Covering graphs by monochromatic trees and Helly-type results in hypergraphs

Matija Bucić
joint work with Daniel Korándi and Benny Sudakov

ETH Zürich

Covering by trees

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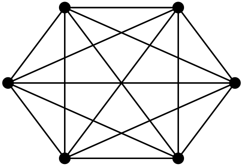

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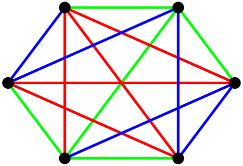

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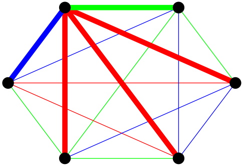

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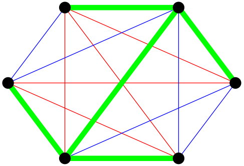

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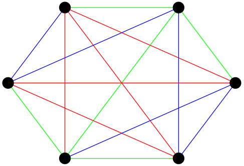

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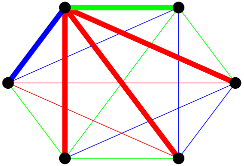

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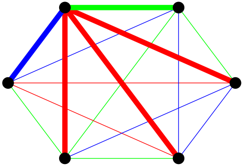

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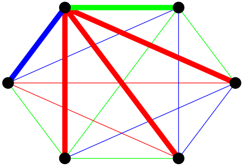

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- For specific trees: Aharoni; Fujita, Furuya, Gyárfás, and Tóth, Gyárfás; Gerencsér and Gyárfás; Pokrovskiy; Haxell and Kohayakawa; Gyárfás, Ruszinkó, Sárközy and Szemerédi; Erdős, Gyárfás and Pyber.

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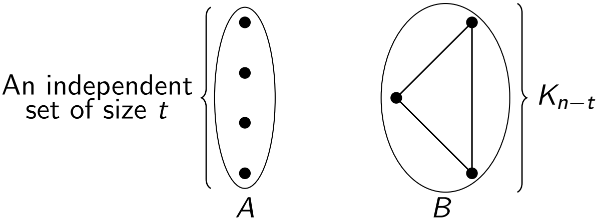

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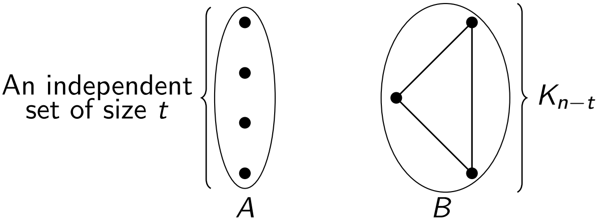

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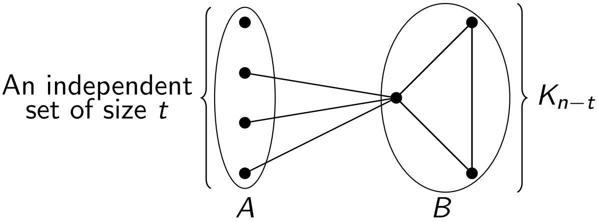

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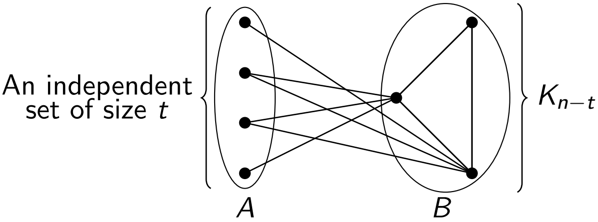

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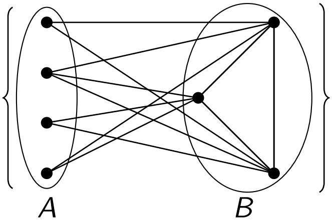

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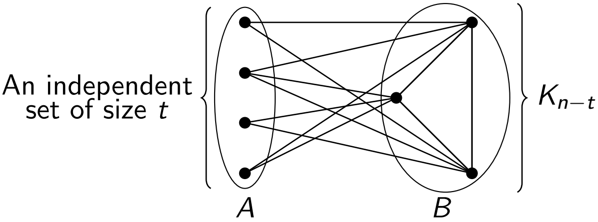

