Unit and distinct distances in typical norms

Matija Bucić

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based on joint work with Noga Alon and Lisa Sauermann
Question (Erdős, 1946)

What is the maximum number of unit distances defined by $n$ points in $(\mathbb{R}^2, \| \cdot \|_2)$?
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What is the maximum number of unit distances defined by \( n \) points in \( \mathbb{R}^2 \)?

Let \( U_{\|\cdot\|_2}(n) \) denote the answer.
**Question (Erdős, 1946)**

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Let \( U_{\|\cdot\|_2}(n) \) denote the answer.

\[
U_{\|\cdot\|_2}(n) \leq O(n^{4/3})
\]

Spencer, Szemerédi, Trotter ’84
Question (Erdős, 1946)

What is the maximum number of unit distances defined by $n$ points in $(\mathbb{R}^2, \|\cdot\|_2)$?

Let $U_{\|\cdot\|_2}(n)$ denote the answer.

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\Omega \left( n^{1 + \frac{1}{\log \log n}} \right) \leq U_{\|\cdot\|_2}(n) \leq O(n^{4/3})
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In 2D:

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In 3D:

\[
\frac{n^4}{3} + o(1) \leq U_{\|\cdot\|_2}(n) \leq O(n^{3/2 - \varepsilon})
\]

Zahl ’19

In 4 and more D:

\[U_{\|\cdot\|_2}(n) = \Theta(n^2)\]
Erdős unit distance problem

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What is the maximum number of unit distances defined by \( n \) points in \((\mathbb{R}^d, \| \cdot \|_2)\)?

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Erdős ’46

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In 3D: \[ n^{4/3 + o(1)} \leq U_{\|\cdot\|_2}(n) \leq O(n^{3/2 - \varepsilon}) \]

Erdős ’60

Zahl ’19
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General normed spaces

Question (Erdős, Ulam 1980)

What is the max number $U_{||.||}(n)$ of unit distances defined by $n$ points in $(\mathbb{R}^d, ||.||)$?
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Matija Bucić (IAS and Princeton)  Unit and distinct distances in typical norms  Budapest, April 2023
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- Brass conjectured that $\ell_\infty$ maximizes $U_{||.||} (n)$ among $\mathbb{R}^d$ norms $||.||$. 
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- Swanepoel 2018: $U_{\| \cdot \|}(n) \leq (1 + o(1)) \cdot (1 - 2^{1-d}) \cdot \frac{n^2}{2}$ for any $\mathbb{R}^d$-norm $\| \cdot \|$

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Maximization problem among strictly convex norms

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Typical normed spaces

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Theorem (Alon, B., Sauermann, 2023+)

For “most” $\mathbb{R}^d$-norms $U_{\|\|}(n) \leq \frac{d}{2} \cdot n \log_2 n$
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**Theorem (Alon, B., Sauermann, 2023+)**

For “most” $\mathbb{R}^d$-norms $U_{\| \cdot \|}(n) \leq \frac{d}{2} \cdot n \log_2 n$

For all $\mathbb{R}^d$-norms $U_{\| \cdot \|}(n) \geq \frac{d - 1 - o(1)}{2} \cdot n \log_2 n$
Question (Erdős, 1946)

What is the min number of distinct distances defined by n points in \((\mathbb{R}^d, \|\cdot\|_2)\)?
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Let \(D_{\|\cdot\|_2}(n)\) denote the answer.
Erdős distinct distances problem

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Let \( D_{\| . \|_2}(n) \) denote the answer.

In 2D:

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D_{\| . \|_2}(n) \leq O \left( \frac{n}{\sqrt{\log n}} \right)
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Erdős ’46
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Guth, Katz ’15

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In 3 and more D:

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D_{\| \cdot \|_2}(n) \leq O(n^{2/d})
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Question (Swanepoel 1997)

What is the min $\# D_{\|\cdot\|}(n)$ of distinct distances defined by $n$ points in $(\mathbb{R}^d, \|\cdot\|)$?
General normed spaces

**Question (Swanepoel 1997)**

What is the min \( \# D_{||\cdot||}(n) \) of distinct distances defined by \( n \) points in \( (\mathbb{R}^d, ||\cdot||) \)?

- \( D_{||\cdot||}(n) \leq n - 1 \) for any \( ||\cdot|| \).

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- Our result on unit distance problem $\Rightarrow D_{\|\cdot\|}(n) \geq \frac{n-1}{d \log n}$ for most $\mathbb{R}^d$-norms $\|\cdot\|$.
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Theorem (Alon, B., Sauermann, 2023+)

For most \( \mathbb{R}^d \)-norms

\[
D_{\| \cdot \|}(n) = n - o(n)
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What do we mean by “most”
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- A norm on $\mathbb{R}^d$ $\leftrightarrow$ convex, compact, 0-symmetric body in $\mathbb{R}^d$
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- Hausdorff distance: maximum distance between a point of one of the bodies to the other
What do we mean by “most”

- A norm on $\mathbb{R}^d$ corresponds to a convex, compact, 0-symmetric body in $\mathbb{R}^d$

- Hausdorff distance: maximum distance between a point of one of the bodies to the other

- Defines a metric (and hence a topology) on the space of $\mathbb{R}^d$-norms
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  complements of meagre sets are dense
Proof strategy

What makes a norm "special"?

An $\mathbb{R}^d$-norm is special if $\exists$ non-parallel unit vectors $u_1, \ldots, u_{kd+1}$ s.t. $\forall i, u_i \in \text{Span}(u_1, \ldots, u_k)$.

Step 1: Show the set of special norms is meagre

Step 2: Show that non-special norms have $U\parallel .\parallel(\mathbb{n}) \leq d^2 n \log n$. 

