# Large cliques and independent sets all over the place

Matija Bucić

joint work with Noga Alon and Benny Sudakov

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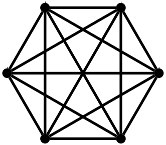

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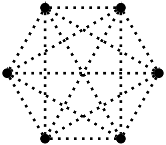

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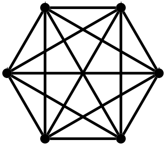

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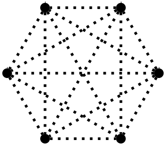

**Theorem (Erdős-Szekeres 1935)**

*Any $n$-vertex graph contains a clique or an independent set of order $\frac{1}{2}\log n$.*

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  **Example** (Erdős and Szemeredi, 1972):
  $c \log n$-Ramsey graphs have density bounded away from 0 and 1.

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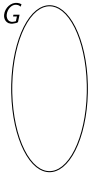

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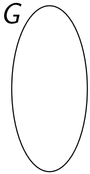


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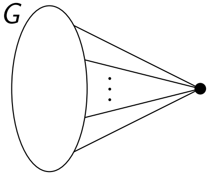


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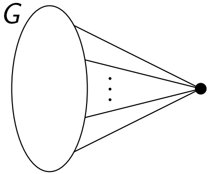


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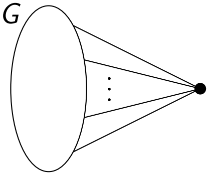


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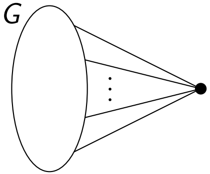


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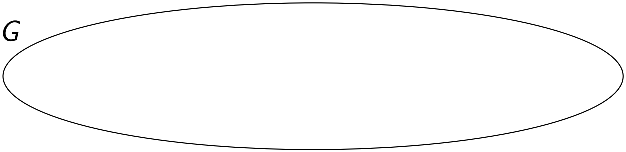

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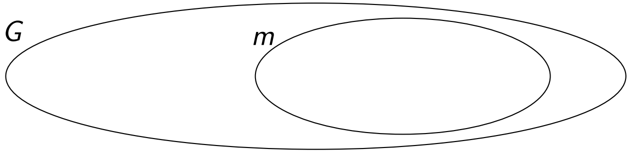

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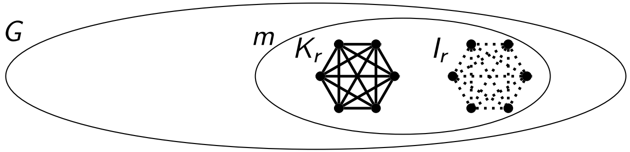

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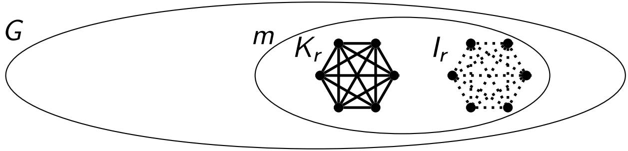


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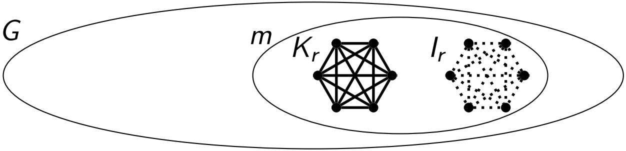


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**Definition (Erdős and Hajnal, 1988)**

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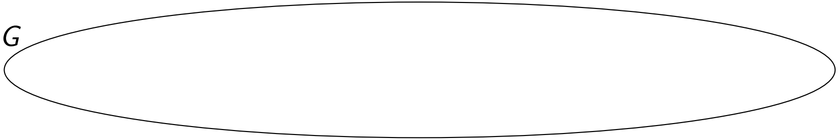

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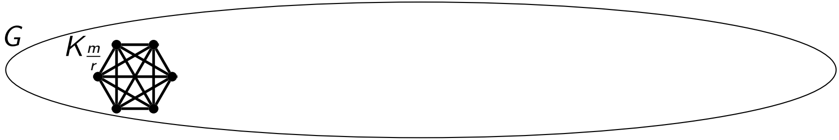

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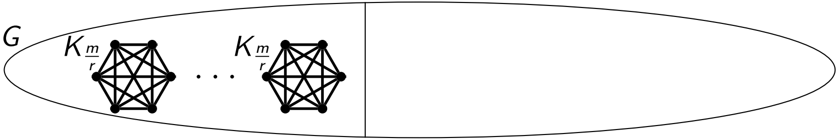

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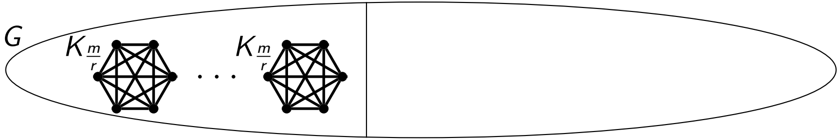

