Towards the Erdős-Gallai Cycle Decomposition Conjecture

Matija Bucić

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joint work with Richard Montgomery
General graph decomposition question:

Walecki 1883: $K_{2n+1}$ can be decomposed into $n$ cycles.

Veblen 1912: Any graph with all degrees even decomposes into cycles.
General graph decomposition question:

Can we decompose a graph into few graphs with some “nice” properties?

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Graph decomposition problems

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Conjecture (Erdős-Gallai 1960s)

Every \( n \)-vertex graph can be decomposed into \( O(n) \) cycles and edges.

▶ Any graph with all odd degrees needs at least \( \frac{n}{2} \) edges.
▶ Gallai 1966: one needs at least \((4/3 - o(1))n\) cycles and edges.
▶ Erdős 1983: one needs at least \((3/2 - o(1))n\) cycles and edges.

Lovász 1968: True for paths in place of cycles
Pyber 1985: Precise solution for the covering version.
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Conjecture

Every Eulerian $n$-vertex graph can be decomposed into at most $O(n)$ cycles.
**Erdős-Gallai Conjecture**

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*Every n-vertex graph can be decomposed into $O(n)$ cycles and edges.*

**Conjecture (Hajós 1960s)**

*Every Eulerian n-vertex graph can be decomposed into at most $\lfloor n/2 \rfloor$ cycles.*

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- True for graphs with linear minimum degree.
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- Folklore: $O(n \log n)$ cycles and edges always suffice.
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Theorem (B., Montgomery 2022+)

Any $n$-vertex graph can be decomposed into $O(n \log^* n)$ cycles and edges.
Theorem (Lovász, 1968)

Every \( n \)-vertex graph can be decomposed into at most \( \frac{n}{2} \) paths and cycles.

Proof:

1. Add an auxiliary vertex \( v \) and join it to all even degree vertices.
2. Lovász' Theorem gives a decomposition into paths and cycles.
3. \( n \) odd degree vertices \( \Rightarrow \) it is a path decomposition and
4. each original vertex is an endpoint of exactly one path.
5. Removing \( v \) gives the desired decomposition.
Theorem (Lovász, 1968)

Every $n$-vertex graph can be decomposed into at most $\frac{n}{2}$ paths and cycles.

Corollary

Every $n$-vertex graph can be decomposed into at most $n$ paths in such a way that no vertex is an endpoint of more than two paths.
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- each original vertex is an endpoint of exactly one path.
- Removing $v$ gives the desired decomposition.
Robust sublinear expansion

**Definition (Expansion)**

An $n$-vertex graph $G$ is a $\lambda$-expander if for all $U \subseteq V(G)$ s.t. $|U| \leq \frac{n}{2\lambda}$ we have:

$$|N(U)| > \lambda |U|.$$
Definition (Sublinear expansion)

An $n$-vertex graph $G$ is a sublinear expander if $\forall U \subseteq V(G)$ s.t. $|U| \leq \frac{n}{2}$ we have:

$$|N(U)| > \frac{1}{\log^2 n} \cdot |U|$$
Definition (Robust sublinear expansion)

An \( n \)-vertex graph \( G \) is a \( d \)-robust expander if \( \forall U \subseteq V(G) : |U| \leq \frac{n}{2} \) we have

\[
1^\circ : \quad |N(U)| > d \cdot |U| \quad \text{or} \quad 2^\circ : \quad N(U) \text{ has } > \frac{|U|}{\log^2 n} \text{ vtcs with } \geq d \log^2 n \text{ neighbours in } U.
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We prove a number of properties of this weaker notion of expander graphs:
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We prove a number of properties of this weaker notion of expander graphs:

- An (almost) partitioning of an arbitrary graph into expanders lemma.
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**Definition (Robust sublinear expansion)**

An $n$-vertex graph $G$ is a $d$-robust expander if $\forall U \subseteq V(G) : |U| \leq \frac{n}{2}$ we have

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We prove a number of properties of this weaker notion of expander graphs:

- An (almost) partitioning of an arbitrary graph into expanders lemma.
  - partition all but $n \log^C n$ edges into expanders $H_1, \ldots, H_t$
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  - such that $|H_1| + \ldots + |H_t| \leq 2n$
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We prove a number of properties of this weaker notion of expander graphs:

- An (almost) partitioning of an arbitrary graph into expanders lemma.
- An edge-subsampling lemma.
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- We prove a number of properties of this weaker notion of expander graphs:
  - An (almost) partitioning of an arbitrary graph into expanders lemma.
  - An edge-subsampling lemma.
    - Can be used to “parallelize” some of our arguments.
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  - A very strong connectivity property

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    - We can join any collection of disjoint pairs of vtcs with short edge disjoint paths

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  - Also true if we insist paths go “through” a random subset of vtcs
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- Existence of an expanding “skeleton”.
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- We prove a number of properties of this weaker notion of expander graphs:
  - An (almost) partitioning of an arbitrary graph into expanders lemma.
  - An edge-subsampling lemma.
  - A very strong connectivity property even under vertex subsampling.
  - Existence of an expanding “skeleton”.

