

Turán numbers of sunflowers

Matija Bucić

Institute for Advanced Study and Princeton University

joint work with Domagoj Bradač and Benny Sudakov

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$

Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

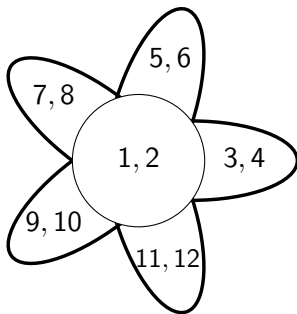
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

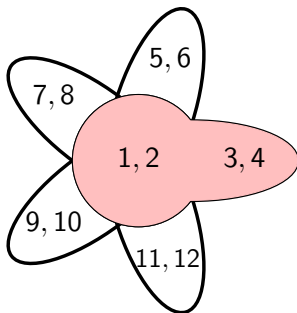
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

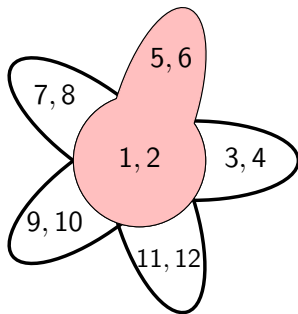
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

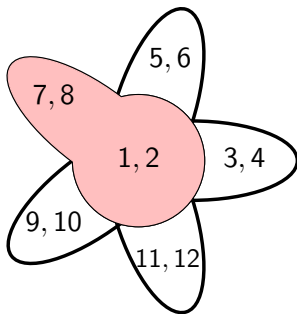
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

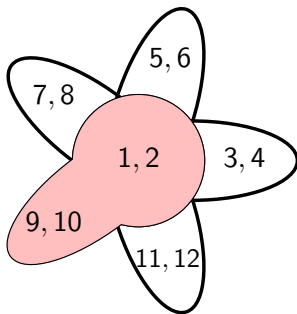
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

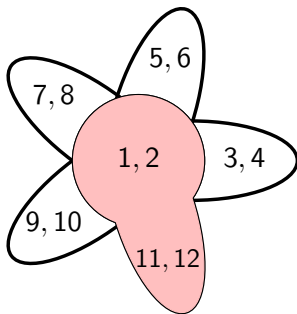
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

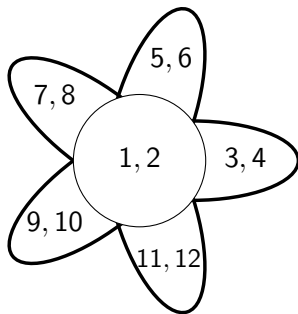
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



- The common intersection is the *kernel* of the sunflower.

Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

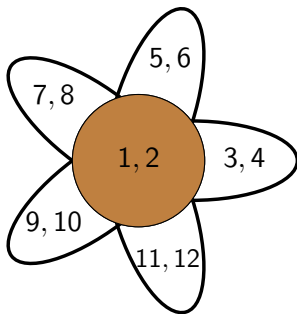
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



- The common intersection is the *kernel* of the sunflower.

Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

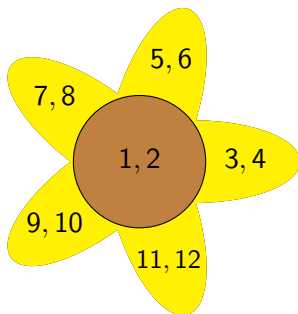
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



- The common intersection is the *kernel* of the sunflower.

Sunflowers

Definition

A collection of distinct sets is called a sunflower if the intersection of any pair of sets equals the common intersection of all the sets

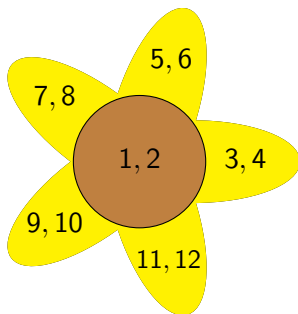
$\{1, 2, 3, 4\}$

$\{1, 2, 5, 6\}$

$\{1, 2, 7, 8\}$

$\{1, 2, 9, 10\}$

$\{1, 2, 11, 12\}$



- The common intersection is the *kernel* of the sunflower.
- *r*-uniform if all sets have size *r*.

Question (Erdős-Rado, 1960)

What is the max size of a family of r -sets without a k petal sunflower?

Question (Erdős-Rado, 1960)

What is the max size of a family of r -sets without a k petal sunflower?

- Denote the answer by $f_r(k)$.

Question (Erdős-Rado, 1960)

What is the max size of a family of r -sets without a k petal sunflower?

- Denote the answer by $f_r(k)$.
- Erdős-Rado sunflower lemma:

$$f_r(k) \leq (k-1)^r \cdot r!.$$

Question (Erdős-Rado, 1960)

What is the max size of a family of r -sets without a k petal sunflower?

- Denote the answer by $f_r(k)$.
- Erdős-Rado sunflower lemma: $(k - 1)^r \leq f_r(k) \leq (k - 1)^r \cdot r!$.

Question (Erdős-Rado, 1960)

What is the max size of a family of r -sets without a k petal sunflower?

- Denote the answer by $f_r(k)$.
- Erdős-Rado sunflower lemma: $(k - 1)^r \leq f_r(k) \leq (k - 1)^r \cdot r!$.
- Best known upper bound is: $f_r(k) \leq O(k \log r)^r$.

Erdős-Rado sunflower conjecture

Question (Erdős-Rado, 1960)

What is the max size of a family of r -sets without a k petal sunflower?

