1. Main result

Let K be a perfect field, G a connected reductive algebraic group defined over K and V an K-representation of G such that $G \times V \to V, (g, v) \mapsto g \cdot v$ is a morphism over K.

Theorem 1.1 (Kempf (1978))**.** *Let $v \in V$. Then the following are equivalent:*

(1) *There exists a morphism $\lambda\colon G_m \to G$ defined over K such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$.*
(2) *There exists a morphism $\lambda\colon G_m \to G$ defined over $\overline{K}$ such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$.*
(3) *We have $0 \in \overline{G(\overline{K}) \cdot v}$.*

Here $G_m = \{t \mid t \neq 0\} = \operatorname{Spec}(K[x, y]/(xy - 1))$ is an algebraic group defined over K. A morphism $\lambda\colon G_m \to G$ is called a *one-parameter subgroup.* Recall that we have the following result.

Theorem 1.2. *Let V be a representation of G_m (such that $G_m \times V \to V$ is a morphism defined over K). Then any vector $v \in V$ can be written unique as*

$$v = \sum_{i\in I(v)} v_i, \quad v_i \in V \setminus \{0\}$$

where $I(v)$ is a finite subset of $\mathbb{Z}$ and $t \cdot v = \sum_{i\in I(v)} t^i v_i$.

Definition 1.3. Let $\lambda\colon G_m \to G$ be a morphism and $v \in V$. Then $\lambda(t) \cdot v = \sum_{i\in I} w_i$ for some finite subset $I \subseteq \mathbb{Z}$.

(1) We write $\lim_{t\to 0} \lambda(t) \cdot v = 0$ when $I \subseteq \mathbb{Z}_{\geq 1}$.
(2) We write $\lim_{t\to 0} \lambda(t) \cdot v = w_0$ when $I \subseteq \mathbb{Z}_{\geq 0}$ and $0 \in I$.
(3) We say that $\lim_{t\to 0} \lambda(t) \cdot v$ does not exist when $I \not\subseteq \mathbb{Z}_{\geq 0}$.

◆

We have (1)⇒(2) and (2)⇒(3) since $\lim_{t\to 0} \lambda(t) \cdot v \in \overline{\lambda(G_m(\overline{K})) \cdot v} \subseteq \overline{G(\overline{K}) \cdot v}$ when the limit exists.

Theorem 1.4 (Mumford (1965))**.** *(3)⇒(2) holds.*

Proof. Assume that 0 lies in the closure of $G(\overline{K}) \cdot v$. Then there in fact exists a curve in $G(\overline{K}) \cdot v$ that has 0 in its closure. So there exists a rational map $p\colon C \to G$ defined over $\overline{K}$ and a $\sigma \in C(\overline{K})$ such that $\lim_{c\to\sigma} p(c) \cdot x = 0$. The completion of the local ring of C at σ is $\overline{K}[\![t]\!]$, this gives an element $q(t) \in G(\overline{K}[\![t]\!])$ with $\lim_{t\to 0} q(t) \cdot v = 0$. By the Cartan-Iwahori decomposition we have

$$q(t) = f(t)^{-1} \cdot \overline{\mu}(t) \cdot h(t)$$

for $f(t), \overline{\mu}(t), h(t) \in G(\overline{K}[\![t]\!])$ where $\overline{\mu}$ is obtained by by taking the Laurent series expansion at 0 of a morphism $\mu\colon G_m \to G$ defined over $\overline{K}$. Take $\lambda\colon G_m \to G, t \mapsto \mu(t) \cdot h(0)$. We now have

$$\lim_{t\to 0} \lambda(t) \cdot v = \lim_{t\to 0} \mu(t) \cdot h(0) \cdot v = \lim_{t\to 0} \mu(t) \cdot h(t) \cdot v = \lim_{t\to 0} f(t) \cdot q(t) \cdot v = f(0) \lim_{t\to 0} q(t) \cdot v = 0.$$

□

Idea of proof of (2)⇒(1). Assume that a one-parameter subgroup $\lambda\colon G_m \to G$ defined over $\overline{K}$ exists with $\lim_{t\to 0} \lambda(t) \cdot v = 0$. Now show that among all such one-parameter subgroups there is a unique one where $\lambda(t) \cdot v \to 0$ the "fastest" as $t \to 0$. All Galois conjugates are just as fast, hence $\lambda(t)$ is actually defined over K. □

We will write the set of one-parameter subgroups such that $\lim_{t\to 0} \lambda(t) \cdot v = 0$ as

$$\{\lambda \in \Gamma^*(G) \mid \mu(\Xi, \lambda) > 0\}.$$

We need to define the speed of convergence $\mu(\Upsilon, \lambda)$ of a one-parameter subgroup λ in this set. We need to define an "norm" on one-parameter subgroups.

