Strength of polynomials via polynomial functors

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The strength of polynomials

Let $f$ be a homogeneous polynomial of degree $d \geq 2$.

**Definition**

The *strength* of $f$ is the minimal number $\text{str}(f) := r \geq 0$ such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d - 1$.

Defined by Ananyan and Hochster in order to prove Stillman’s Conjecture. Used by Erman, Sam and Snowden in their work on big polynomial rings. Plays a big role when studying the geometry of polynomial functors. Has also been defined for sections of line bundles over algebraic varieties by Ballico and Ventura.
The strength of polynomials

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\]

with \( g_1, h_1, \ldots, g_r, h_r \) homogeneous polynomials of degree \( \leq d - 1 \).

**Theorem (Ballico-B-Oneto-Ventura)**

The set

\[
    \{ f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \text{str}(f) \leq 3 \}
\]

is not Zariski-closed for \( n \gg 0 \).
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Example \((d = 2)\)

Let

\[
f = (x_1, \ldots, x_n) \cdot A \cdot (x_1, \ldots, x_n)^\top, \quad A \in \mathbb{C}^{n \times n} \text{ with } A^\top = A
\]

be a homogeneous polynomial of degree 2. By applying a coordinate transformation (or replacing \(A\) be a congruent matrix), we may assume that \(A = \text{Diag}(1_k, 0_{n-k})\) and \(f = x_1^2 + \ldots + x_k^2\).

If \(f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r\) with

\[
g_j = (x_1, \ldots, x_n) \cdot v_j^\top \text{ and } h_j = w_j \cdot (x_1, \ldots, x_n)^\top,
\]

then \(A = (v_1^\top w_1 + w_1^\top v_1) + \ldots + (v_r^\top w_r + w_r^\top v_r)\). So \(k \leq 2r\).

As \(x_j^2 + x_{j+1}^2 = (x_j + ix_{j+1})(x_j - ix_{j+1})\), we have \(\text{str}(f) = \lceil k/2 \rceil\).
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Definition

The slice rank of $f$ is the minimal number $\text{slrk}(f) := r \geq 0$ such that

$$f = g_1 \cdot \ell_1 + \ldots + g_r \cdot \ell_r$$

with $g_1, \ldots, g_r$ of degree $d - 1$ and $\ell_1, \ldots, \ell_r$ linear.

Proposition (Tao-Sawin, Derksen-Eggermont-Snowden)

The set

$$\{ f \in \mathbb{C}[x_1, \ldots, x_n]_d \mid \text{slrk}(f) \leq k \}$$

is Zariski-closed for all $d \geq 2$, $n \geq 1$ and $k \geq 0$.

Proof.

It is the cone of the projection of

$$\{ ([f], V) \in \mathbb{P}(\mathbb{C}[x_1, \ldots, x_n]_d) \times \text{Gr}(n - k, n) \mid f(V) = 0 \}$$
The strength of polynomials

**Definition**

The *slice rank* of $f$ is the minimal number $\text{slrk}(f) := r \geq 0$ such that

$$f = g_1 \cdot \ell_1 + \ldots + g_r \cdot \ell_r$$

with $g_1, \ldots, g_r$ of degree $d - 1$ and $\ell_1, \ldots, \ell_r$ linear.

**Theorem**

For $d \geq 3$ and $n \geq 1$, the generic slice rank in $\mathbb{C}[x_1, \ldots, x_n]_d$ is

$$\text{slrk}_{d,n}^\circ := \min \left\{ r \in \mathbb{Z} \left| \ r(n - r) \geq \binom{d - r + n - 1}{d} \right. \right\}.$$.  

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Conjecture

The generic strength and generic slice rank coincide.

Example (Fermat polynomials)

Take \( f = x_1^d + \ldots + x_n^d \) with \( d \geq 2 \).

As \( x_j^d + x_{j+1}^d \) is reducible, we have \( \text{str}(f) \leq \lceil n/2 \rceil \).

Ananyan-Hochster Trick:

If \( f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r \), then

\[
\text{Sing}\{f = 0\} = \{0\}
\]

contains the variety defined by \( g_1, h_1, \ldots, g_r, h_r \) and hence has codimension \( \leq 2r \). So we find \( \text{str}(f) \geq \lceil n/2 \rceil \).
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Theorem (Ballico-B-Oneto-Ventura)

The set
\[ \{ f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \text{str}(f) \leq 3 \} \]
is not Zariski-closed for \( n \gg 0 \).