- Denote the answer by $f_r(k)$.
- Erdős-Rado sunflower lemma: $(k-1)^r \leq f_r(k) \leq (k-1)^r \cdot r!$.
- Best known upper bound is: $f_r(k) \leq O(k \log r)^r$.

Conjecture (Sunflower conjecture, Erdős-Rado, 1960)

$$f_r(k) \leq O(k)^r$$

Erdős-Rado sunflower conjecture

Question (Erdős-Rado, 1960)

What is the max size of a family of r -sets without a k petal sunflower?

- Denote the answer by $f_r(k)$.
- Erdős-Rado sunflower lemma: $(k-1)^r \leq f_r(k) \leq (k-1)^r \cdot r!$.
- Best known upper bound is: $f_r(k) \leq O(k \log r)^r$.

Conjecture (Sunflower conjecture, Erdős-Rado, 1960)

$$f_r(k) \leq O(k)^r$$

- Even $k = 3$ case is open and very interesting.

Question (Erdős-Rado, 1960)

What is the max size of a family of r -sets without a k petal sunflower?

- Denote the answer by $f_r(k)$.
- Erdős-Rado sunflower lemma: $(k-1)^r \leq f_r(k) \leq (k-1)^r \cdot r!$.
- Best known upper bound is: $f_r(k) \leq O(k \log r)^r$.

Conjecture (Sunflower conjecture, Erdős-Rado, 1960)

$$f_r(k) \leq O(k)^r$$

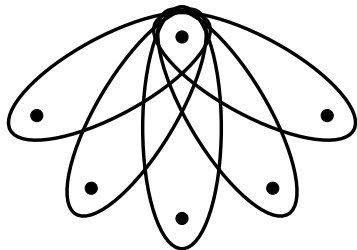
- Even $k = 3$ case is open and very interesting.
- Relations to many topics in computer science and probability theory.

Specific sunflowers

- Let $\mathcal{S}_t^{(r)}(k)$ be the r -uniform sunflower with k petals and kernel of size t .

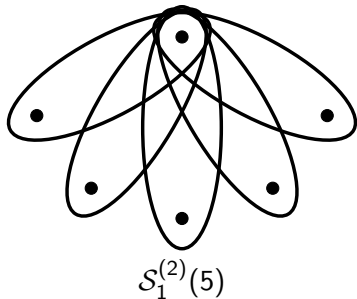
Specific sunflowers

- Let $\mathcal{S}_t^{(r)}(k)$ be the r -uniform sunflower with k petals and kernel of size t .



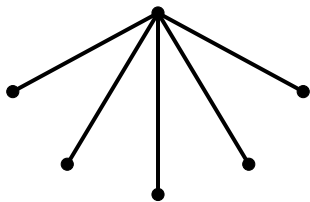
Specific sunflowers

- Let $\mathcal{S}_t^{(r)}(k)$ be the r -uniform sunflower with k petals and kernel of size t .



Specific sunflowers

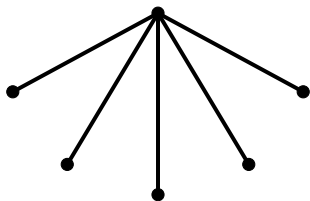
- Let $\mathcal{S}_t^{(r)}(k)$ be the r -uniform sunflower with k petals and kernel of size t .



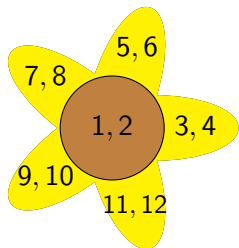
$$\mathcal{S}_1^{(2)}(5)$$

Specific sunflowers

- Let $\mathcal{S}_t^{(r)}(k)$ be the r -uniform sunflower with k petals and kernel of size t .



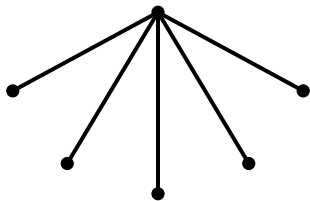
$\mathcal{S}_1^{(2)}(5)$



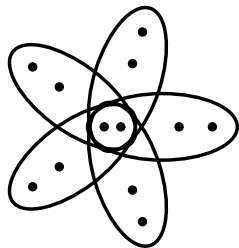
$\mathcal{S}_2^{(4)}(5)$

Specific sunflowers

- Let $\mathcal{S}_t^{(r)}(k)$ be the r -uniform sunflower with k petals and kernel of size t .



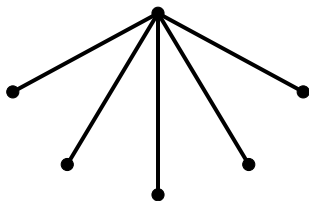
$\mathcal{S}_1^{(2)}(5)$



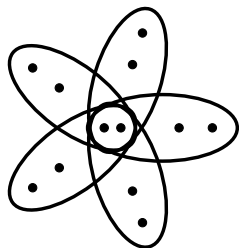
$\mathcal{S}_2^{(4)}(5)$

Specific sunflowers

- Let $\mathcal{S}_t^{(r)}(k)$ be the r -uniform sunflower with k petals and kernel of size t .