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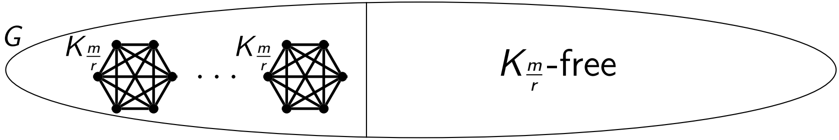

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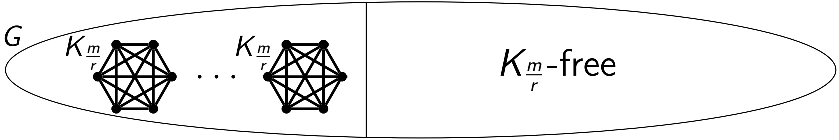

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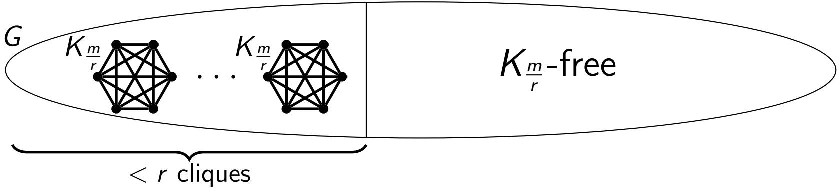

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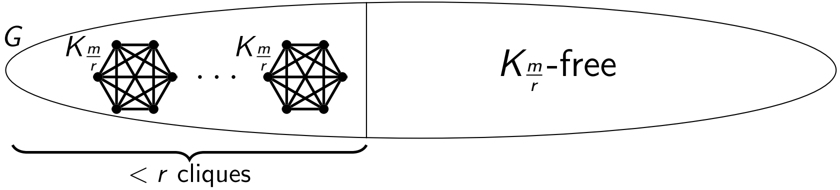

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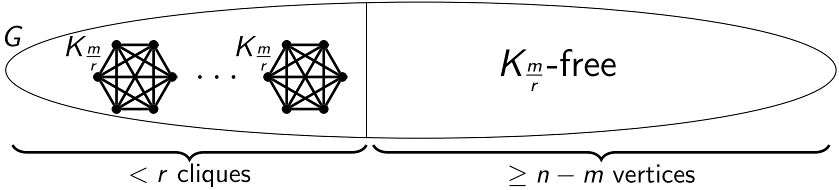

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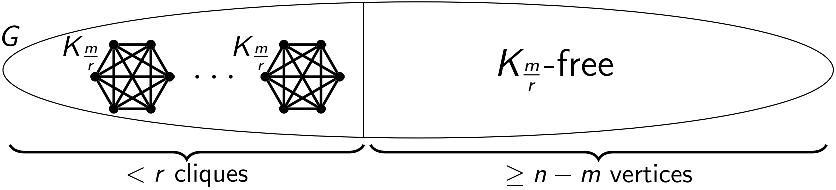

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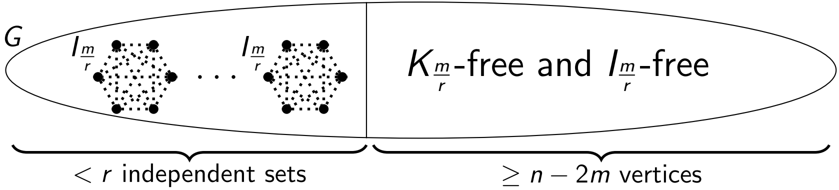

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**Corollary (Erdős, 1988)**

*There is no $m$-locally $\log n$-Ramsey graph with $m < \frac{1}{2}(\log n)^2$.*

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**Theorem (Alon, B., Sudakov, 2020+)**

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**Theorem (Alon, B., Sudakov, 2020+)**

*There exists a $2^{(\log n)^{1/2+o(1)}}$-locally $\log n$-Ramsey graph on $n$ vertices.*

- Gives rise to $2^{(\log n)^{1/2+o(1)}}$-Ramsey graphs very different from $\mathcal{G}(n, 1/2)$.

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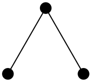

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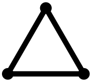

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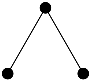

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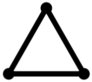

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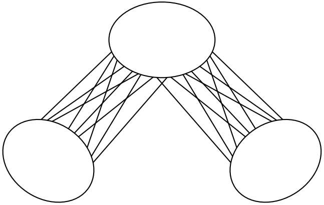

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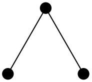

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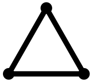

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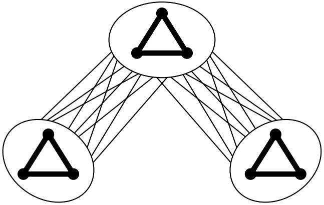

