Towards the Erdős-Hajnal Conjecture

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based on joint works with Pablo Blanco and

Tung Nguyen, Alex Scott, and Paul Seymour

Erdős-Hajnal Conjecture

Conjecture (Erdős-Hajnal, 1977)

$\forall$ *graph* H $\exists \varepsilon > 0$: *any* H*-free graph* G $\omega(G)$ *or* $\alpha(G) \geq |G|^{\varepsilon}$.

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- Chudnovsky, Safra 2006: True for the Bull graph
- Chudnovsky, Scott, Seymour, Spirkl 2021: True for C_5.

Improvements towards Erdős-Hajnal

- Erdős, Hajnal, 1977: H-free $\implies \omega(G)$ or $\alpha(G) \geq 2^{c\sqrt{\log |G|}}$

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- We get the same improvement for an infinite family of graphs.

Pure pairs, cographs

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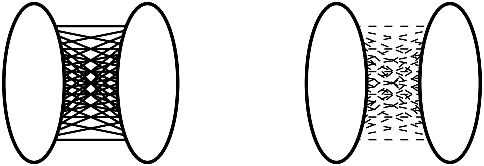

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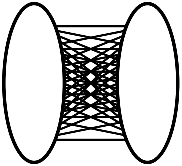

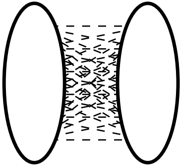

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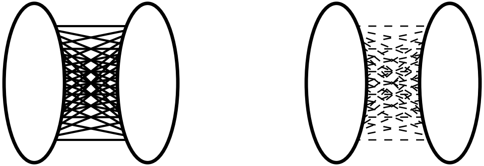

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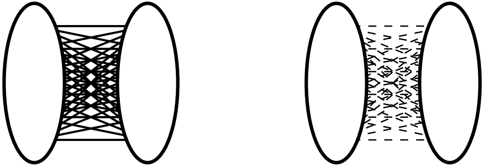

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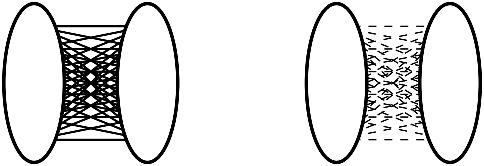

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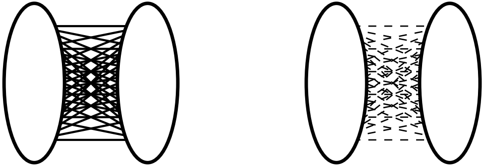

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- $\mu(G) :=$ the maximum order of a cograph in G.

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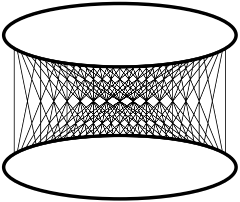

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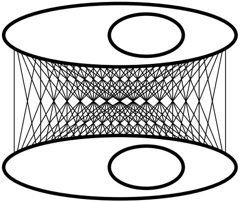

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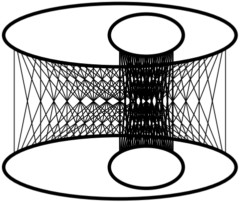

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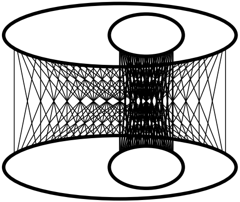

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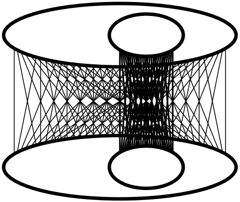

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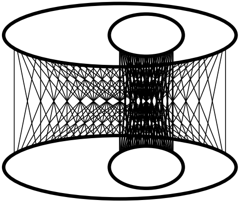

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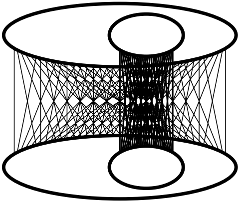