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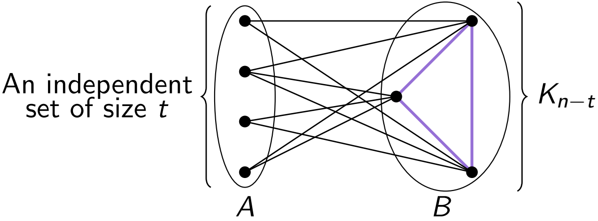

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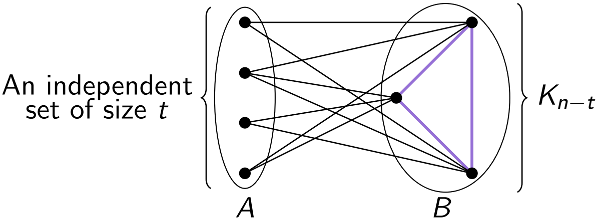

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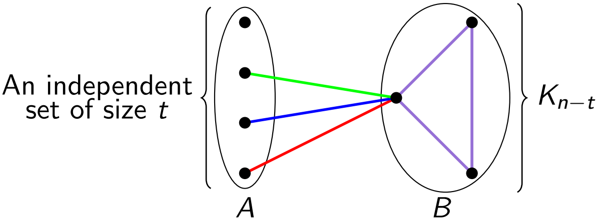

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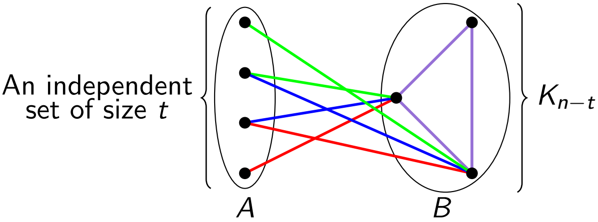

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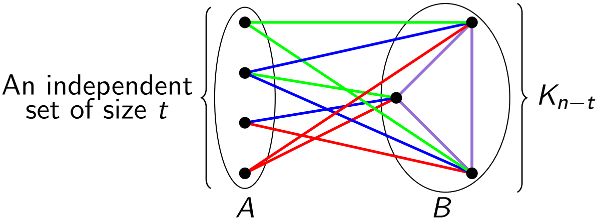

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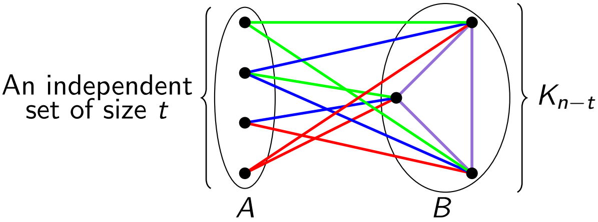

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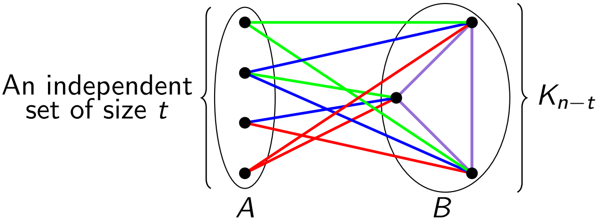

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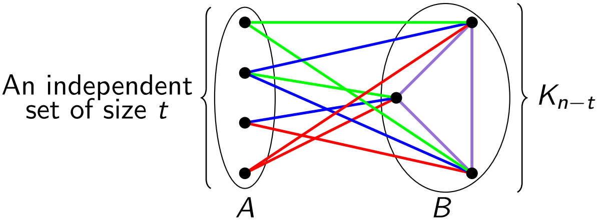