2. One-parameter subgroups

In this section, assume that $K = \overline{K}$. Let $\Gamma(G)$ be the set of one-parameter subgroups of G and $\Gamma^*(G) := \Gamma(G) \setminus \{t \mapsto 1\}$ the set of non-trivial one-parameter subgroups of G. Let $X(G) := \mathrm{Hom}(G, G_m)$ be the set of characters of G. We have a pairing $\langle -, - \rangle$ between $X(G)$ and $\Gamma(T)$ such that $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$. When T is a torus of rank r, then $X(T) \cong \Gamma(T) \cong \mathbb{Z}^r$ and $\langle -, - \rangle$ is a perfect pairing. For general G, the group $G(K)$ acts on $\Gamma(G)$ by conjugation. Also define $m \cdot \lambda := t \mapsto \lambda(t^m)$ and say that λ is indivisible when it is not of the form $m \cdot \mu$ with $m > 1$.

Let T be a maximal torus of G and ler N be the normailizer of T. Then the Weyl group $W = N/T$ acts on $\Gamma(T)$ by conjugation.

Lemma 2.1.

1. *All maximal tori of G are conjugate.*
2. *For any maximal torus T of G, the inclusion $\Gamma(T) \subseteq \Gamma(G)$ induces a bijection $\Gamma(T)/W \to \Gamma(G)/G$.*
3. *For any W-invariant integral-valued positive definite bilinear form $(-, -)$ on $\Gamma(T)$, there exists a unique nonnegative function $||-||$ on $\Gamma(G)$ such that $||\lambda||^2 = (\lambda, \lambda)$ for $\lambda \in \Gamma(T)$ and $||g \cdot \lambda|| = ||\lambda||$ for all $\lambda \in \Gamma(G)$ and $g \in G(K)$.*

Example 2.2. Suppose that $G = \mathrm{SL}_n$ and $K = \mathbb{C}$. Then we can take T to be the set of all diagonal matrices in SL_n. We get $W = S_n$ and we have

$$\Gamma(T) = \left\{ t \mapsto \mathrm{Diag}(t^{a_1}, \ldots, t^{a_n}) \;\middle|\; \sum_i a_i = 0 \right\}$$

with W-invariant integral-valued positive definite bilinear form

$$(t \mapsto \mathrm{Diag}(t^{a_1}, \ldots, t^{a_n}), t \mapsto \mathrm{Diag}(t^{b_1}, \ldots, t^{b_n})) = \sum_i a_i b_i$$

In this case, the function $||-||$ is given by

$$||\lambda||^2 = \mathrm{Tr}\left(\left(\frac{d}{dt}\lambda(t)\bigg|_{t=1} \right)^2 \right).$$

♠

The parabolic subgroup $P(\lambda)$ of G consists of all elements p such that $\lim_{t \to 0} \lambda(t) \cdot p \cdot \lambda(t)^{-1}$ exists. A state assigns to every torus R of G a nonempty subset $\Xi(R) \subseteq X(R)$ such that $R_1 \subseteq R_2$ implies $\{\chi|_{R_1} \mid \chi \in \Xi(R_2)\} = \Xi(R_1)$. We define

$$\mu(\Xi, \lambda) := \min_{\chi \in \Xi(\mathrm{im}\,\lambda)} \langle \chi, \lambda \rangle$$

For $g \in G$, we define $G_* \Xi$ by

$$g_* \Xi(R) := \{ r \mapsto \chi(g^{-1} r g) \mid \chi \in \Xi(g^{-1} R g) \}.$$

A state Ξ is called bounded when $\bigcup_{g \in G} g_* \Xi(R)$ is a finite subset of $X(R)$ for all R. A state Ξ is called admissible when $\mu(\Xi, g \lambda g^{-1}) = \mu(\Xi, \lambda)$ for all $\lambda \in \Gamma(G)$ and $g \in G$.

Theorem 2.3. *Let Ξ, Υ be admissible bounded states such that $\{\lambda \in \Gamma^*(G) \mid \mu(\Xi, \lambda) > 0\}$ is nonempty.*

1. *The function $\mu(\Upsilon, \lambda)/||\lambda||$ achieves a maximum on this set.*
2. *This maximum is achieved by an indivisible one-parameter subgroup.*
3. *All such indivisible one-parameter subgroups have the same parabolic subgroup $P(\Xi, \Upsilon)$ of G.*
4. *Any maximal torus of $P(\Xi, \Upsilon)$ contains a unique indivisible one-parameter subgroup $\lambda \in \Gamma^*(G)$ with $\mu(\Xi, \lambda) > 0$ that achieves the maximum.*