$\mathcal{S}_1^{(2)}(5)$

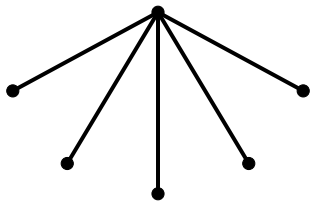


$\mathcal{S}_2^{(4)}(5)$

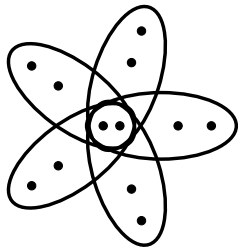
- Sunflower problem: What is the max number of edges in an r -graph without any of $\mathcal{S}_0^{(r)}(k), \mathcal{S}_1^{(r)}(k), \dots, \mathcal{S}_{r-1}^{(r)}(k)$?

Specific sunflowers

- Let $\mathcal{S}_t^{(r)}(k)$ be the r -uniform sunflower with k petals and kernel of size t .



$\mathcal{S}_1^{(2)}(5)$



$\mathcal{S}_2^{(4)}(5)$

- Sunflower problem: What is the max number of edges in an r -graph without any of $\mathcal{S}_0^{(r)}(k), \mathcal{S}_1^{(r)}(k), \dots, \mathcal{S}_{r-1}^{(r)}(k)$?

Question (Duke and Erdős 1977)

What is the max number of edges in an n -vertex r -graph without $\mathcal{S}_t^{(r)}(k)$?

Turán problem for sunflowers

Question (Duke and Erdős 1977)

What is the max number of edges in an n -vertex r -graph without $\mathcal{S}_t^{(r)}(k)$?

Turán problem for sunflowers

Question (Duke and Erdős 1977)

What is the max number of edges in an n -vertex r -graph without $\mathcal{S}_t^{(r)}(k)$?

- The answer is called the *Turán number* of $\mathcal{S}_t^{(r)}(k)$, denoted $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$

Turán problem for sunflowers

Question (Duke and Erdős 1977)

What is the max number of edges in an n -vertex r -graph without $\mathcal{S}_t^{(r)}(k)$?

- The answer is called the *Turán number* of $\mathcal{S}_t^{(r)}(k)$, denoted $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$
- Captures several classical problems:

Turán problem for sunflowers

Question (Duke and Erdős 1977)

What is the max number of edges in an n -vertex r -graph without $\mathcal{S}_t^{(r)}(k)$?

- The answer is called the *Turán number* of $\mathcal{S}_t^{(r)}(k)$, denoted $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$
- Captures several classical problems:
 - ▶ Case $t = 0$ corresponds to the Erdős matching conjecture

Question (Duke and Erdős 1977)

What is the max number of edges in an n -vertex r -graph without $\mathcal{S}_t^{(r)}(k)$?

- The answer is called the *Turán number* of $\mathcal{S}_t^{(r)}(k)$, denoted $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$
- Captures several classical problems:
 - ▶ Case $t = 0$ corresponds to the Erdős matching conjecture
 - ▶ Case $k = 2$ corresponds to the forbidden intersection problem

Question (Duke and Erdős 1977)

What is the max number of edges in an n -vertex r -graph without $\mathcal{S}_t^{(r)}(k)$?

- The answer is called the *Turán number* of $\mathcal{S}_t^{(r)}(k)$, denoted $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$
- Captures several classical problems:
 - ▶ Case $t = 0$ corresponds to the Erdős matching conjecture
 - ▶ Case $k = 2$ corresponds to the forbidden intersection problem
- Many results and bounds in various regimes.

Question (Duke and Erdős 1977)

What is the max number of edges in an n -vertex r -graph without $\mathcal{S}_t^{(r)}(k)$?

- The answer is called the *Turán number* of $\mathcal{S}_t^{(r)}(k)$, denoted $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$
- Captures several classical problems:
 - ▶ Case $t = 0$ corresponds to the Erdős matching conjecture
 - ▶ Case $k = 2$ corresponds to the forbidden intersection problem
- Many results and bounds in various regimes.
- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?

Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

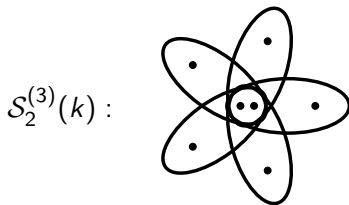
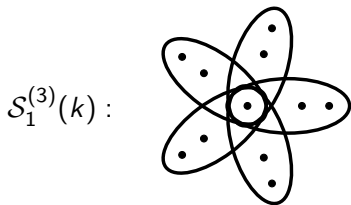
- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size

Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size

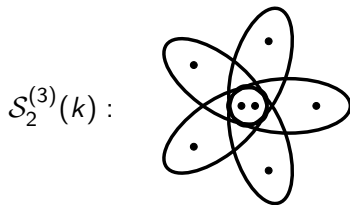
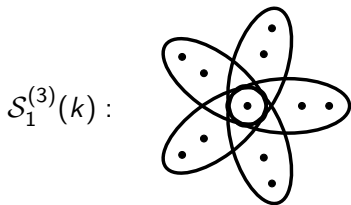


Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size



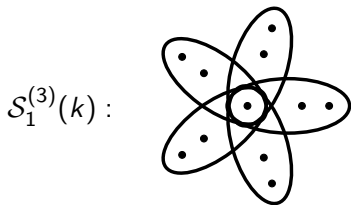
$$\text{ex}(n, \mathcal{S}_2^{(3)}(k)) \approx n^2 k$$

