Decomposition problems

Matija Bucić

Institute for Advanced Study and Princeton University
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Such problems occur naturally across mathematics and beyond:

- Design theory
- Geometry
- Group theory
- Error-correcting codes
- Distributed computing
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  - Design theory
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- Scientific study dates back to the work of Euler in the 18th century.
Rota’s basis conjecture

- Given $n$ bases of an $n$-dimensional vector space a basis which intersects each of them in exactly one element is called a transversal basis.
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*One can decompose elements of any \( n \) disjoint bases of an \( n \)-dimensional vector space into \( n \) transversal bases.*
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- Connections with quite diverse other topics e.g.
  - Rota’s bracket theoretic approach to representation theory
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  - Lies just beyond the boundary of what we know about matroids
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What is known?

True for many special classes of matroids.
True for vector spaces over characteristic zero when dimension is $p \pm 1$ for any prime $p > 2$.

Aharoni-Berger 2006: decomposition into $2^n$ "partial" transversals.
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Theorem (B., Kwan, Pokrovskiy, Sudakov 2020)

Given any $n$ bases in an $n$-dimensional vector space there exist $(1/2 - o(1))n$ disjoint transversal bases.
Proof ideas

- We develop a new “algorithmic” approach.
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- We use a local switching idea.
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- Used to prove a conjecture of Erdős and Lovász from 1973.
General graph decomposition question:

- Walecki 1883: $K_{2n+1}$ can be decomposed into $n$ cycles.

- Veblen 1912: Any graph with all degrees even decomposes into cycles.
General graph decomposition question:

Can we decompose a graph into few graphs with some “nice” properties?

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Conjecture (Erdős-Gallai 1960s)

Every $n$-vertex graph can be decomposed into $O(n)$ cycles and edges.

Erdős 1983: one needs at least $(\frac{3}{2} - o(1))n$ cycles and edges.

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What do we know?

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Theorem (B., Montgomery 2022+)

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Concluding remarks

- Decomposition problems make some of the most classical well-studied problems with many interesting connections and applications.
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