Proof. The case where $G = T$ is not very interesting as $\Gamma(T)$ is a lattice inside a real vector space. For the general case, we know that $\mu(\Upsilon, \lambda)/||\lambda||$ has a maximum on $\{\lambda \in \Gamma^*(g^{-1} T g) \mid \mu(\Xi, \lambda) > 0\}$ for all $g \in G$ where this set is nonempty. This maximum is the same as the maximum of $\mu(g_* \Upsilon, \lambda)/||\lambda||$ on $\{\lambda \in \Gamma^*(T) \mid \mu(g_* \Xi, \lambda) > 0\}$. By the boundedness of Ξ, Υ, there are only finitely many values these maxima can take. The highest of these values is the maximum of $\mu(\Upsilon, \lambda)/||\lambda||$ on $\{\lambda \in \Gamma^*(G) \mid \mu(\Xi, \lambda) > 0\}$. This shows (1). Part (2) is easy. □

3. Limits

We still assume that $K = \overline{K}$. Let V be a representation of G and $v \in V \setminus \{0\}$. We write

$$\begin{aligned} |V, v| &:= \{\lambda \in \Gamma(G) \mid \lim_{t\to 0} \lambda(t)\cdot v \text{ exists}\} \\ |V, v|_0 &:= \{\lambda \in \Gamma(G) \mid \lim_{t\to 0} \lambda(t)\cdot v = 0\} \end{aligned}$$

Assume that $\lambda \in |V, v|$. Then we get a morphism $M_\lambda \colon \mathbb{A}^1 \to V, 0 \neq t \mapsto \lambda(t)\cdot v$. Now $M_\lambda^{-1}(0)$ is a divisor on $\mathbb{A}^1$ supported at 0 and we denote $a_v(\lambda)$ be its degree. We think of $a_v(\lambda)$ as the speed at which $\lambda(t)\cdot v$ converges to 0.

We now define a state $\Xi_{v,V}$ of G. Let R be a torus of G. Then we have a decomposition $V = \bigoplus_\chi V_\chi$ where R acts on V_χ via the character χ of R. We define $\Xi_{v,V}(R)$ to be the set of all characters χ of R such that $v_\chi \neq 0$. One can check that this defined as state.

Lemma 3.1. *The following hold:*

1. *We have $|V, v| = \{\lambda \in \Gamma(G) \mid \mu(\Xi_{v,V}, \lambda) \geq 0\}$.*
2. *We have $|V, v|_0 = \{\lambda \in \Gamma(G) \mid \mu(\Xi_{v,V}, \lambda) > 0\}$.*
3. *If $\lambda \in |V, v|$, then $a_v(\lambda) = \mu(\Xi_{v,V}, \lambda)$.*
4. *We have $g_*\Xi_{v,V} = \Xi_{g\cdot v,V}$ for all $g \in G$.*
5. *The state $\Xi_{v,V}$ is admissible and bounded.*

Proof. Let $\lambda \in \Gamma(G)$ and write $v = \sum_i v_i$ such that each v_i is nonzero and $\lambda(t)\cdot v = \sum_{i\in I} t^i v_i$. Then $\lambda \in |V, v|$ if and only if $I \subseteq \mathbb{Z}_{\geq 0}$ and $\lambda \in |V, v|_0$ if and only if $I \subseteq \mathbb{Z}_{\geq 1}$. Let R be any torus of G wich contains the image of λ. Then we have $I = \{\langle \chi, \lambda\rangle \mid \chi \in \Xi_{v,V}(R)\}$. So $I \subseteq \mathbb{Z}_{\geq 0}$ if and only if $\mu(\Xi_{v,V}, \lambda) \geq 0$ and $I \subseteq \mathbb{Z}_{\geq 1}$ if and only if $\mu(\Xi_{v,V}, \lambda) > 0$. This shows (1) and (2).

For (3), let $\lambda \in |V, v|$ and note that $M_\lambda(t) = \sum_{i\in I} t^i v_i$ where $v_i \neq 0$ and $I \subseteq \mathbb{Z}_{\geq 0}$. In this case, the degree of the divisor $M_\lambda^{-1}(0)$ is $\min I = \mu(\Xi_{v,V}, \lambda)$.