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$$u_i \in \text{Span}(u_1, \ldots, u_k)$$

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Step 2: Show that non-special norms have

$$\|U\|_\infty(n) \leq d^2 n \log n.$$
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- **Step 2:** Show that non-special norms have $U_{\|\cdot\|}(n) \leq \frac{d}{2} n \log n.$
Typical norms have small unit distance function

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- An $\mathbb{R}^d$-norm is special if $\exists$ non-parallel unit vectors $u_1, \ldots, u_{kd+1}$ s.t. $\forall i$
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- Any $k$ of the vectors $u_i$ span (over $\mathbb{Q}$) at most $kd$ of the vectors
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Given $n$ points in $\mathbb{R}^d$, suppose $> \frac{d}{2} n \log n$ pairs are at unit distance.

Define a graph with points as vertices and unit distance pairs as edges.

Let $u_1, \ldots, u_m$ be the unit directions appearing as edges.

Any $k$ of the vectors $u_i$ span (over $\mathbb{Q}$) at most $kd$ of the vectors.

Edmonds matroid decomposition thm $\Rightarrow$ can partition the vectors into $d$ $\mathbb{Q}$-independent sets.

There exist $u_{i_1}, \ldots, u_{i_t}$ which are: 1. $\mathbb{Q}$-independent and 2. account for $\frac{1}{d}$-fraction of the edges.
An $\mathbb{R}^d$-norm is special if $\exists$ non-parallel unit vectors $u_1, \ldots, u_{kd+1}$ s.t. $\forall i$ 

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Given $n$ points in $\mathbb{R}^d$, suppose $\frac{d}{2} \cdot n \log n$ pairs are at unit distance.

Define a graph with points as vertices and unit distance pairs as edges.

Let $u_1, \ldots, u_m$ be the unit directions appearing as edges.

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- Relabel so that $\mathbf{u}_1, \ldots, \mathbf{u}_t$ are: 1. $\mathbb{Q}$-independent and
  2. account for $\geq \frac{1}{2} n \log n$ edges
Typical norms have small unit distance function

Given $n$ points in $\mathbb{R}^d$ and $\mathbb{Q}$-independent unit directions $u_1, \ldots, u_t$ s.t.

- there are $\frac{1}{2} n \log n$ pairs at unit distance along these directions
Typical norms have small unit distance function

- Given $n$ points in $\mathbb{R}^d$ and $\mathbb{Q}$-independent unit directions $u_1, \ldots, u_t$ s.t.
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Typical norms have small unit distance function

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Can assume $G$ is connected
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• Define a graph $G$ with points as vertices and such pairs as edges

• Can assume $G$ is connected

• Can embed this graph into the grid graph $\mathbb{Z}^t$
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Matija Bucić (IAS and Princeton)  Unit and distinct distances in typical norms  Budapest, April 2023
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- Bollobás-Leader edge-isoperimetric inequality for the grid \( \Rightarrow \)
  \( G \) can have at most \( \frac{1}{2}n \log n \) edges.
Concluding remarks and open problems

What happens for typical norms in other classical problems?

For example, Hadwiger-Nelson problem

Theorem (Alon, B., Sauermann)

Chromatic number of the unit distance graph of $\mathbb{R}^2$ is $4$ for most norms.

In $\mathbb{R}^d$ we get an upper bound of $2^d$.

Question

Is $\chi$ of the unit distance graph of $\mathbb{R}^d$ subexponential for most norms?
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Matija Bucić (IAS and Princeton)

Unit and distinct distances in typical norms

Budapest, April 2023
Special norms are meagre

- An $\mathbb{R}^d$-norm is special if $\exists$ non-parallel unit vectors $u_1, \ldots, u_{kd+1}$ s.t. $\forall i$
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- Fix the “dependencies”: - a $(kd + 1) \times k$ rational matrix $A$ and
  - a rational angle $\eta > 0$
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- Fix the “dependencies”: - a \((kd + 1) \times k\) rational matrix \( A \) and
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**Goal:** for any fixed $A, \eta$ the set of $(A, \eta)$-special norms is nowhere dense
Special norms are meagre

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- **Goal**: for any fixed $A, \eta$ the set of $(A, \eta)$-special norms is nowhere dense

- Fix a norm $\| \cdot \|$ with unit ball $B$
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- Fix a norm $\|\cdot\|$ with unit ball $B$.

- We need to find an open set close to $\|\cdot\|$ not containing any $(A, \eta)$-special norm.
Special norms are meagre

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- Approximate $B$ by a convex 0-symmetric polytope with small facets
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- If there are $2f$ facets for each $x \in [-\varepsilon, \varepsilon]^f$ we define $B(x)$ to be the polytope obtained by translating $i$-th facet pair by $x_i$
Special norms are meagre

- An $\mathbb{R}^d$-norm is $(A, \eta)$-special if $\exists$ non-parallel unit vectors $u_1, \ldots, u_{kd+1}$ s.t.
  - $u_j = \sum_{i=1}^{k} A_{ji} u_i$ for all $j = 1, \ldots, dk + 1$.
  - $\angle(u_i, u_j) > \eta$, holds for all distinct $i$ and $j$.

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- An $(A, \eta)$-special $B(x)$ must have its bad $u_1, \ldots, u_{kd+1}$ on different facets.
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- If we fix which facets they belong to, this allows us to express $kd + 1$ of $x_i$’s as linear functions of $dk$ variables given by the coordinates of $u_1, \ldots, u_k$. 

Matija Bucić (IAS and Princeton)
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- A tiny open ball around the centre of the subbox has no $(A, \eta)$-special norms.