Large sunflowers

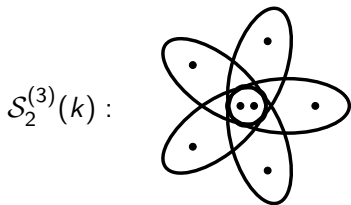
- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size



$$\text{ex}(n, \mathcal{S}_1^{(3)}(k)) \approx nk^2$$



$$\text{ex}(n, \mathcal{S}_2^{(3)}(k)) \approx n^2 k$$

Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size
 - ▶ Duke and Erdős; Frankl: $\text{ex}(n, \mathcal{S}_1^{(3)}(k)) \approx nk^2$ and $\text{ex}(n, \mathcal{S}_2^{(3)}(k)) \approx n^2k$

Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size
 - ▶ Duke and Erdős; Frankl: $\text{ex}(n, \mathcal{S}_1^{(3)}(k)) \approx nk^2$ and $\text{ex}(n, \mathcal{S}_2^{(3)}(k)) \approx n^2k$
 - ▶ Chung determined $\text{ex}(n, \mathcal{S}_1^{(3)}(k))$ up to lower order terms.

Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size
 - ▶ Duke and Erdős; Frankl: $\text{ex}(n, \mathcal{S}_1^{(3)}(k)) \approx nk^2$ and $\text{ex}(n, \mathcal{S}_2^{(3)}(k)) \approx n^2k$
 - ▶ Chung determined $\text{ex}(n, \mathcal{S}_1^{(3)}(k))$ up to lower order terms.
 - ▶ Chung and Frankl determined $\text{ex}(n, \mathcal{S}_1^{(3)}(k))$ precisely.

Large sunflowers

- Frankl and Füredi 1985: For fixed r and k we have

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_{r,k} n^{\max\{r-t+1, t\}}.$$

- Chung, Erdős, Graham 1980's: What if we let k grow with n ?
- If $r = 2$ it is trivial to see $\text{ex}(n, \mathcal{S}_1^{(2)}(k)) \approx nk$
- If $r = 3$ there are two types of sunflowers depending on kernel size
 - ▶ Duke and Erdős; Frankl: $\text{ex}(n, \mathcal{S}_1^{(3)}(k)) \approx nk^2$ and $\text{ex}(n, \mathcal{S}_2^{(3)}(k)) \approx n^2k$
 - ▶ Chung determined $\text{ex}(n, \mathcal{S}_1^{(3)}(k))$ up to lower order terms.
 - ▶ Chung and Frankl determined $\text{ex}(n, \mathcal{S}_1^{(3)}(k))$ precisely.
- The $r = 4$ case solved approximately by B., Draganić, Sudakov and Tran.

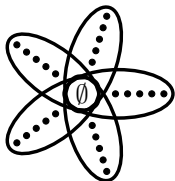
Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

Main result

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

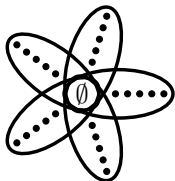


$$\text{ex}(n, \mathcal{S}_0^{(5)}(k)) \approx n^4 k$$

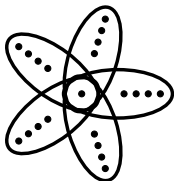
Main result

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$



$$\text{ex}(n, \mathcal{S}_0^{(5)}(k)) \approx n^4 k$$

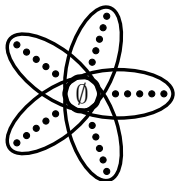


$$\text{ex}(n, \mathcal{S}_1^{(5)}(k)) \approx n^3 k^2$$

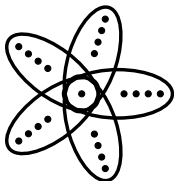
Main result

Theorem (Bradač, B. and Sudakov)

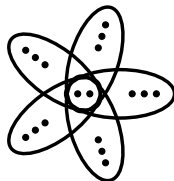
$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$



$$\text{ex}(n, \mathcal{S}_0^{(5)}(k)) \approx n^4 k$$



$$\text{ex}(n, \mathcal{S}_1^{(5)}(k)) \approx n^3 k^2$$

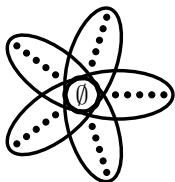


$$\text{ex}(n, \mathcal{S}_2^{(5)}(k)) \approx n^2 k^3$$

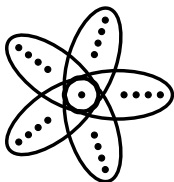
Main result

Theorem (Bradač, B. and Sudakov)

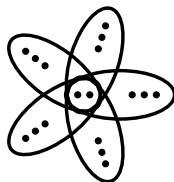
$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$



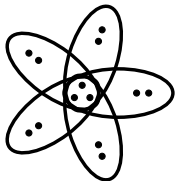
$$\text{ex}(n, \mathcal{S}_0^{(5)}(k)) \approx n^4 k$$



$$\text{ex}(n, \mathcal{S}_1^{(5)}(k)) \approx n^3 k^2$$



$$\text{ex}(n, \mathcal{S}_2^{(5)}(k)) \approx n^2 k^3$$

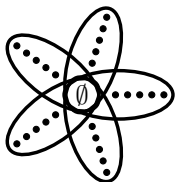


$$\text{ex}(n, \mathcal{S}_3^{(5)}(k)) \approx n^3 k^2$$

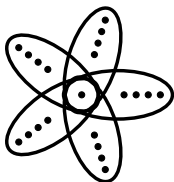
Main result

Theorem (Bradač, B. and Sudakov)

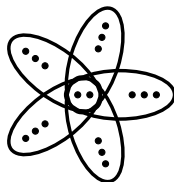
$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$



