# Covering random graphs by monochromatic trees

Matija Bucić

based on joint works with Daniel Korándi and Benny Sudakov and Domagoj Bradač

Institute for Advanced Study and Princeton University

## Definition

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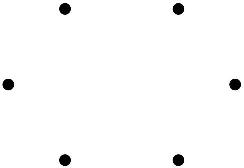

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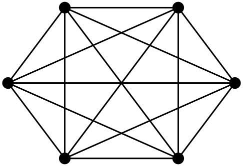

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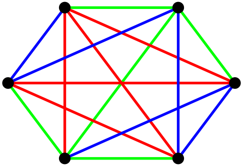

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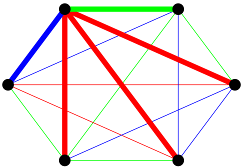

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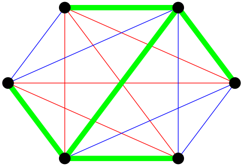

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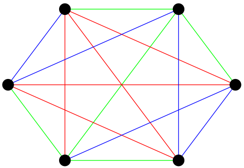

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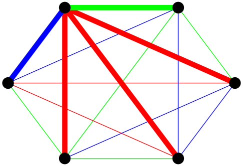

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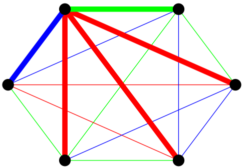

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- Related results by:
  - Aharoni;
  - Fujita, Furuya, Gyárfás, Tóth;
  - Gyárfás;
  - Gerencsér and Gyárfás;
  - Pokrovskiy;
  - Haxell and Kohayakawa;
  - Gyárfás, Ruszinkó, Sárközy, Szemerédi;
  - Erdős, Gyárfás and Pyber.

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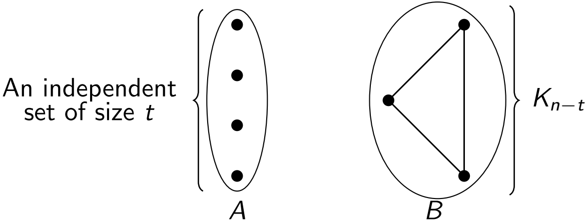

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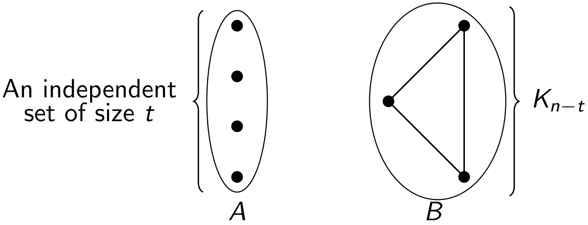


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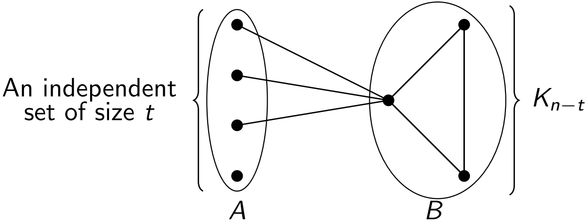


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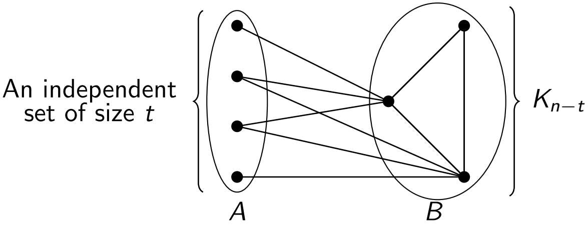


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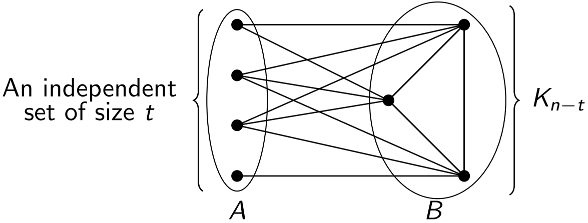