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Theorem (B., Montgomery 2022+)

Any $n$-vertex graph $H$ can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

Proof sketch.

Step 1: Set aside an expanding skeleton $A$ of $H$.

Step 2: Decompose $H-A$ into paths using the corollary to Lovász' theorem.

Step 3: Use short paths in $A$ to close the paths into cycles.

Step 4: Split $A$ into expanding skeletons $A_1, A_2, A_3$.

Partition $V(H)$ into $V_1, V_2, V_3$.

Decompose $H-A$ into $H_1, H_2, H_3$. 

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Proof:

- Step 1: Reduce the proof to the case of $H$ being a robust expander by
  - Decomposing all but $O(n \log^C n)$ edges into expanders $H_1, \ldots, H_t$ s.t.
    
    $$|H_1| + \ldots + |H_t| \leq 2n$$

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    \[ |H_1| + \ldots + |H_t| \leq 2n \]
  - Apply the theorem to each $H_i$ to get a decomposition into
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  - Decomposing all but $O(n \log^C n)$ edges into expanders $H_1, \ldots, H_t$ s.t.
    $$|H_1| + \ldots + |H_t| \leq 2n$$
  - Apply the theorem to each $H_i$ to get a decomposition into
    $$3|H_1| + \ldots + 3|H_t| \leq 6n$$
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**Theorem (B., Montgomery 2022+)**

*Any $n$-vertex graph $H$ can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.*

**Proof:**

- **Step 1:** Reduce the proof to the case of $H$ being a robust expander by
  - Decomposing all but $O(n \log^C n)$ edges into expanders $H_1, \ldots, H_t$ s.t.
    
    \[ |H_1| + \ldots + |H_t| \leq 2n \]

  - Apply the theorem to each $H_i$ to get a decomposition into
    
    \[ 3|H_1| + \ldots + 3|H_t| \leq 6n \quad \text{cycles and} \]
    
    \[ O(|H_1| \log^C |H_1|) + \ldots + O(|H_t| \log^C |H_t|) \leq O(n \log^C n) \quad \text{edges.} \]
Proof sketch.

Theorem (B., Montgomery 2022+)

Any $n$-vertex graph $H$ can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

Proof:

1. Step 1: Assume $H$ is a robust sublinear expander.
Proof sketch.

**Theorem (B., Montgomery 2022+)**

*Any n-vertex graph $H$ can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.*

**Proof:**

- **Step 1:** Assume $H$ is a robust sublinear expander.
- **Step 2:** Set aside an expanding skeleton $A$ of $H$.

![Graph $H - A$]
Proof sketch.

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- Step 1: Assume $H$ is a robust sublinear expander.
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Split \( A \) into expanding skeletons \( A_1, A_2, A_3 \)
Proof sketch.

**Theorem (B., Montgomery 2022+)**

*Any* $n$-*vertex graph* $H$ *can be decomposed into* $6n$ *cycles and* $O(n \log^C n)$ *edges.*

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H - A

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Proof sketch.

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- Split $A$ into expanding skeletons $A_1, A_2, A_3$
- Partition $V(H)$ into $V_1, V_2, V_3$ u.a.r.
Proof sketch.

**Theorem (B., Montgomery 2022+)**

*Any* $n$-vertex graph $H$ *can be decomposed into* $6n$ *cycles and* $O(n \log^C n)$ *edges.*

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Partition $V(H)$ into $V_1, V_2, V_3$ u.a.r.

Decompose $H - A$ into $H_1, H_2, H_3$. 
Proof sketch.

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Partition $V(H)$ into $V_1, V_2, V_3$ u.a.r.

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\[
\begin{array}{c}
V_1 \\
H - A \\
V_2 \\
V_3
\end{array}
\]

- Split $A$ into expanding skeletons $A_1, A_2, A_3$
- Partition $V(H)$ into $V_1, V_2, V_3$ u.a.r.
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Theorem (B., Montgomery 2022+)

Any \( n \)-vertex graph \( H \) can be decomposed into \( 6n \) cycles and \( O(n \log^C n) \) edges.