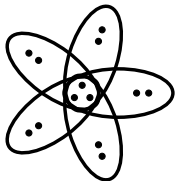
$$\text{ex}(n, \mathcal{S}_0^{(5)}(k)) \approx n^4 k$$



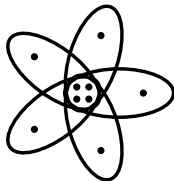
$$\text{ex}(n, \mathcal{S}_1^{(5)}(k)) \approx n^3 k^2$$



$$\text{ex}(n, \mathcal{S}_2^{(5)}(k)) \approx n^2 k^3$$



$$\text{ex}(n, \mathcal{S}_3^{(5)}(k)) \approx n^3 k^2$$



$$\text{ex}(n, \mathcal{S}_4^{(5)}(k)) \approx n^4 k$$

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

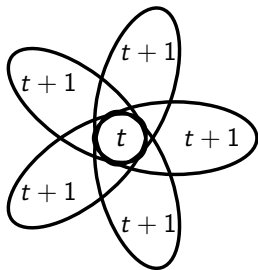
Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

A balanced sunflower:



Upper bounds: overview

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

- **Step 2:** Reduce the balanced case to an existence problem for $(t+1, t)$ -systems

Upper bounds: overview

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

- **Step 2:** Reduce the balanced case to an existence problem for $(t+1, t)$ -systems

Definition

$\mathcal{A} \subseteq \mathcal{P}([N])$ is a $(t+1, t)$ -system if:

Upper bounds: overview

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

- **Step 2:** Reduce the balanced case to an existence problem for $(t+1, t)$ -systems

Definition

$\mathcal{A} \subseteq \mathcal{P}([N])$ is a $(t+1, t)$ -system if:

- \mathcal{A} is intersection closed, i.e. $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,

Upper bounds: overview

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

- **Step 2:** Reduce the balanced case to an existence problem for $(t+1, t)$ -systems

Definition

$\mathcal{A} \subseteq \mathcal{P}([N])$ is a $(t+1, t)$ -system if:

- \mathcal{A} is intersection closed, i.e. $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size at most t is contained in some set in \mathcal{A} and

Upper bounds: overview

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

- **Step 2:** Reduce the balanced case to an existence problem for $(t+1, t)$ -systems

Definition

$\mathcal{A} \subseteq \mathcal{P}([N])$ is a $(t+1, t)$ -system if:

- \mathcal{A} is intersection closed, i.e. $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size at most t is contained in some set in \mathcal{A} and
- $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t+1}$.

Upper bounds: overview

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

- **Step 2:** Reduce the balanced case to an existence problem for $(t+1, t)$ -systems

Definition

$\mathcal{A} \subseteq \mathcal{P}([N])$ is a $(t+1, t)$ -system if:

- \mathcal{A} is intersection closed, i.e. $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size at most t is contained in some set in \mathcal{A} and
- $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t+1}$.

- Nägele, Sudakov, Zenklusen: no $(t+1, t)$ -system exists if $t+1$ is a prime power

Upper bounds: overview

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

- **Step 2:** Reduce the balanced case to an existence problem for $(t+1, t)$ -systems

Definition

$\mathcal{A} \subseteq \mathcal{P}([N])$ is a $(t+1, t)$ -system if:

- \mathcal{A} is intersection closed, i.e. $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size at most t is contained in some set in \mathcal{A} and
- $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t+1}$.

- Nägele, Sudakov, Zenklusen: no $(t+1, t)$ -system exists if $t+1$ is a prime power
- Brakensiek, Gopi, Guruswami: $(t+1, t)$ -systems exist otherwise

Upper bounds: overview

- **Step 1:** Use induction to reduce to the balanced case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

- **Step 2:** Reduce the balanced case to an existence problem for $(t+1, t)$ -systems

Definition

$\mathcal{A} \subseteq \mathcal{P}([N])$ is a $(t+1, t)$ -system if:

- \mathcal{A} is intersection closed, i.e. $\forall A, B \in \mathcal{A}$ we also have $A \cap B \in \mathcal{A}$,
- any subset of $[N]$ of size at most t is contained in some set in \mathcal{A} and
- $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t+1}$.

- Nägele, Sudakov, Zenklusen: no $(t+1, t)$ -system exists if $t+1$ is a prime power
- Brakensiek, Gopi, Guruswami: $(t+1, t)$ -systems exist otherwise

- **Step 3:** Show there are no $(t+1, t)$ -systems on ground set of size $N = 2t+1$

Reduction to the existence problem for a $(t + 1, t)$ -system

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies \text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies \text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform, $\mathcal{S}_t^{(2t+1)}(k)$ -free hypergraph

Reduction to the existence problem for a $(t + 1, t)$ -system

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform, $\mathcal{S}_t^{(2t+1)}(k)$ -free hypergraph
- For every set S of t vertices there is a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

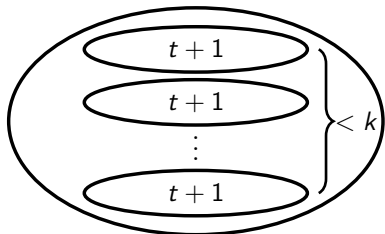
- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform, $\mathcal{S}_t^{(2t+1)}(k)$ -free hypergraph
- For every set S of t vertices there is a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
 - ▶ Among $t + 1$ -sets extending S into an edge there are no k vertex disjoint ones