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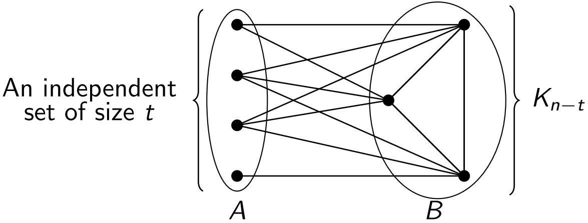


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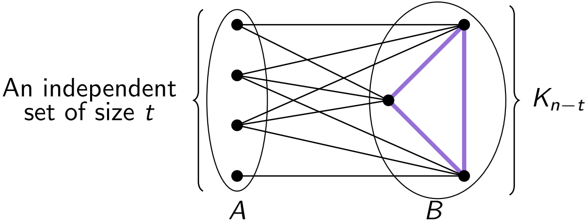


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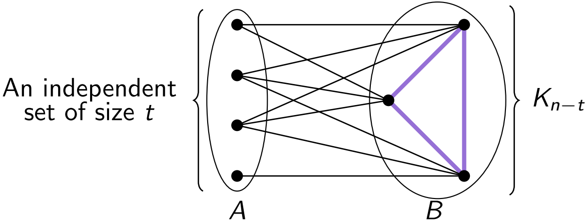


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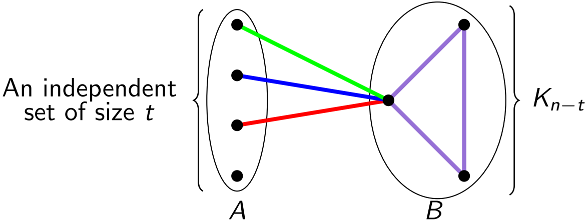


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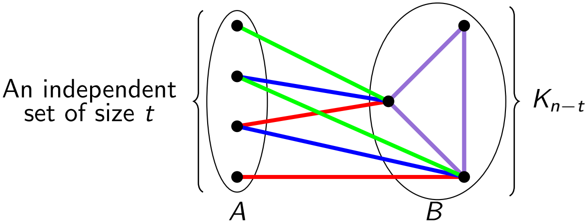


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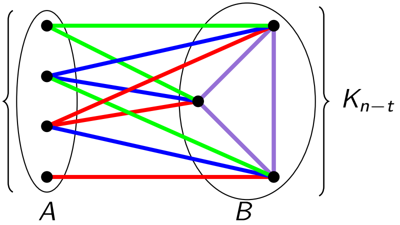


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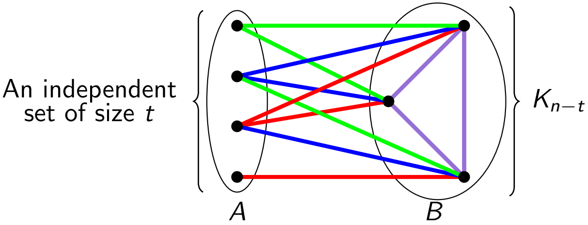


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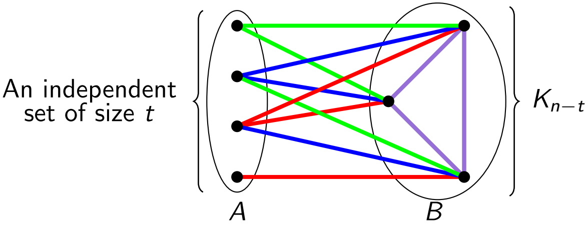


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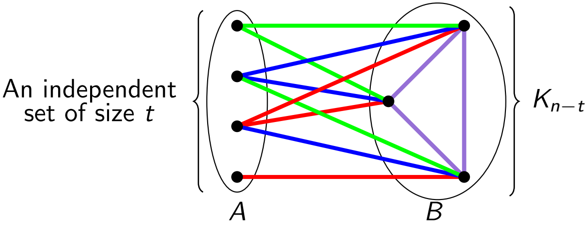


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*How big is it close to the threshold?*

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## Theorem (B., Korándi, Sudakov 2021)

*If* $\left(\frac{\log n}{n}\right)^{1/r} \ll p$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \le 3r^2$.