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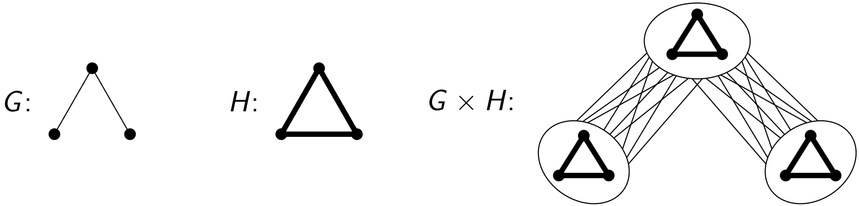


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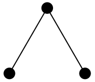

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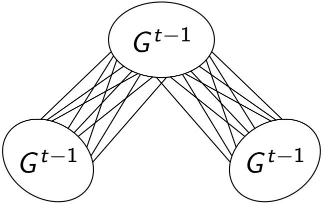


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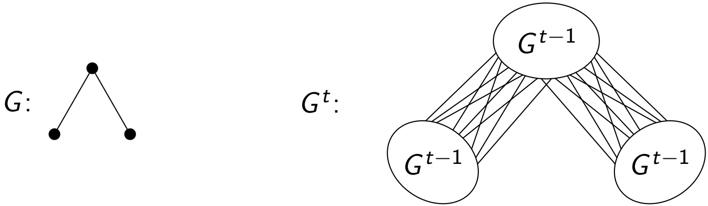


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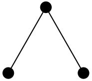
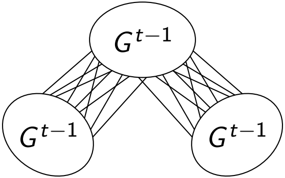
$G$: 

$G^t$: 

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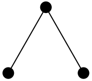
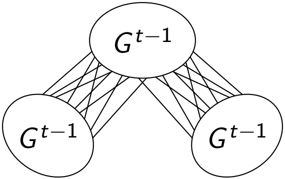
$G$: 

$G^t$: 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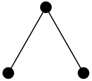

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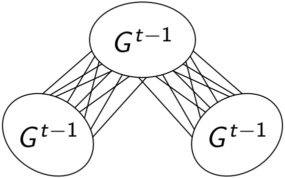


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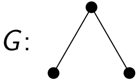


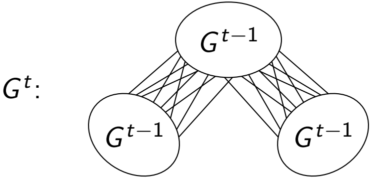


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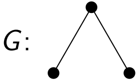


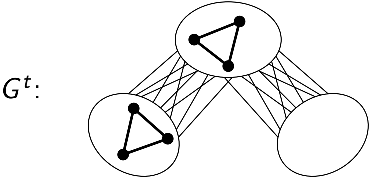


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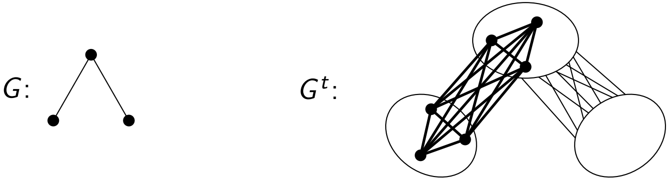


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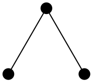

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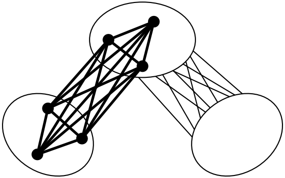

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  Use claim to join the cliques.

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$$m \to \frac{m}{4\log n} \text{ but also } t \to t-1.$$

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  **Goal:** find a big clique among vertices of $S$.

- Let $S_v$ consist of elements in $S$ with first coordinate $v$
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- Case 2: There are at least $2\log n$ vertices $v$ with non-empty $S_v$.
  $\exists\, u \sim_G v$ with non-empty $S_u, S_v$, so with $|S_u|, |S_v| \geq \frac{m}{2n}$.
  Find big cliques inside both $S_u$ and $S_v$ and join them to find one of size:

$$\text{double of what we find for } m \to \frac{m}{2n} \text{ and } t \to t-1$$

# Proof

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- Case 1: $m \to \frac{m}{4\log n}$ and $t \to t-1$.
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$$(2n)^{\log\log V} \approx 2^{\sqrt{\log V}\log\log V} = \sqrt{m}$$

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**Theorem (Alon, B., Sudakov, 2020+)**

*$G^t$ is a $V$-vertex graph which is $2^{(\log V)^{1/2+o(1)}}$-locally $\log V$-Ramsey.*

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## Proposition

$\exists$ *an m-locally r-Ramsey n-vertex graph* $\Longrightarrow$ $\exists$ *an n-vertex graph which is:*
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  The clique number is at most $\omega(\mathcal{G}(n, 1-p)) = \frac{\log n}{p} = r$

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  Case 3: at least $m$ – can combine $r$ cliques.

# Concluding remarks

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**Problem**

*Improve bounds on $m_n(\log n)$.*


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danke
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
faafetai lava
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dhanyavad
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mercé
мерси
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ευχαριστώ
merci
תודה רבה