

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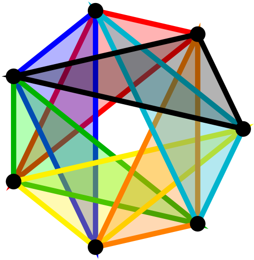

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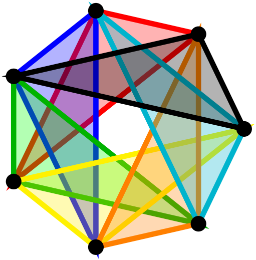

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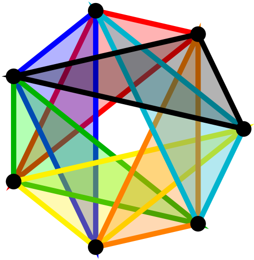

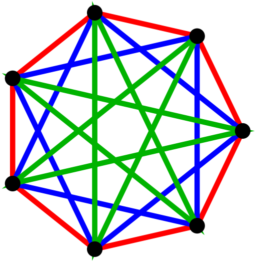

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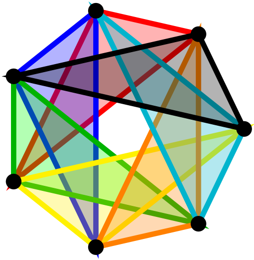

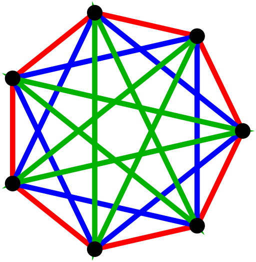

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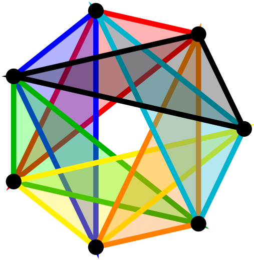

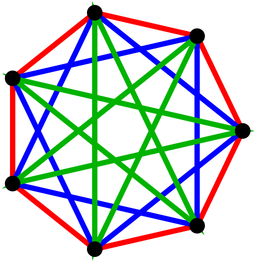

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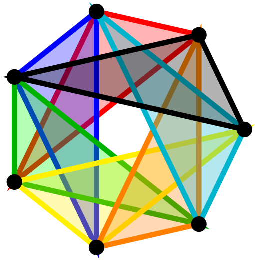

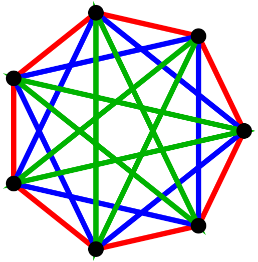

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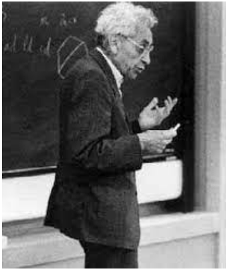
- Can we decompose any graph into cycles? No if $\exists$ an odd degree vertex.
- Veblen 1912: Any graph with all degrees even decomposes into cycles.

Erdős-Gallai Conjecture

Conjecture (Erdős-Gallai 1960s)

Every n-vertex graph can be decomposed into $O(n)$ cycles and edges.



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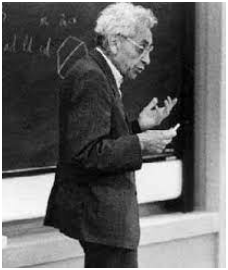

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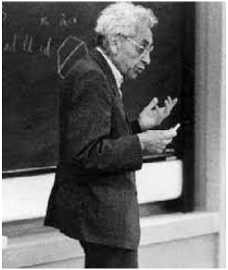

Conjecture (Hajós 1960s)

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- Pyber 1985: Precise solution for the covering version.

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  - Glock, Kühn and Osthus 2016: exact result.

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Theorem (B., Montgomery 2022+)

Any n-vertex graph can be decomposed into $O(n \log^{\star} n)$ cycles and edges.