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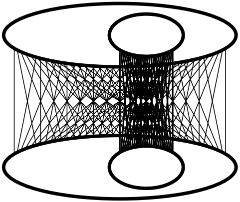

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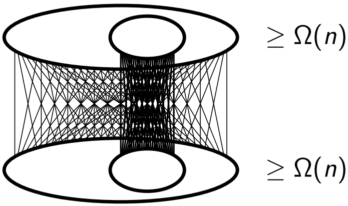

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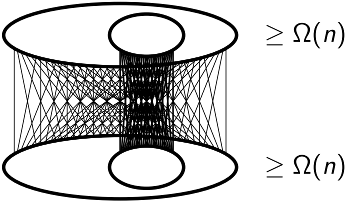

- Strong Erdős-Hajnal property
- Not true for most forbidden graphs H

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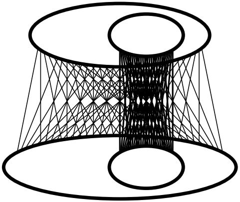

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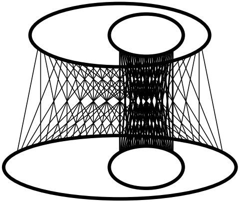

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- Works with $\varepsilon = \sqrt{\frac{\log \log n}{\log n}}$

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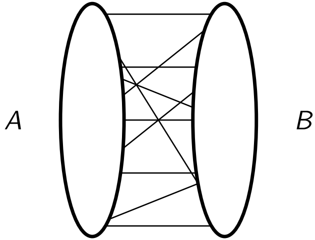

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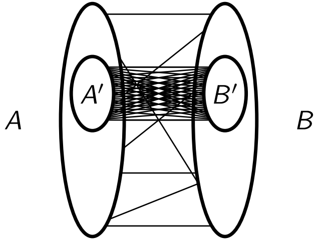

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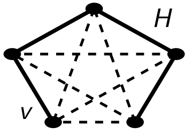

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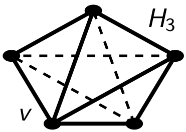

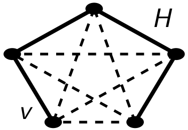

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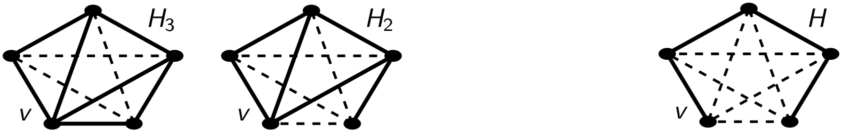

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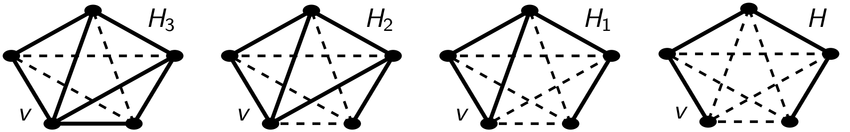

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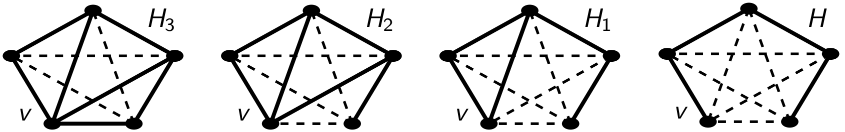

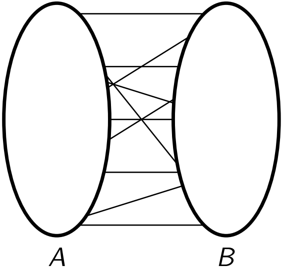

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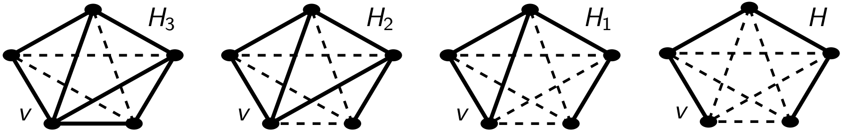