Question

Is the set
\[ \{ f \in \mathbb{C}[x_1, \ldots, x_n]_d \mid \text{str}(f) \leq 2 \} \]
Zariski-closed for all \( d \geq 2 \) and \( n \geq 1 \)?

Proof is non-constructive and uses polynomial functors.
Let $\text{Vec}$ be the category of finite-dimensional vector spaces.

**Definition**
A functor $P : \text{Vec} \to \text{Vec}$ sends

$$
V \mapsto P(V) \\
(\ell : V \to W) \mapsto (P(\ell) : P(V) \to P(W))
$$

such that $P(\text{id}_V) = \text{id}_{P(V)}$ and $P(\ell_1 \circ \ell_2) = P(\ell_1) \circ P(\ell_2)$.

**Examples**
Take $U \in \text{Vec}$ fixed.

- $C_U : V \mapsto U, \ell \mapsto \text{id}_U$
- $T : V \mapsto V, \ell \mapsto \ell$
Polynomial functors: Definition

You can add and multiply two functors $P, Q : \text{Vec} \to \text{Vec}$.

$$(P \oplus Q)(V) = P(V) \oplus Q(V), \quad (P \otimes Q)(V) = P(V) \otimes Q(V)$$

$$(P \oplus Q)(\ell) = P(\ell) \oplus Q(\ell), \quad (P \otimes Q)(\ell) = P(\ell) \otimes Q(\ell)$$

You can take subfunctors and quotients:
We have $Q \subseteq P$ when $Q(V) \subseteq P(V)$ and $P(\ell)$ restricts to $Q(\ell)$.
In this case, we also get $P/Q$.

Definition
A polynomial functor is a functor $\text{Vec} \to \text{Vec}$ obtained from $T$ and the $C_U$ via addition, multiplication, subfunctors and quotients.

Examples
- Square matrices: $V \mapsto V \otimes V$
- Tensors: $V \mapsto V \otimes \cdots \otimes V$
- Polynomials: $V \mapsto S^dV$
**Definition**

Let $P, Q$ be polynomial functors. A morphism $\alpha: Q \rightarrow P$ is a family $(\alpha_V: Q(V) \rightarrow P(V))_{V \in \text{Vec}}$ of polynomial maps such that

\[
\begin{array}{ccc}
Q(V) & \xrightarrow{\alpha_V} & P(V) \\
\downarrow Q(\ell) & & \downarrow P(\ell) \\
Q(W) & \xrightarrow{\alpha_W} & P(W)
\end{array}
\]

commutes for all linear maps $\ell: V \rightarrow W$.

**Definition**

A (closed) subset $X \subseteq P$ sends

\[
V \mapsto \text{(closed) subset } X(V) \subseteq P(V)
\]

such that $P(\ell)(X(V)) \subseteq X(W)$ for all linear maps $\ell: V \rightarrow W$. 

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Example

We have a morphism $C_{\mathbb{C}^{n \times (n-1)}} \oplus T^{n-1} \to T^n$ defined by:

$$\mathbb{C}^{n \times (n-1)} \oplus V^{n-1} \ni (A, v_1, \ldots, v_{n-1}) \mapsto A \cdot (v_1, \ldots, v_n) \top \in V^n$$

Its image is the closed subset of $T^n$ consisting of all linearly dependent $n$-tuples of vectors.

Example

We have a morphism $T^{2k} \to T \otimes T$ defined by:

$$V^{2k} \ni (v_1, w_1, \ldots, v_k, w_k) \mapsto v_1 \otimes w_1 + \ldots + v_k \otimes w_k \in V \otimes V$$

Its image is the closed subset of $T \otimes T$ consisting of all matrices of rank $\leq k$. 

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Example

We have a morphism $(S^1)^r \oplus (S^{d-1})^r \rightarrow S^d$ defined by:

$$(\ell_1, \ldots, \ell_r, g_1, \ldots, g_r) \mapsto \ell_1 \cdot g_1 + \ldots + \ell_r \cdot g_r$$

Its image is the closed subset of $S^d$ consisting of all homogeneous polynomials of degree $d$ and slice rank $\leq r$.

Example

The subset of $T^\otimes n$ consisting of tensors with tensor rank $\leq k$.

Example

The subset of $S^d$ consisting of polynomials with strength $\leq r$. 