Path decompositions with well-spread endvertices

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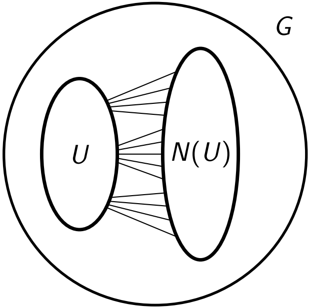
- n odd degree vertices $\implies$ it is a path decomposition and
- each original vertex is an endpoint of exactly one path.
- Removing v gives the desired decomposition.

Robust sublinear expansion

Definition (Expansion)

An n-vertex graph G is a λ-expander if for all $U \subseteq V(G)$ s.t. $|U| \leq \frac{n}{2\lambda}$ we have:

$$|N(U)| > \lambda |U|.$$



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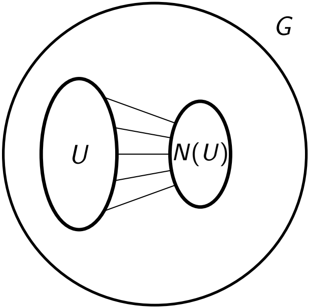

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An n-vertex graph G is a d-robust expander if $\forall U \subseteq V(G) : |U| \le \frac{n}{2}$ *we have*

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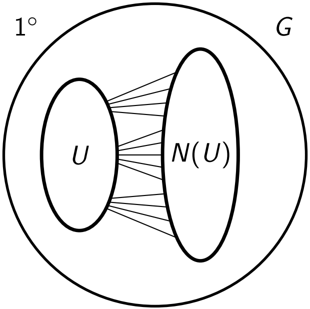

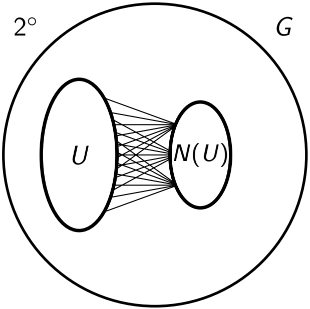

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    - such that $|H_1| + \ldots + |H_t| \leq 2n$

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    - Can be used to "parallellize" some of our arguments.

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  - Existence of an expanding "skeleton".

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  - A very strong connectivity property even under vertex subsampling
  - Existence of an expanding "skeleton".

Proof sketch.

Theorem (B., Montgomery 2022+)

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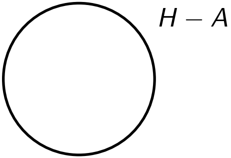

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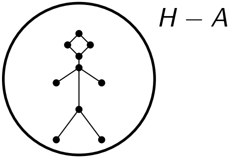

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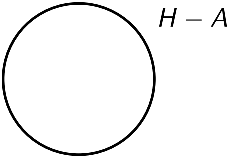

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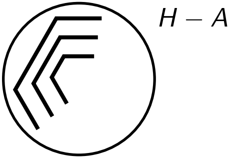

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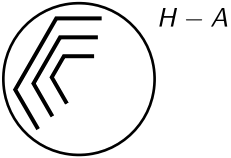

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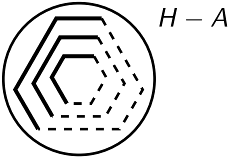

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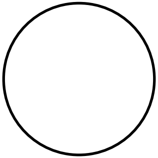

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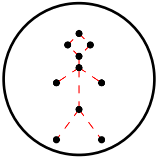

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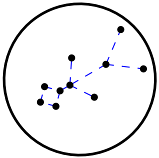

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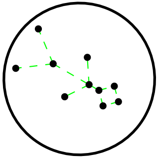

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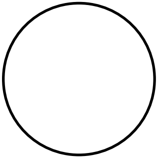

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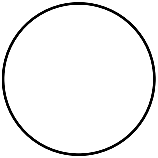