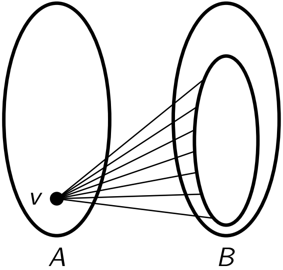

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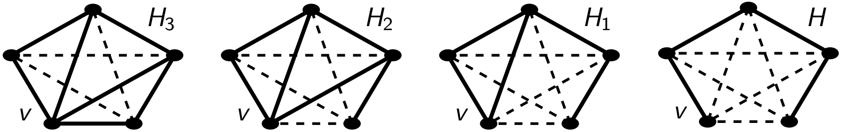

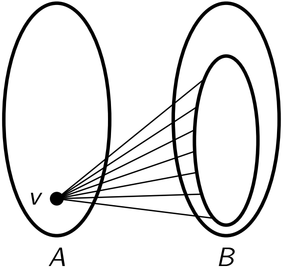

- On average $|N_B(v)| \geq \frac{|B|}{\mu} \geq \frac{n}{\mu^{O(1)}}$

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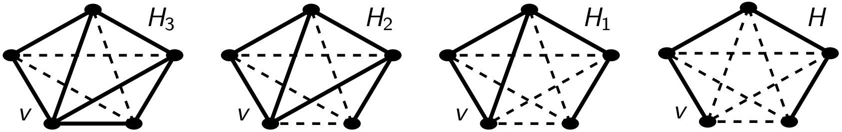

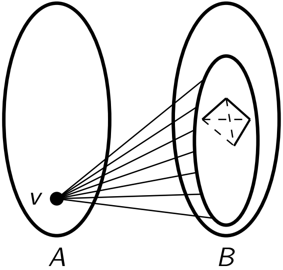

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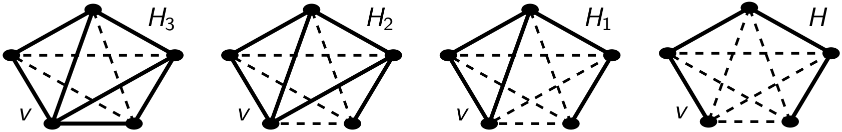

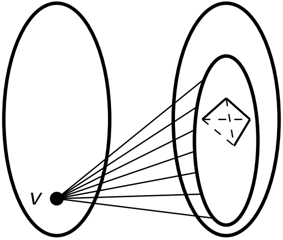

- On average $|N_B(v)| \geq \frac{|B|}{\mu} \geq \frac{n}{\mu^{O(1)}}$
- Supersaturation $\Rightarrow \frac{n^{|H|-1}}{\mu^{O(1)}}$ copies of $H \setminus v \subseteq N_B(v)$

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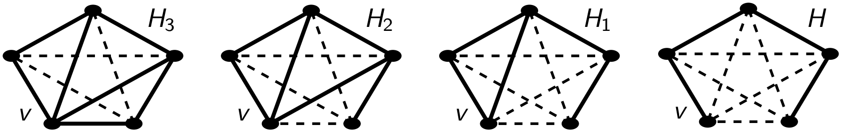

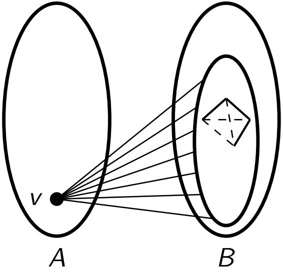

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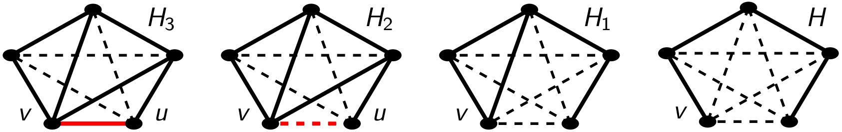