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If $\left(\frac{\log n}{n}\right)^{\frac{1}{k}} \ll p \ll \left(\frac{\log n}{n}\right)^{\frac{1}{k+1}}$ *then w.h.p.* $\frac{r^2}{20\log k} \leq \mathrm{tc}_r(\mathcal{G}(n,p)) \leq \frac{16r^2\log r}{\log k}$

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Definition

Let $\mathrm{hp}_r(k)$ *be the maximum possible size of a minimum cover of an r-partite, r-uniform H in which any k edges have a transversal cover.*

The connection in the random case

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Graphs with high minimum degree

Conjecture (Bal and DeBiasio)

Let G be an r-coloured graph on n vertices with $\delta(G) \geq (1 - \frac{1}{2^r})n$. Then vertices of G can be covered by monochromatic trees of distinct colours.

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Concluding remarks and open problems

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Is the $\log r$ *necessary? Same question without* r*-partiteness condition.*


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
спасибо
dziękuję
bedankt
sukriya
kop khun krap
go raibh maith agat
mochchakkeram
grazie
arigatô
dakujem
terima kasih
tapadh leat
dank je
hvala
mauruuru
sagolun
takk
taiku
мерси
감사합니다
תודה רבה
ευχαριστώ
ありがとう
рахмат
Баярлалаа
faafetai lava
kiitos
dankie
nanni
nandri
dhanyavad
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gràcie
köszönöm
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sobodi
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спасибі
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misaotra
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asante
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tenki
murakoze
mamnun
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djiere dieuf
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akun dankon
aciu
chnorakaloutioun
gratias ago
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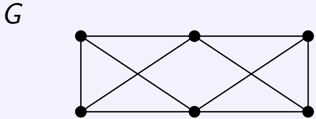

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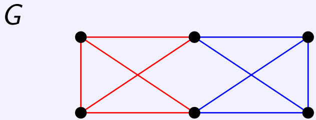

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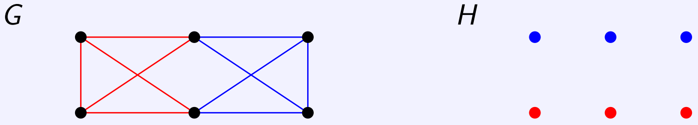

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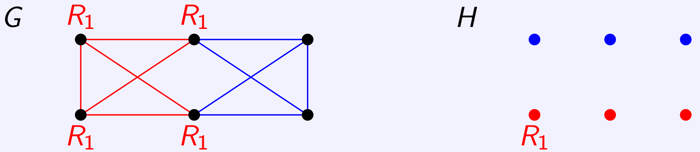

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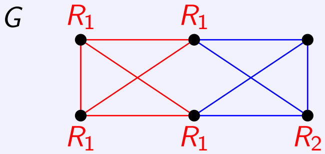

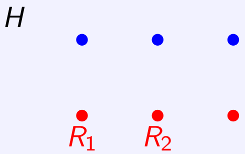

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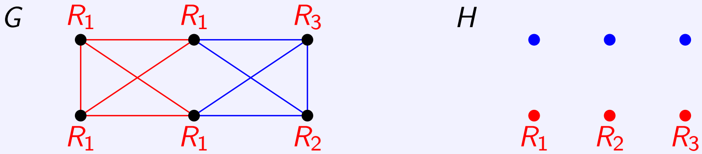

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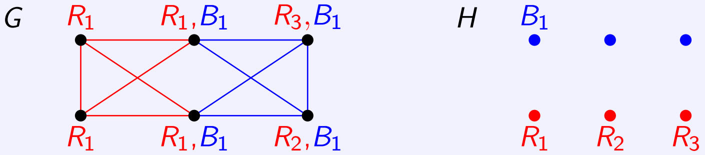

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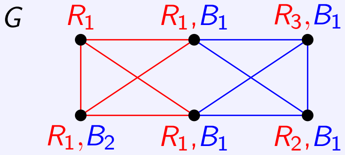

