Unit and distinct distances in typical norms

Matija Bucić

Institute for Advanced Study and Princeton University

based on joint work with Noga Alon and Lisa Sauermann

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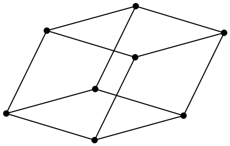

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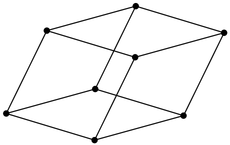

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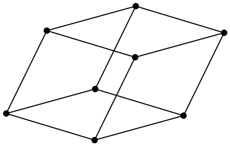

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- **Step 2:** Show that non-special norms have $U_{\|.\|}(n) \leq \frac{d}{2} n \log n$.

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- Given n points in $\mathbb{R}^d$, suppose $> \frac{d}{2} n \log n$ pairs are at unit distance.

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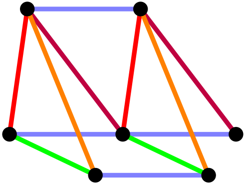

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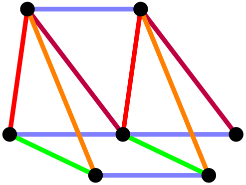

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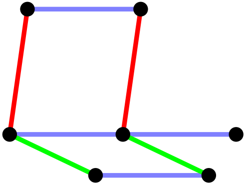

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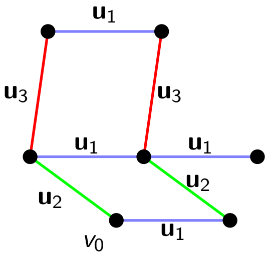

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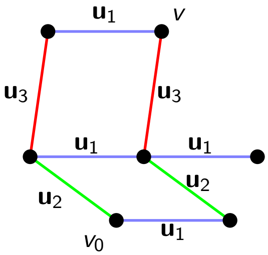

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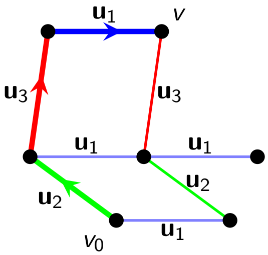

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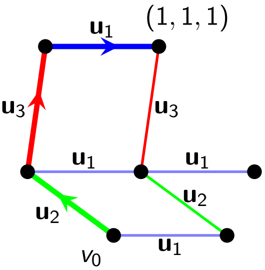

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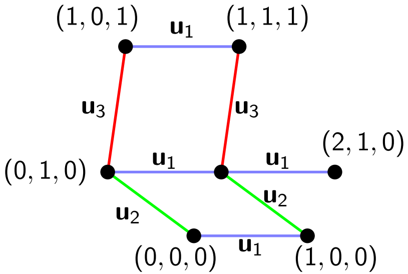

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Question

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рахмат
danke
謝謝
ngiyabonga
spas
teşekkür ederim
Баярлалаа
mersi
barka
welalin
tack
tapadh leat
спасибо
faafetai lava
vinaka
kia ora
dank je
misaotra
matondo
paldies
grazzi
mahalo
kiitos
dankie
спасибі
blagodaram
nanni
nandri
dhanyavad
thank you
gracias
хвала
asante
manana
bedankt
bayarlalaa
gràcie
hvala
mauruuru
köszönöm
aciu
obrigada
tenki
enkosi
dankon
akun
djiere dieuf
tau
mochchakkeram
murakoze
dziękuję
дякую
mamnun
chnorakaloutioun
gratias ago
gràcies
sulpáy
go raibh maith agat
chokrane
sobodi
děkuji
sagolun
sukriya
kop khun krap
taiku
obrigado
mèsi
ありがとう
arigatô
takk
dakujem
trugarez
didi madloba
najis tuke
tanemirt
grazie
kam sah hamnida
terima kasih
rahmet
diolch
dhanyavadagalu
shukriya
mercê
мерси
rahmat
তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
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- **Goal:** for any fixed A, η the set of (A, η)-special norms is nowhere dense

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- **Goal:** for any fixed A,η the set of (A,η)-special norms is nowhere dense
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- Approximate B by a convex 0-symmetric polytope with small facets

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- If we fix which facets they belong to, this allows us to express $kd+1$ of x_i's as linear functions of dk variables given by the coordinates of $\mathbf{u}_1,\ldots,\mathbf{u}_k$

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- A tiny open ball around the centre of the subbox has no (A,η)-special norms