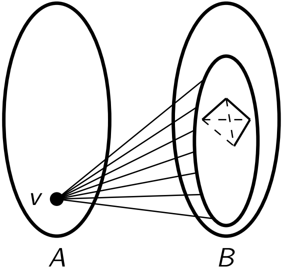

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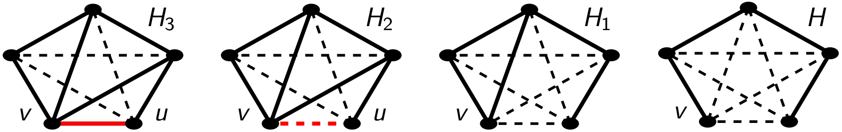

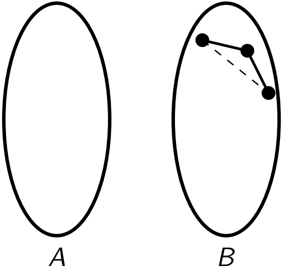

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- average $H \setminus \{v,u\} \subseteq B$ has $\frac{n^2}{\mu^{O(1)}}$ H_3-extensions

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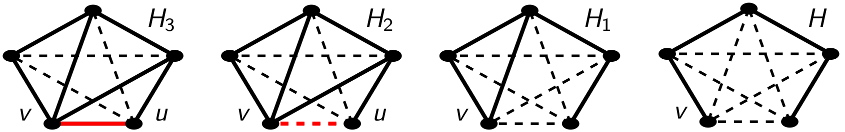

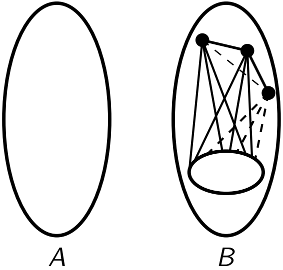

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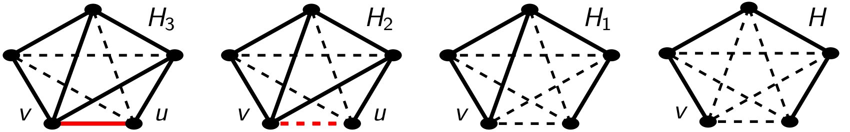

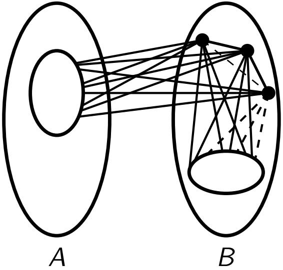

- On average $|N_B(v)| \geq \frac{|B|}{\mu} \geq \frac{n}{\mu^{O(1)}}$
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Goal: for a pair (A, B) with density $\geq \frac{1}{\mu}$ such that $|A|, |B| \geq \frac{n}{\mu^{O(1)}}$ then

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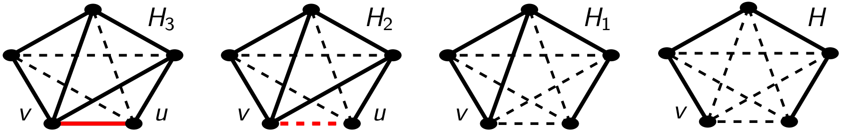

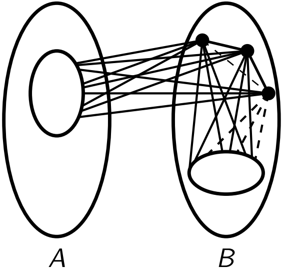

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- $\frac{n^{|H|}}{\mu^{O(1)}}$ copies of H_3 with $v \in A$ and rest in B
- average $H \setminus \{v, u\} \subseteq B$ has $\frac{n^2}{\mu^{O(1)}}$ H_3-extensions
- $\exists \frac{n^{|H|}}{\mu^{O(1)}}$ copies of H_2 with $v \in A$ rest in B or we win

- For a pair (A, B) with density $\geq \frac{1}{\mu}$ such that $|A|, |B| \geq \frac{n}{\mu^{O(1)}}$ then $\exists$ subpair (A', B') with density $\geq 1 - \frac{1}{\mu^{O(1)}}$ and $|A'| \geq \frac{|A|}{\mu^{O(1)}}$ and $|B'| \geq \frac{|B|}{\mu^{O(1)}}$

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  - We win if $\Omega(n)$ vertices are A-dense or A-sparse.