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*When does* $\mathrm{tc}_r(\mathcal{G}(n,p))$ *become bounded by a function of* $r$*?*

$$\begin{aligned} p \ll \left(\frac{\log n}{n}\right)^{1/r} \quad &\implies \quad \mathrm{tc}_r(\mathcal{G}(n,p)) \to \infty \\ \left(\frac{\log n}{n}\right)^{1/r} \ll p \ll \left(\frac{\log n}{n}\right)^{1/(r+1)} \quad &\implies \quad r+1 \leq \mathrm{tc}_r(\mathcal{G}(n,p)) \leq 3r^2 \\ \left(\frac{\log n}{n}\right)^{1/(r+1)} \ll p \quad &\implies \quad \mathrm{tc}_r(\mathcal{G}(n,p)) \leq r^2 \end{aligned}$$

## Theorem (B., Korándi, Sudakov 2021)

*If* $\left(\frac{\log n}{n}\right)^{1/r} \ll p$ *then w.h.p.* $\mathrm{tc}_r(\mathcal{G}(n,p)) \leq 3r^2$.

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- Answers a question of Lang and Lo.

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*If* $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ *then w.h.p.* $\frac{r^2}{20\log k} \leq \mathrm{tc}_r(\mathcal{G}(n,p)) \leq \frac{16r^2\log r}{\log k}$

# Local to global covering problem for hypergraphs

**Question (Erdős, Hajnal and Tuza '90)**

*Given an $r$-uniform hypergraph $H$ in which any $k$ edges have a cover of size at most $\ell$, how big can a minimum cover of $H$ be?*

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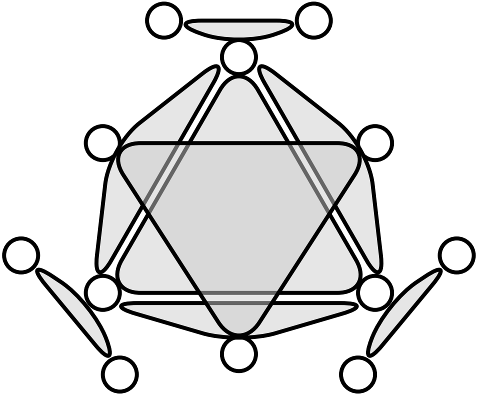

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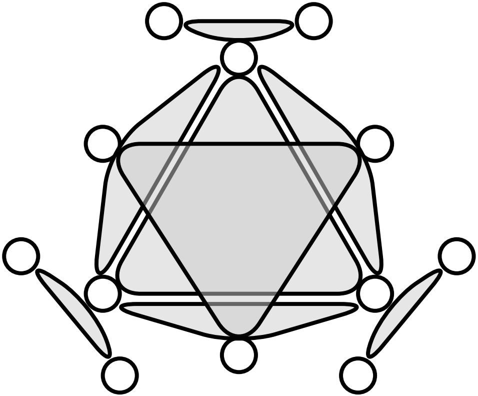

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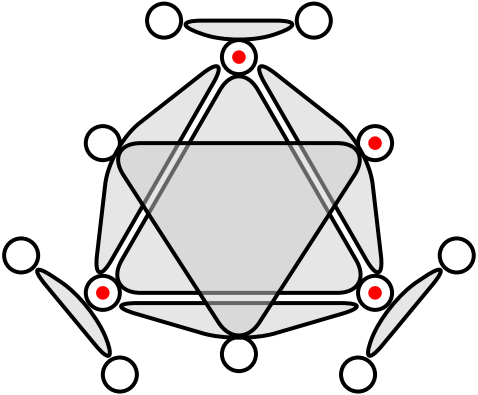