Strength of polynomials via polynomial functors

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Polynomial functors: The dichotomy

Let $P, Q$ be polynomial functors. Write $Q \prec P$ when $Q_d$ is a quotient of $P_d$ for $d$ maximal with $Q_d \not= P_d$.

**Dichotomy Theorem (B-Draisma-Eggermont-Snowden)**

Let $X \subseteq P$ be a closed subset. Then $X = P$ or there are polynomial functors $Q_1, \ldots, Q_k \prec P$ and $\alpha_i : Q_i \to P$ such that $X \subseteq \bigcup_i \text{im}(\alpha_i)$.

**Consequence**

Any closed subset of $T \otimes T$ consists of rank $\leq k \leq \infty$ matrices.

**Consequence (B-Draisma-Eggermont)**

Any closed subset of $S^d$ consists of strength $\leq k$ polynomials.

**Consequence (Draisma)**

Any polynomial functor $P$ is Noetherian.
Back to our goal

The homogeneous polynomials of degree 4 and strength $\leq 3$ form a subset of $S^4$. This subset is the union of the images of the morphisms

$$\alpha_k: (S^1 \oplus S^3)^k \oplus (S^2 \oplus S^2)^{3-k} \rightarrow S^4$$

$$((\ell_i, q_i)_i, (g_j, h_j)_j) \mapsto \sum_{i=1}^k \ell_i \cdot q_i + \sum_{j=1}^{3-k} g_j \cdot h_j$$

over $k = 0, 1, 2, 3$.

Goal

Prove that the subset $\bigcup_{k=0}^3 \text{im}(\alpha_k)$ of $S^4$ is not closed.

Idea

Consider polynomials of the form

$$x^2 f + y^2 g + u^2 h + v^2 q$$

with $x, y, u, v \in S^1$ and $f, g, h, q \in S^2$. 
Consider the morphism
\[ \beta : (S^1)^{\oplus 4} \oplus (S^2)^{\oplus 4} \rightarrow S^4 \]
\[ (x, y, u, v, f, g, h, q) \mapsto x^2 f + y^2 g + u^2 h + v^2 q \]

**Lemma**

We have \( \text{im}(\beta) \subseteq \text{im}(\alpha_0) \).

**Proof.**

The family of strength \( \leq 3 \) polynomials

\[
\frac{1}{t} \left( (x^2 + tg)(y^2 + tf) - (u^2 - tq)(v^2 - th) - (xy + uv)(xy - uv) \right)
\]

converges to \( x^2 f + y^2 g + u^2 h + v^2 q \) as \( t \to 0 \).

**Goal**

Prove that \( \text{im}(\beta) \not\subseteq \bigcup_{k=1}^{3} \text{im}(\alpha_k) \).
Polynomial functors: Inverse limits

Let $P$ be a polynomial functor.

**Definition**

We define $P_{\infty}$ as the inverse limit of the sequence

$$
\ldots \xrightarrow{P(\pi_4)} P(C^4) \xrightarrow{P(\pi_3)} P(C^3) \xrightarrow{P(\pi_2)} P(C^2) \xrightarrow{P(\pi_1)} P(C^1)
$$

where $\pi_n : C^{n+1} \to C^n$ is the projection forgetting the last coordinate.

**Example**

Take $P = T^n$. Then $P_{\infty} = (C^N)^n$.

**Example**

Take $P = T \otimes T$. Then $P_{\infty} = C^N \times N$. 
Polynomial functors: Inverse limits

A morphism \( \alpha: Q \to P \) induces a map \( \alpha_\infty: Q_\infty \to P_\infty \).

Example

The morphism \( T^{2k} \to T \otimes T \) defined by
\[
(v_1, w_1, \ldots, v_k, w_k) \mapsto v_1 \otimes w_1 + \ldots + v_k \otimes w_k
\]
induces a map \( (\mathbb{C}^N)^{2k} \to \mathbb{C}^N \times \mathbb{N} \) (defined the same).

Let \( p \in P_\infty \) be a point with projections \( p_n \in P(\mathbb{C}^n) \).

Lemma

We have \( p \in \text{im}(\alpha_\infty) \) if and only if \( p_n \in \text{im}(\alpha_{\mathbb{C}^n}) \) for all \( n \geq 1 \).

Proof.

Follows from a theorem by Lang stating that a countable system of polynomial equations over an uncountable field, any finite subsystem of which has a solution, has a solution.
Polynomial functors: Systems of variables

Let $P$ be a polynomial functor and $p \in P_\infty$ be a point.