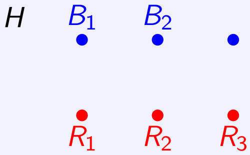

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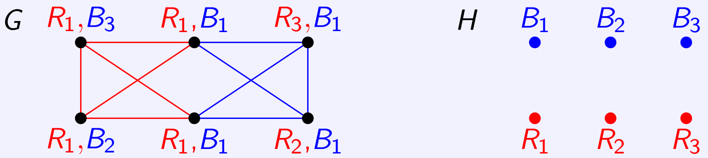

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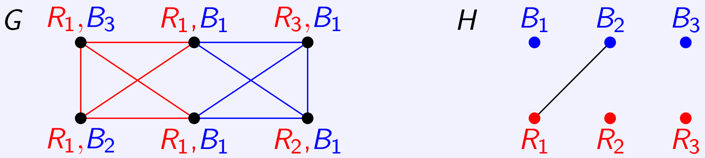

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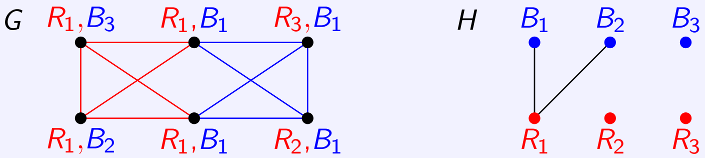

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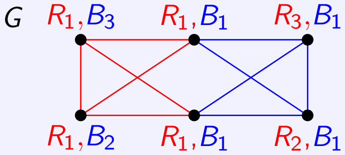

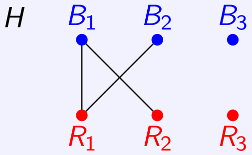

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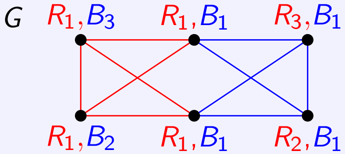

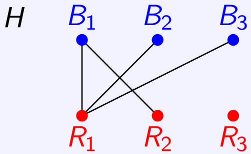

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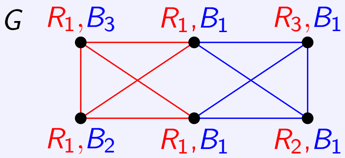

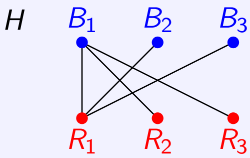

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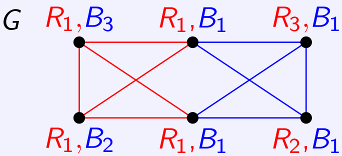

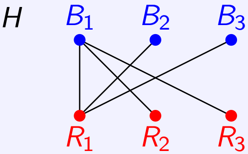

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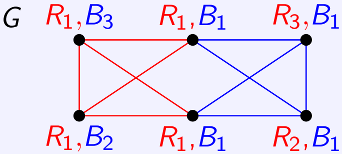

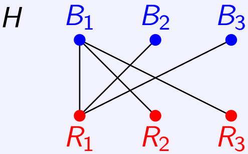

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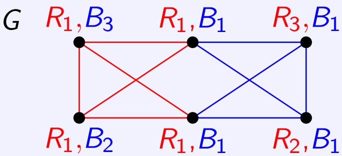

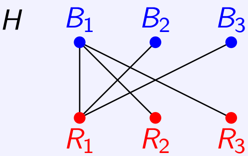

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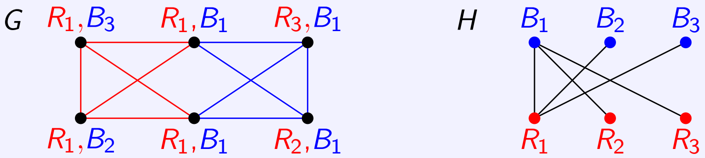

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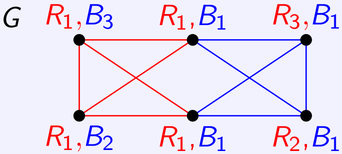

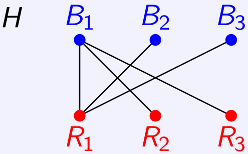

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