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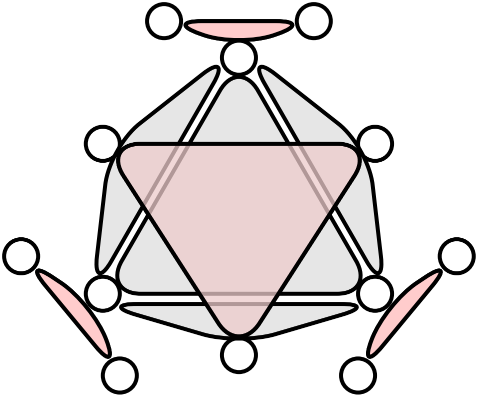

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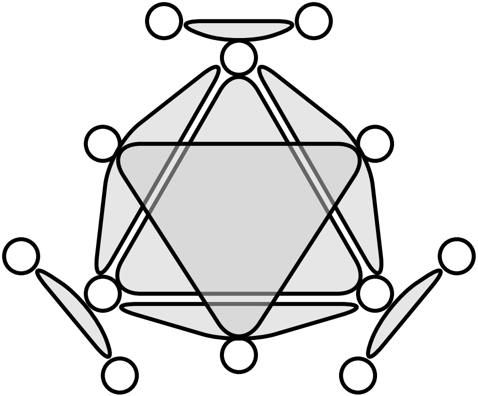

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- Any 3 edges have a cover of size 3

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- Results by:
  - Erdős, Hajnal and Tuza (1990);
  - Erdős, Fon-Der-Flaass, Kostochka and Tuza (1991);
  - Fon-Der-Flaass, Kostochka and Woodall (1999);
  - Kostochka (2001).

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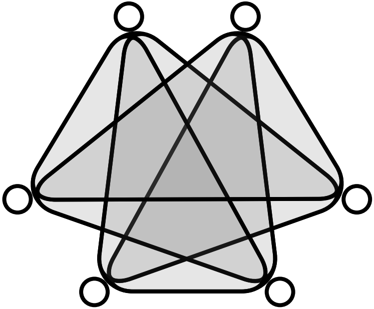

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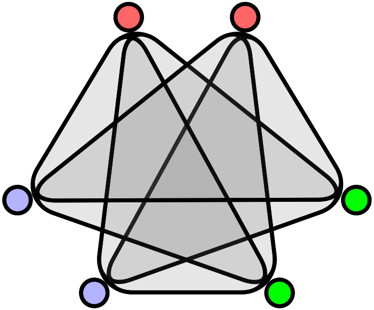

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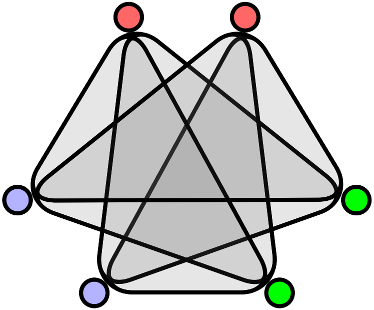

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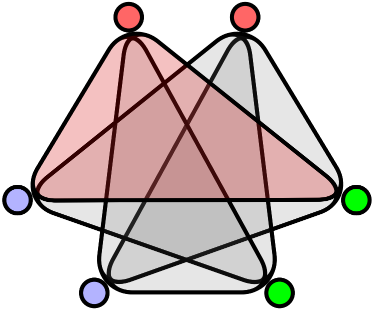

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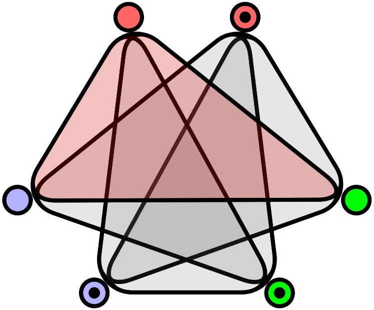