Reduction to the existence problem for a $(t + 1, t)$ -system

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies \text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform, $\mathcal{S}_t^{(2t+1)}(k)$ -free hypergraph
- For every set S of t vertices there is a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
- ▶ Among $t + 1$ -sets extending S into an edge there are no k vertex disjoint ones

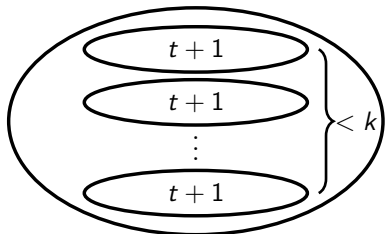


Reduction to the existence problem for a $(t + 1, t)$ -system

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies \text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform, $\mathcal{S}_t^{(2t+1)}(k)$ -free hypergraph
- For every set S of t vertices there is a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
 - ▶ Among $t + 1$ -sets extending S into an edge there are no k vertex disjoint ones
 - ▶ Taking the union of a maximal vertex disjoint collection gives τ_S .



Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies \text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.

Reduction to the existence problem for a $(t + 1, t)$ -system

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
- Let e_X be the number of edges containing X . So $|E| \leq \sum_{S \subseteq V, |S|=t} e_S$.

Reduction to the existence problem for a $(t + 1, t)$ -system

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

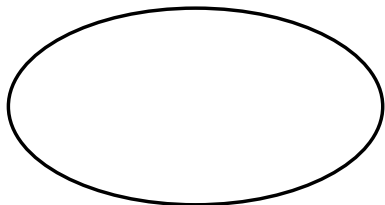
- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
- Let e_X be the number of edges containing X . So $|E| \leq \sum_{S \subseteq V, |S|=t} e_S$.
- Let S be a t -set then $e_S \leq \sum_{v \in \tau_S} e_{S \cup \{v\}}$

Reduction to the existence problem for a $(t + 1, t)$ -system

Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
- Let e_X be the number of edges containing X . So $|E| \leq \sum_{S \subseteq V, |S|=t} e_S$.
- Let S be a t -set then $e_S \leq \sum_{v \in \tau_S} e_{S \cup \{v\}}$



Reduction to the existence problem for a $(t + 1, t)$ -system

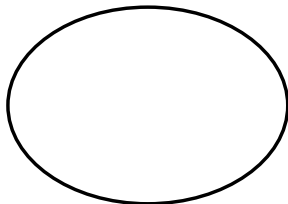
Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
- Let e_X be the number of edges containing X . So $|E| \leq \sum_{S \subseteq V, |S|=t} e_S$.
- Let S be a t -set then $e_S \leq \sum_{v \in \tau_S} e_{S \cup \{v\}}$



•
 $(t + 1)k$
choices

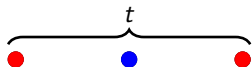


Reduction to the existence problem for a $(t + 1, t)$ -system

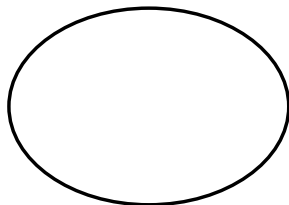
Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
- Let e_X be the number of edges containing X . So $|E| \leq \sum_{S \subseteq V, |S|=t} e_S$.
- Let S be a t -set then $e_S \leq \sum_{v \in \tau_S} e_{S \cup \{v\}}$



$(t + 1)k$
choices

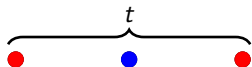


Reduction to the existence problem for a $(t + 1, t)$ -system

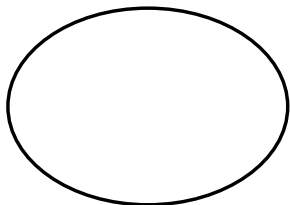
Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
- Let e_X be the number of edges containing X . So $|E| \leq \sum_{S \subseteq V, |S|=t} e_S$.
- Let S be a t -set then $e_S \leq \sum_{v \in \tau_S} e_{S \cup \{v\}}$
- For any X if $\exists t$ -set $S \subseteq X$ such that $\tau_S \cap X = \emptyset$ then $e_X \leq \sum_{v \in \tau_S} e_{X \cup \{v\}}$



$(t + 1)k$
choices

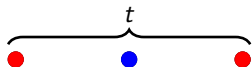




Reduction to the existence problem for a $(t + 1, t)$ -system

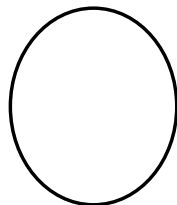
Lemma

No $(t + 1, t)$ -system on $2t + 1$ points $\implies ex(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1})$

- Let $H = (V, E)$ be an n -vertex, $2t + 1$ -uniform hypergraph
- For every set S of t vertices **fix** a set of vertices τ_S disjoint from S which intersects all edges containing S and has size $|\tau_S| \leq (t + 1)k$.
- Let e_X be the number of edges containing X . So $|E| \leq \sum_{S \subseteq V, |S|=t} e_S$.
- Let S be a t -set then $e_S \leq \sum_{v \in \tau_S} e_{S \cup \{v\}}$
- For any X if $\exists t$ -set $S \subseteq X$ such that $\tau_S \cap X = \emptyset$ then $e_X \leq \sum_{v \in \tau_S} e_{X \cup \{v\}}$



 $(t + 1)k$ choices
 $(t + 1)k$ choices



Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$.