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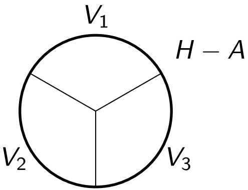

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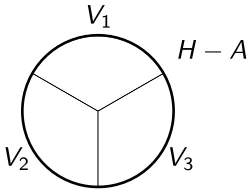

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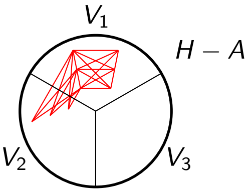

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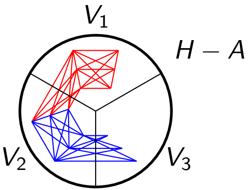

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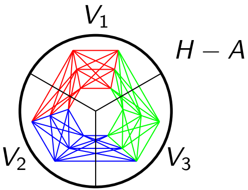

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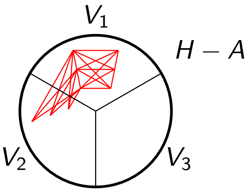

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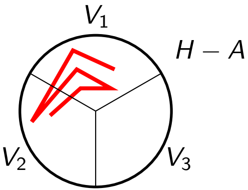

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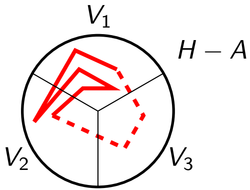

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Iteration.

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Theorem 1

Any n-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

Theorem 2

Any n-vertex graph with average degree d can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.

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Theorem 2

Any n-vertex graph with average degree d can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.

- Step 1: Remove any cycle of length $\geq d$.

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Any n-vertex graph can be decomposed into $6n$ *cycles and* $O(n \log^C n)$ *edges.*

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Any n-vertex graph with average degree d *can be decomposed into* $13n$ *cycles and* $O(n \log^C d)$ *edges.*

- Step 1: Remove any cycle of length $\geq d$.
  - removes at most n cycles.

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- Step 1: Remove any cycle of length $\geq d$.
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- Step 2: **Fully** decompose the remainder into sublinear expanders $H_1, \ldots, H_t$.

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  - $\forall i \ \ |H_i| \leq d \log^4 d$
  - Use Theorem 1 to decompose each H_i to get a decomposition into at most:

$$6|H_1| + \ldots + 6|H_t| \leq 12n \qquad \text{cycles and}$$

$$\sum O(|H_i| \log^C |H_i|) \leq \sum O(|H_i| \log^C d) \leq O(n \log^C d) \qquad \text{edges.}$$

Concluding remarks

Theorem

Any n-vertex graph of average degree d can be decomposed into $O(n)$ cycles and $O(n \log^{O(1)} d)$ edges.

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For any constant k any n-vertex graph of average degree d can be decomposed into $O(kn)$ cycles and $O(n \underbrace{\log \log \ldots \log}_{k} d)$ edges.

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Any n-vertex graph of average degree d can be decomposed into $O(n \log^{\star} d)$ cycles and edges.

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Any n-vertex graph of average degree d can be decomposed into $O(n \log^{\star} d)$ cycles and edges.

Conjecture (Erdős-Gallai, 1960s)

Every n-vertex graph can be decomposed into $O(n)$ cycles and edges.


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
спасибо
Баярлалаа
рахмат
obrigado
dziękuję
bedankt
merci
go raibh maith agat
mochchakkeram
sukriya
kop khun krap
terima kasih
grazie
arigatô
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תודה רבה
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ありがとう
hvala
mauruuru
sagolun
takk
taiku
faafetai lava
vinaka
спасибі
blagodaram
mersi
kia ora
barka
welalin
tack
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misaotra
matondo
paldies
grazzi
mahalo
хвала
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sobodi
dėkuji
mèsi
didi madloba
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rahmat
tanemirt
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xièxie
diolch
dhanyavadagalu
shukriya
mercé
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