- For a pair (A, B) with density $\geq \frac{1}{\mu}$ such that $|A|, |B| \geq \frac{n}{\mu^{O(1)}}$ then $\exists$ subpair (A', B') with density $\geq 1 - \frac{1}{\mu^{O(1)}}$ and $|A'| \geq \frac{|A|}{\mu^{O(1)}}$ and $|B'| \geq \frac{|B|}{\mu^{O(1)}}$

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- Let $V(H) = [|H|]$ and suppose we found disjoint $A_1, \ldots, A_i : |A_i| \geq \frac{n}{\mu^{O(1)}}$

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- Let $V(H) = [|H|]$ and suppose we found disjoint $A_1, \ldots, A_i : |A_i| \geq \frac{n}{\mu^{O(1)}}$
  - If $vu \in E(H)$ then (A_v, A_u) is almost complete

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  - If $vu \notin E(H)$ then (A_v, A_u) is almost anticomplete

- For a pair (A, B) with density $\geq \frac{1}{\mu}$ such that $|A|, |B| \geq \frac{n}{\mu^{O(1)}}$ then $\exists$ subpair (A', B') with density $\geq 1 - \frac{1}{\mu^{O(1)}}$ and $|A'| \geq \frac{|A|}{\mu^{O(1)}}$ and $|B'| \geq \frac{|B|}{\mu^{O(1)}}$
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  - If $vu \in E(H)$ then (A_v, A_u) is almost complete
  - If $vu \notin E(H)$ then (A_v, A_u) is almost anticomplete
- There are $\Omega(n)$ vertices which are neither A_j-sparse nor A_j-dense

- For a pair (A, B) with density $\geq \frac{1}{\mu}$ such that $|A|, |B| \geq \frac{n}{\mu^{O(1)}}$ then $\exists$ subpair (A', B') with density $\geq 1 - \frac{1}{\mu^{O(1)}}$ and $|A'| \geq \frac{|A|}{\mu^{O(1)}}$ and $|B'| \geq \frac{|B|}{\mu^{O(1)}}$

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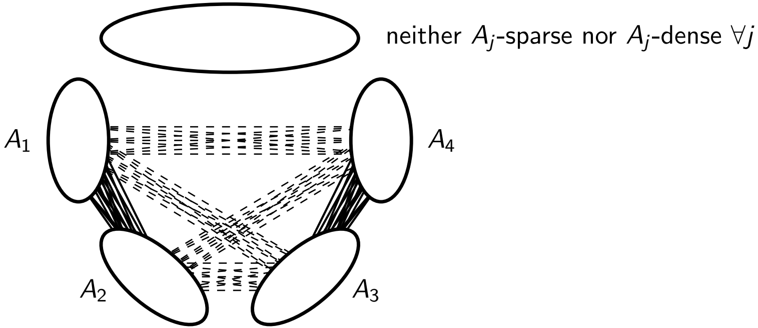

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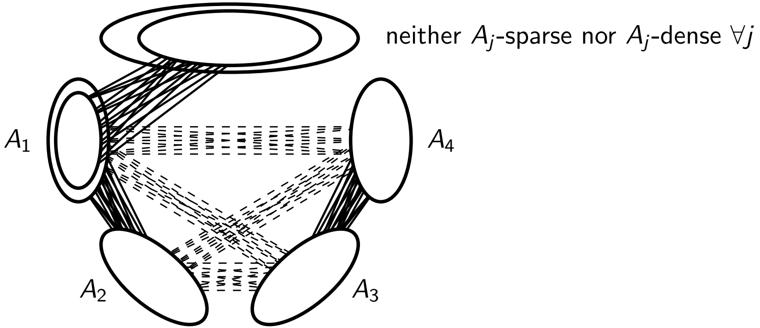

