Decomposition problems

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- Scientific study dates back to the work of Euler in the 18th century.

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B_5 :	$b_{5,1}$	$b_{5,2}$	$b_{5,3}$	$b_{5,4}$	$b_{5,5}$	$b_{5,6}$
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  - Lies just beyond the boundary of what we know about matroids

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- Aharoni-Berger 2006: decomposition into $2n$ "partial" transversals.

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Theorem (B., Kwan, Pokrovskiy, Sudakov 2020)

Given any n bases in an n-dimensional vector space there exist $(1/2 - o(1))n$ disjoint transversal bases.

Proof ideas

- We develop a new "algorithmic" approach.

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  - Used to prove a conjecture of Erdős and Lovász from 1973.

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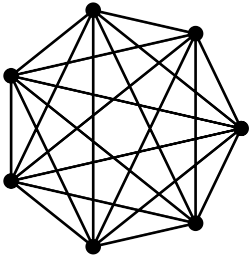

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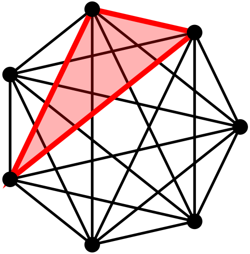

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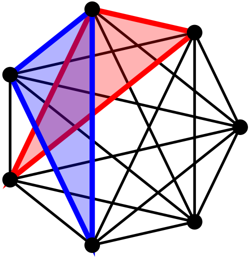

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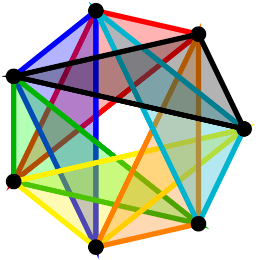

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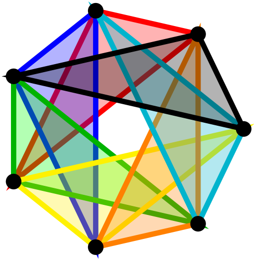

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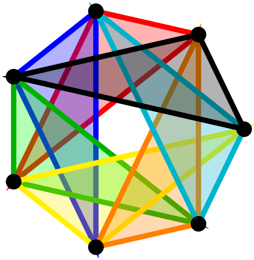

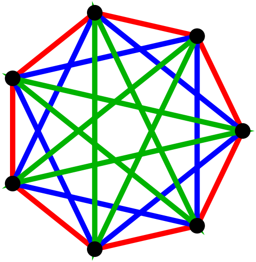

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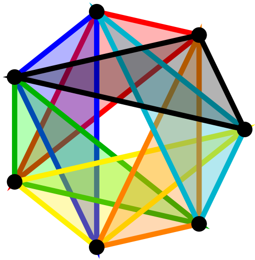

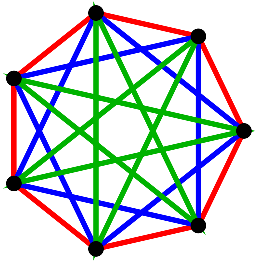

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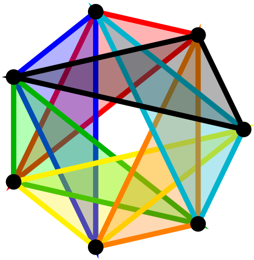

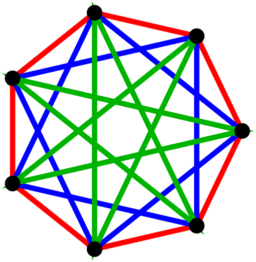

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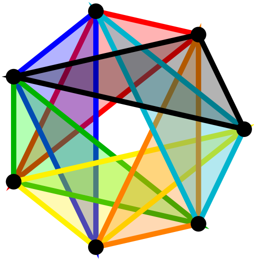

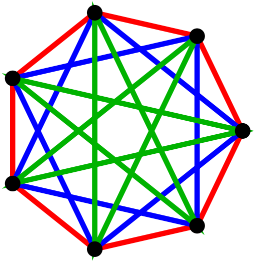

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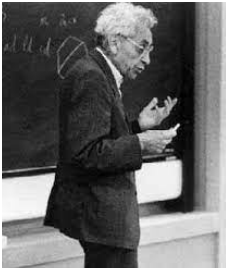
- Can we decompose any graph into cycles? No if $\exists$ an odd degree vertex.
- Veblen 1912: Any graph with all degrees even decomposes into cycles.

Conjecture (Erdős-Gallai 1960s)

Every n-vertex graph can be decomposed into $O(n)$ cycles and edges.



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- Pyber 1985: Precise solution for the covering version.

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Theorem (B., Montgomery 2022+)

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