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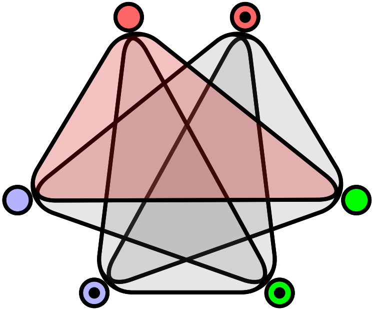

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# General connection

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*If any $k$ vertices in $G$ have a common neighbour then* $\mathrm{tc}_r(G) \leq \mathrm{hp}_r(k)$.

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$\square$

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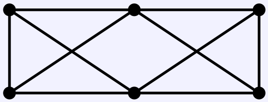

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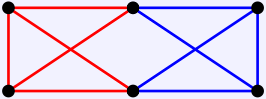

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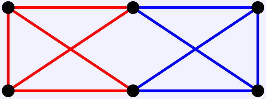

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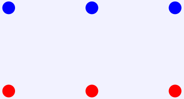

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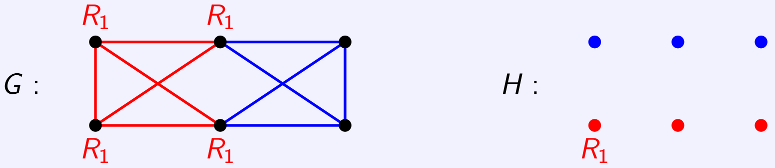


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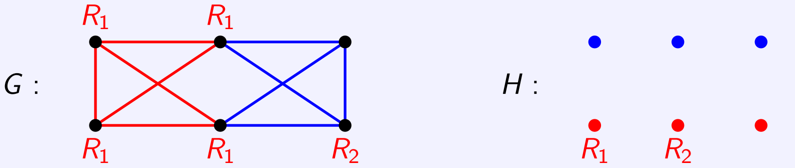


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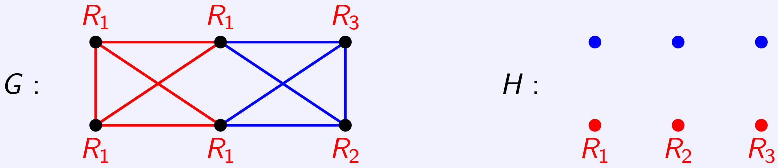


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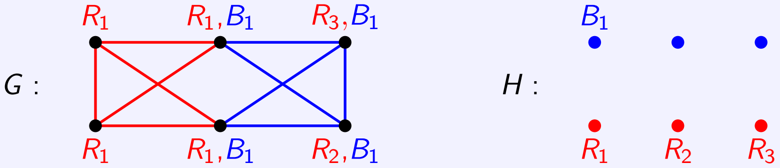


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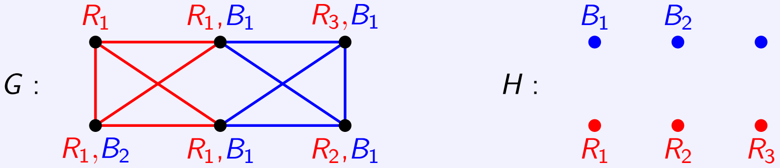


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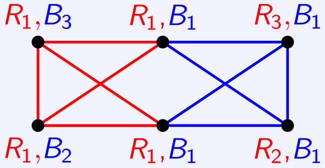


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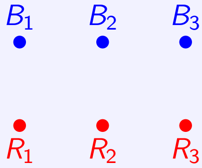


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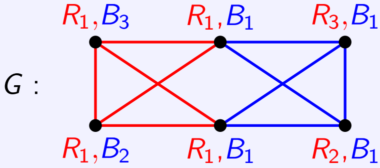