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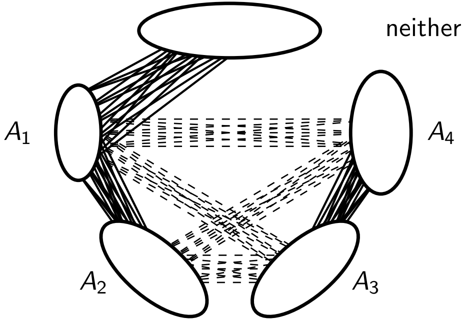

neither A_j-sparse nor A_j-dense $\forall j$

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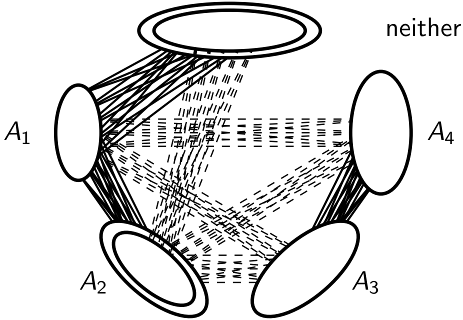

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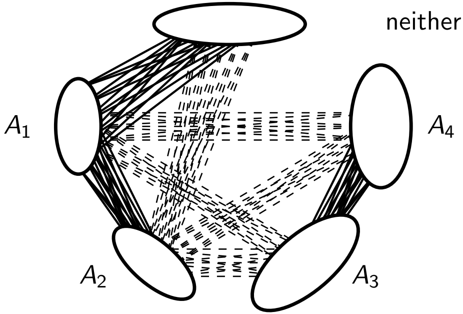
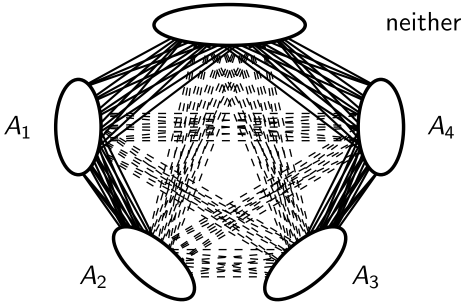


neither A_j-sparse nor A_j-dense $\forall j$

- For a pair (A, B) with density $\geq \frac{1}{\mu}$ such that $|A|, |B| \geq \frac{n}{\mu^{O(1)}}$ then
  $\exists$ subpair (A', B') with density $\geq 1 - \frac{1}{\mu^{O(1)}}$ and $|A'| \geq \frac{|A|}{\mu^{O(1)}}$ and $|B'| \geq \frac{|B|}{\mu^{O(1)}}$
- Let $A \subseteq V(G)$ s.t. $|A| \geq \frac{n}{\mu^{O(1)}}$
  - we say $v \in V(G)$ is A-dense if $|N_A(v)| \geq (1 - \frac{1}{\mu}) \cdot |A|$ and
  - we say $v \in V(G)$ is A-sparse if $|N_A(v)| \leq \frac{1}{\mu} \cdot |A|$.
  - We win if $\Omega(n)$ vertices are A-dense or A-sparse.




thank you
danke
рахмат
謝謝
ngiyabonga
teşekkür ederim
Баярлалаа
спасибо
gracias
tapadh leat
merci
obrigado
bedankt
dziękuję
hvala
mauruuru
sukriya
kop khun krap
go raibh maith agat
mochchakkeram
arigatô
dakujem
grazie
terima kasih
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ありがとう
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תודה רבה
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didi madloba
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