Proof:

- Step 1: Assume \( H \) is a robust sublinear expander.
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- Step 3: Decompose \( H - A \) into paths using the corollary to Lovász’ theorem.
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- Split \( A \) into expanding skeletons \( A_1, A_2, A_3 \)
- Partition \( V(H) \) into \( V_1, V_2, V_3 \) u.a.r.
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Theorem 1

Any $n$-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.
**Theorem 1**

Any $n$-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

**Theorem 2**

Any $n$-vertex graph with average degree $d$ can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.
Theorem 1

Any $n$-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

Theorem 2

Any $n$-vertex graph with average degree $d$ can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.

- Step 1: Remove any cycle of length $\geq d$. 
Theorem 1

Any $n$-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

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Any $n$-vertex graph with average degree $d$ can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.

- Step 1: Remove any cycle of length $\geq d$.
  - removes at most $n$ cycles.
Theorem 1

Any $n$-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

Theorem 2

Any $n$-vertex graph with average degree $d$ can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.

- Step 1: Remove any cycle of length $\geq d$.
  - removes at most $n$ cycles.
- Step 2: **Fully** decompose the remainder into sublinear expanders $H_1, \ldots, H_t$. 
Theorem 1

Any $n$-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

Theorem 2

Any $n$-vertex graph with average degree $d$ can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.

- **Step 1:** Remove any cycle of length $\geq d$.
  - removes at most $n$ cycles.
- **Step 2:** **Fully** decompose the remainder into sublinear expanders $H_1, \ldots, H_t$.
  - $|H_1| + \ldots + |H_t| \leq 2n$
Theorem 1

Any $n$-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

Theorem 2

Any $n$-vertex graph with average degree $d$ can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.

- **Step 1:** Remove any cycle of length $\geq d$.
  - Removes at most $n$ cycles.

- **Step 2:** **Fully** decompose the remainder into sublinear expanders $H_1, \ldots, H_t$.
  - $|H_1| + \ldots + |H_t| \leq 2n$
  - $\forall i \ |H_i| \leq d \log^4 d$
Iteration.

**Theorem 1**

Any $n$-vertex graph can be decomposed into $6n$ cycles and $O(n \log^C n)$ edges.

**Theorem 2**

Any $n$-vertex graph with average degree $d$ can be decomposed into $13n$ cycles and $O(n \log^C d)$ edges.

- **Step 1:** Remove any cycle of length $\geq d$.
  - Removes at most $n$ cycles.
- **Step 2:** **Fully** decompose the remainder into sublinear expanders $H_1, \ldots, H_t$.
  - $|H_1| + \ldots + |H_t| \leq 2n$
  - $\forall i \ |H_i| \leq d \log^4 d$
  - Use Theorem 1 to decompose each $H_i$ to get a decomposition into at most:
    
    $6|H_1| + \ldots + 6|H_t| \leq 12n$ cycles and
    
    $\sum O(|H_i| \log^C |H_i|) \leq \sum O(|H_i| \log^C d) \leq O(n \log^C d)$ edges.
Concluding remarks

**Theorem**

Any $n$-vertex graph of average degree $d$ can be decomposed into $O(n)$ cycles and $O(n \log^{O(1)} d)$ edges.

**Conjecture (Erdős-Gallai, 1960s)**

Every $n$-vertex graph can be decomposed into $O(n)$ cycles and edges.
Concluding remarks

Theorem

Any \( n \)-vertex graph of average degree \( d \) can be decomposed into \( O(n) \) cycles and \( O(n \log^{O(1)} d) \) edges.

Theorem

For any constant \( k \) any \( n \)-vertex graph of average degree \( d \) can be decomposed into \( O(kn) \) cycles and \( O(n \log \log \ldots \log d) \) edges. 

\( k \)
Theorem
Any $n$-vertex graph of average degree $d$ can be decomposed into $O(n)$ cycles and $O(n \log^{O(1)} d)$ edges.

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Theorem
Any $n$-vertex graph of average degree $d$ can be decomposed into $O(n \log^* d)$ cycles and edges.
Concluding remarks

**Theorem**

Any $n$-vertex graph of average degree $d$ can be decomposed into $O(n)$ cycles and $O(n \log^{O(1)} d)$ edges.

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For any constant $k$ any $n$-vertex graph of average degree $d$ can be decomposed into $O(kn)$ cycles and $O(n \log \log \ldots \log d)$ edges.

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Any $n$-vertex graph of average degree $d$ can be decomposed into $O(n \log^* d)$ cycles and edges.

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Thank you