**Definition**

We say that the point $p$ is $\text{GL}_\infty$-generic if $\text{GL}_\infty \cdot p = P_\infty$. Otherwise, the point is called degenerate.

**Lemma**

For $d \geq 2$, the set $\Omega_d$ of degenerate points in $S^d_\infty$ equals the subspace of points with finite strength.

**Proof.**

Follows from the Dichotomy Theorem.

**Definition**

A system of variables consists of a basis of $S^d_\infty / \Omega_d$ over all $d \geq 1$. 
Let $R, Q, P$ be direct sums of copies of $S^d$ with $d \geq 1$. Let $\beta : Q \to P$ and $\alpha : R \to P$ be morphisms. Let $q \in Q_{\infty}$ and $r \in R_{\infty}$ be points.

**Lemma**

Suppose that $q$ is $\text{GL}_{\infty}$ generic and $p := \beta_{\infty}(q) = \alpha_{\infty}(r)$. Then $\beta = \alpha \circ \gamma$ for some morphism $\gamma : Q \to R$.

**Proof.**

Extend $q$ to a system of variables. Express $r$ in these variables:

$$r = \delta(q, q'), \quad \delta : Q \oplus Q' \to R, \quad q' \in Q'_{\infty}$$

We have $\beta_{\infty}(q) = p = (\alpha \circ \delta)_{\infty}(q, q')$. So $p = (\alpha \circ \delta)_{\infty}(q, 0)$.

Take $\gamma = \delta(-, 0)$. Then $\beta = \alpha \circ \gamma$ since this holds on $\text{GL}_{\infty} \cdot q$. $\square$
The proof

We have the morphisms

\[ \alpha_k : (S^1 \oplus S^3)^{\oplus k} \oplus (S^2 \oplus S^2)^{\oplus 3-k} \rightarrow S^4 \]

\[ \left( (\ell_i, q_i)_i, (g_j, h_j)_j \right) \mapsto \sum_{i=1}^{k} \ell_i \cdot q_i + \sum_{j=1}^{3-k} g_j \cdot h_j \]

for \( k = 0, 1, 2, 3 \) and the morphism

\[ \beta : (S^1)^{\oplus 4} \oplus (S^2)^{\oplus 4} \rightarrow S^4 \]

\[ (x, y, u, v, f, g, h, q) \mapsto x^2 f + y^2 g + u^2 h + v^2 q \]

Goal

Prove that \( \beta_{\infty} (x, y, u, v, f, g, h, q) \notin \bigcup_{k=0}^{3} \text{im}(\alpha_{k,\infty}) \).

Enough

Prove that \( \beta = \alpha_k \circ \gamma \) has no solution for \( k = 0, 1, 2, 3 \).
The proof

Lemma
The equation \( \beta = \alpha_0 \circ \gamma \) has no solution.

Proof.
We have to prove that

\[
x^2 f + y^2 g + u^2 h + v^2 q \neq x_1 q_1 + x_2 q_2 + x_3 q_3
\]

with \( x_i, q_i \) polynomials in \( x, y, u, v, f, g, h, q \) of degrees 1, 3.

Coefficients of \( f, g, h, q \) on the left-hand side are \( x^2, y^2, u^2, v^2 \).

Coefficients of \( f, g, h, q \) on right-hand side are contained in the ideal \( (x_1, x_2, x_3) \subseteq k[x, y, u, v] \).

As \( x^2, y^2, u^2, v^2 \in (x_1, x_2, x_3) \) cannot hold, we have inequality. \( \square \)
The proof

Now we know that $\beta = \alpha_k \circ \gamma$ has no solution for $k = 0, 1, 2, 3$.

So $\beta_\infty(x, y, u, v, f, g, h, q) \notin \bigcup_{k=0}^{3} \text{im}(\alpha_{k,\infty})$ for $GL_\infty$-generic $(x, y, u, v, f, g, h, q)$.

So $\beta_{\mathbb{C}^n}(x_n, y_n, u_n, v_n, f_n, g_n, h_n, q_n) \notin \bigcup_{k=0}^{3} \text{im}(\alpha_{k,\mathbb{C}^n})$ for $n \gg 0$.

So the set

\[ \{ f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \text{str}(f) \leq 3 \} \]

is not Zariski-closed for $n \gg 0$.

Thanks for your attention!
Reference

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