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the $\#$ of t -sets covered by some A_i :

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the $\#$ of t -sets covered by some A_i :

$$\binom{N}{t} = \# \text{ of covered } t\text{-subsets}$$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the # of t -sets covered by some A_i :

$$\begin{aligned}\binom{N}{t} &= \# \text{ of covered } t\text{-subsets} \\ &= \binom{|A_1|}{t} + \binom{|A_2|}{t} + \dots + \binom{|A_m|}{t} -\end{aligned}$$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the # of t -sets covered by some A_i :

$$\begin{aligned} \binom{N}{t} &= \# \text{ of covered } t\text{-subsets} \\ &= \binom{|A_1|}{t} + \binom{|A_2|}{t} + \dots + \binom{|A_m|}{t} - \\ &\quad \binom{|A_1 \cap A_2|}{t} - \binom{|A_1 \cap A_3|}{t} - \dots - \binom{|A_{m-1} \cap A_m|}{t} + \end{aligned}$$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the # of t -sets covered by some A_i :

$$\begin{aligned} \binom{N}{t} &= \# \text{ of covered } t\text{-subsets} \\ &= \binom{|A_1|}{t} + \binom{|A_2|}{t} + \dots + \binom{|A_m|}{t} - \\ &\quad \binom{|A_1 \cap A_2|}{t} - \binom{|A_1 \cap A_3|}{t} - \dots - \binom{|A_{m-1} \cap A_m|}{t} + \\ &\quad \dots \\ &\quad (-1)^{m-1} \binom{|A_1 \cap \dots \cap A_m|}{t} \end{aligned}$$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the $\#$ of t -sets covered by some A_i :

$$\binom{N}{t} = \# \text{ of covered } t\text{-subsets} = \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \binom{|\bigcap_{i \in I} A_i|}{t}$$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the $\#$ of t -sets covered by some A_i :

$$\binom{N}{t} = \# \text{ of covered } t\text{-subsets} = \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \binom{|\bigcap_{i \in I} A_i|}{t}$$

- Lucas' theorem implies: $\binom{a}{t} \equiv 0 \pmod{p} \Leftrightarrow a \not\equiv -1 \pmod{p^\alpha}$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the # of t -sets covered by some A_i :

$$\binom{N}{t} = \# \text{ of covered } t\text{-subsets} = \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \binom{|\bigcap_{i \in I} A_i|}{t} \equiv 0 \pmod{p}$$

- Lucas' theorem implies: $\binom{a}{t} \equiv 0 \pmod{p} \Leftrightarrow a \not\equiv -1 \pmod{p^\alpha}$

Non-existence of $(t + 1, t)$ -systems

- Let $t + 1 = p^\alpha$, for p prime and assume that $\mathcal{A} \subseteq \mathcal{P}([N])$ satisfies:
 - ▶ \mathcal{A} is intersection closed
 - ▶ all t -subsets of $[N]$ are covered
 - ▶ $\forall A \in \mathcal{A}$ we have $|A| \not\equiv N \pmod{t + 1}$
- By adding dummy vertices to every $A \in \mathcal{A}$ we may assume $N \equiv -1 \pmod{t + 1}$
- Let $\mathcal{A} = \{A_1, \dots, A_m\}$. Double counting the $\#$ of t -sets covered by some A_i :

$$0 \not\equiv \binom{N}{t} = \# \text{ of covered } t\text{-subsets} = \sum_{\emptyset \neq I \subseteq [m]} (-1)^{|I|-1} \binom{|\bigcap_{i \in I} A_i|}{t} \equiv 0 \pmod{p}$$

- Lucas' theorem implies: $\binom{a}{t} \equiv 0 \pmod{p} \Leftrightarrow a \not\equiv -1 \pmod{p^\alpha}$

Further directions

- We determined the dependency of $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$ on n and k .

Further directions

- We determined the dependency of $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$ on n and k .

Problem 1

What is the dependency on r ?

Further directions

- We determined the dependency of $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$ on n and k .

Problem 1

What is the dependency on r ?

Problem 2

What if we forbid a collection of r -uniform sunflowers?

Further directions

- We determined the dependency of $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$ on n and k .

Problem 1

What is the dependency on r ?

Problem 2

What if we forbid a collection of r -uniform sunflowers?

Problem 3 (Chung-Erdős unavailability problem, 1983)

Among r -uniform hypergraphs with e edges which is hardest to avoid?

Further directions

- We determined the dependency of $\text{ex}(n, \mathcal{S}_t^{(r)}(k))$ on n and k .

Problem 1

What is the dependency on r ?

Problem 2

What if we forbid a collection of r -uniform sunflowers?

Problem 3 (Chung-Erdős unavailability problem, 1983)

Among r -uniform hypergraphs with e edges which is hardest to avoid?

- Known for $r \leq 4$, up to constant factor.



Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

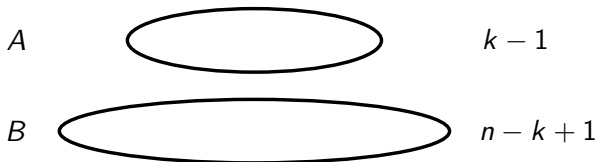
- Partition the vertex set into A and B s.t. $|A| = k - 1$ and $|B| = n - k + 1$

First lower bound

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

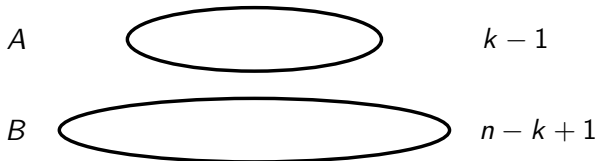
- Partition the vertex set into A and B s.t. $|A| = k - 1$ and $|B| = n - k + 1$



Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

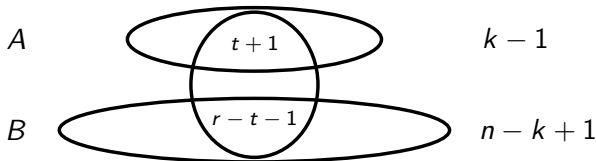
- Partition the vertex set into A and B s.t. $|A| = k - 1$ and $|B| = n - k + 1$
- Choose as an edge any set with $t + 1$ vertices in A and $r - t - 1$ in B .



Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

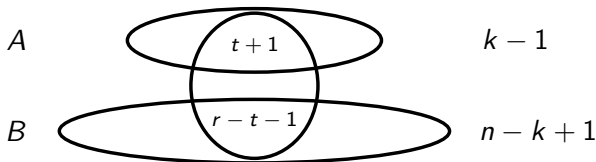
- Partition the vertex set into A and B s.t. $|A| = k - 1$ and $|B| = n - k + 1$
- Choose as an edge any set with $t + 1$ vertices in A and $r - t - 1$ in B .



Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

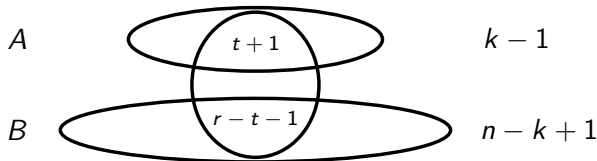
- Partition the vertex set into A and B s.t. $|A| = k - 1$ and $|B| = n - k + 1$
- Choose as an edge any set with $t + 1$ vertices in A and $r - t - 1$ in B .
- No $\mathcal{S}_t^{(r)}(k)$ since every petal needs to have a vertex in A .



Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Partition the vertex set into A and B s.t. $|A| = k - 1$ and $|B| = n - k + 1$
- Choose as an edge any set with $t + 1$ vertices in A and $r - t - 1$ in B .
- No $\mathcal{S}_t^{(r)}(k)$ since every petal needs to have a vertex in A .
- The number of edges is at least $\binom{k-1}{t+1} \binom{n-k+1}{r-t-1} = \Omega_r(n^{r-t-1} k^{t+1})$



Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Partition the vertex set into A and B s.t. $|A| = k - 1$ and $|B| = n - k + 1$
- Choose as an edge any set with $t + 1$ vertices in A and $r - t - 1$ in B .
- No $\mathcal{S}_t^{(r)}(k)$ since every petal needs to have a vertex in A .
- The number of edges is at least $\binom{k-1}{t+1} \binom{n-k+1}{r-t-1} = \Omega_r(n^{r-t-1} k^{t+1}) \implies$

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \geq \Omega_r(n^{r-t-1} k^{t+1}).$$

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Let $m := |\mathcal{S}_t^{(r)}(k)| - 1$ and S be an m -uniform n -vertex hypergraph s.t.

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Let $m := |\mathcal{S}_t^{(r)}(k)| - 1$ and S be an m -uniform n -vertex hypergraph s.t.
 - ▶ any subset of t vertices is contained in precisely one edge of S

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Let $m := |\mathcal{S}_t^{(r)}(k)| - 1$ and S be an m -uniform n -vertex hypergraph s.t.
 - ▶ any subset of t vertices is contained in precisely one edge of S

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Let $m := |\mathcal{S}_t^{(r)}(k)| - 1$ and S be an m -uniform n -vertex hypergraph s.t.
 - ▶ any subset of t vertices is contained in precisely one edge of S
- Choose as edges of our H any r -vertex subset of an edge of S .

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Let $m := |\mathcal{S}_t^{(r)}(k)| - 1$ and S be an m -uniform n -vertex hypergraph s.t.
 - ▶ any subset of t vertices is contained in precisely one edge of S
- Choose as edges of our H any r -vertex subset of an edge of S .
- No $\mathcal{S}_t^{(r)}(k)$ as all its edges must come from the same edge of S .

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Let $m := |\mathcal{S}_t^{(r)}(k)| - 1$ and S be an m -uniform n -vertex hypergraph s.t.
 - ▶ any subset of t vertices is contained in precisely one edge of S
- Choose as edges of our H any r -vertex subset of an edge of S .
- No $\mathcal{S}_t^{(r)}(k)$ as all its edges must come from the same edge of S .
- The number of edges is at least $\binom{n}{t} / \binom{m}{t} \cdot \binom{m}{r} \geq \Omega_r(n^t k^{r-t})$

Theorem (Bradač, B. and Sudakov)

$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1} k^{t+1} & \text{if } t \leq \frac{r-1}{2}, \\ n^t k^{r-t} & \text{if } t > \frac{r-1}{2}. \end{cases}$$

- Let $m := |\mathcal{S}_t^{(r)}(k)| - 1$ and S be an m -uniform n -vertex hypergraph s.t.
 - ▶ any subset of t vertices is contained in precisely one edge of S
- Choose as edges of our H any r -vertex subset of an edge of S .
- No $\mathcal{S}_t^{(r)}(k)$ as all its edges must come from the same edge of S .
- The number of edges is at least $\binom{n}{t} / \binom{m}{t} \cdot \binom{m}{r} \geq \Omega_r(n^t k^{r-t}) \implies$
$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \geq \Omega_r(n^t k^{r-t}).$$