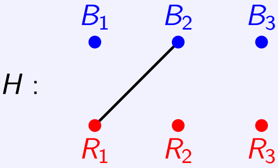


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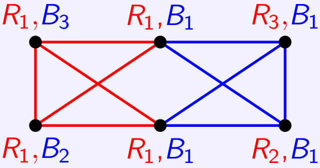


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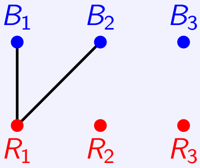


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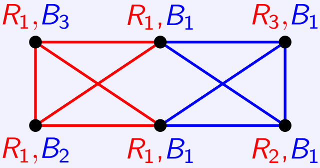


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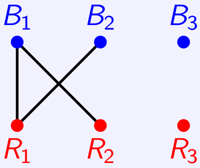


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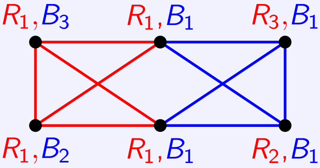


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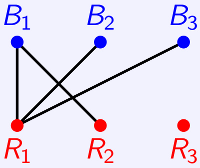


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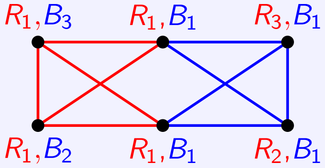


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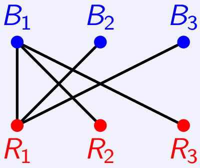


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$$\mathrm{tc}_r(G) \leq \tau(H)$$

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- Given $k$ edges $e_{v_1}, \ldots, e_{v_k}$, let $w$ be a common neighbour of $v_1, \ldots, v_k$.
- $e_w \cap e_{v_i} \neq \emptyset$ so $e_w$ is a transversal cover of $e_{v_1}, \ldots, e_{v_k}$ so $\tau(H) \leq \mathrm{hp}_r(k)$.
- If monochromatic components $C_1, \ldots, C_t$ cover $H$ then $C_1 \cup \cdots \cup C_t = V(G)$
  - For any $v \in V(G)$ $\exists i : C_i \in e_v \implies v \in C_i$.

□

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  $\text{tc}_2(\mathcal{G}(n,p)) = \infty$ if $p \ll \left(\frac{\log n}{n}\right)^{1/2}$ and
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**Theorem (Bradač, B. )**

| $p$ | $\left[0, \left(\frac{\log n}{n}\right)^{1/3}\right)$ | $\left(\left(\frac{\log n}{n}\right)^{1/3}, p^*\right)$ | $\left(p^*, \left(\frac{\log n}{n}\right)^{1/4}\right)$ | $\left(\left(\frac{\log n}{n}\right)^{1/4}, 1-o(1)\right)$ | $(1-o(1), 1]$ |
|---|---|---|---|---|---|
| $\mathrm{tc}_3(\mathcal{G}(n,p))$ | $\infty$ | 5 | 4 | 3 | 2 |

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thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
dziękuję
bedankt
go raibh maith agat
mochchakkeram
sukriya
kop khun krap
terima kasih
grazie
arigatô
dakujem
takk
tapadh leat
спасибо
Баярлалаа
рахмат
maururu
hvala
sagolun
taiku
dank je
mersi
barka
welalin
tack
spas
kia ora
vinaka
спасибі
blagodaram
misaotra
matondo
paldies
grazzi
mahalo
хвала
asante
manana
obrigada
tenki
murakoze
mamnun
chokrane
дякую
tau
djiere dieuf
sulpáy
chnorakaloutioun
gratias ago
gràcies
akun
dankon
aciu
köszönöm
grácie
bayarlalaa
dhanyavad
nandri
nanni
kiitos
dankie
faafetai lava
enkosi
sobodi
dėkuji
mèsi
didi madloba
najis tuke
kam sah hamnida
rahmat
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