For (4), let R be any torus of G, take $g \in G$ and set $T = gRg^{-1}$. Then we have

$$g_*\Xi_{v,V}(T) := \{r \mapsto \chi(g^{-1}rg) \mid \chi \in \Xi_{v,V}(R)\}.$$

Note that $v = \sum_{\chi\in\Xi_{v,V}(R)} v_\chi$ where $v_\chi \neq 0$ and $r\cdot v_\chi = \chi(r)v_\chi$ for all $r \in R$. We have

$$t\cdot(g\cdot v_\chi) = g\cdot(g^{-1}tg\cdot v_\chi) = \chi(g^{-1}tg)(g\cdot v_\chi)$$

and so $g\cdot v = \sum_{\chi\in\Xi_{v,V}(R)} g\cdot v_\chi$ is a decomposition for T summing over $g_*\Xi_{v,V}(T)$. So $g_*\Xi_{v,V} = \Xi_{g\cdot v,V}$.

For (5), note that we can decompose $V = \bigoplus_{\chi\in\Omega(R)} V_\chi$ and hence see that

$$\bigcup_{g\in G} g_*\Xi_{v,V}(R) = \bigcup_{g\in G} \Xi_{g\cdot v,V}(R) \subseteq \Omega(R)$$

is a finite set. So $\Xi_{v,V}$ is bounded. Next we need to check admissibility: let $\lambda \in \Gamma(G)$ and $g \in P(\lambda) = \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}$. We need to check that $\mu(\Xi_{v,V}, g\lambda(t)g^{-1}) = \mu(\Xi_{v,V}, \lambda(t))$. The latter equals the $n \geq 0$ such that $\lim_{t\to 0} t^{-n}\lambda(t)\cdot v$ exists and does not equal 0. We have

$$\lim_{t\to 0} g\lambda(t)g^{-1}\cdot v = \lim_{t\to 0} g(\lambda(t)g^{-1}\lambda(t)^{-1})\cdot(t^{-n}\lambda(t)\cdot v) = g\lim_{t\to 0}\lambda(t)g^{-1}\lambda(t)^{-1}\cdot\lim_{t\to 0} t^{-n}\lambda(t)\cdot v$$

exists and is nonzero. Hence $\mu(\Xi_{v,V}, g\lambda(t)g^{-1}) = n = \mu(\Xi_{v,V}, \lambda(t))$. $\square$

4. The proof of (2)⇒(1)

We no longer assume that $K = \overline{K}$. Assume (2), so that $|V, v|_0 = \{\lambda \in \Gamma^*(G(\overline{K})) \mid \mu(\Xi_{v,V}, \lambda) > 0\}$ is nonempty. We now know the following (over $\overline{K}$):

1. The function $a_v(\lambda)/||\lambda||$ achieves a maximum on this set.
2. This maximum is achieved by an indivisible one-parameter subgroup.
3. All such indivisible one-parameter subgroups have the same parabolic subgroup $P_{v,V}$ of $G(\overline{K})$.
4. Any maximal torus of $P_{v,V}$ contains a unique indivisible one-parameter subgroup $\lambda \in \Gamma^*(G(\overline{K}))$ with $\mu(\Xi_{v,V}, \lambda) > 0$ that achieves the maximum.

Now we need to add "defined over K" in many places and use this uniqueness.

1. We say that $|| - ||$ is defined over K when $||\sigma \cdot \lambda|| = ||\lambda||$ for all $\lambda \in \Gamma(G)$ and $\sigma \in \mathrm{Gal}(\overline{K}/K)$. This exists: Take T any maximal torus of G defined over K. As T is split over a finite extension of K, the group $\mathrm{Gal}(\overline{K}/K)$ acts on $\Gamma(T(\overline{K}))$ by a finite set of automorphisms. Sum any integral-valued positive-definite inner product of $T(\overline{K})$ over all these automorphisms and all elements of the Weyl group to get a $|| - ||$ defined over K.
2. The sets $|V, v|$ and $|V, v|_0$ are $\mathrm{Gal}(\overline{K}/K)$-invariant.
3. We have $a_v(\lambda) = a_v(\sigma \cdot \lambda)$ for all $\lambda \in |V, v|$ and $\sigma \in \mathrm{Gal}(\overline{K}/K)$.
4. The set of indivisible one-parameter subgroup that achieves the maximum is $\mathrm{Gal}(\overline{K}/K)$-invariant.
5. The parabolic subgroup $P_{v,V}$ is $\mathrm{Gal}(\overline{K}/K)$-invariant.
6. Let T be a maximal torus of $P_{v,V}$ defined over K and let $\lambda \in \Gamma^*(G(\overline{K}))$ be the unique one-parameter subgroup of $G(\overline{K})$ with $\mu(\Xi_{v,V}, \lambda) > 0$. Then $\sigma \cdot \lambda = \lambda$ for all $\sigma \in \mathrm{Gal}(\overline{K}/K)$ and hence λ is defined over K. This proves (2)⇒(1).

References

[1] George R. Kempf, *Instability in Invariant Theory*, Annals of Mathematics 108 (1978), No. 2, pp. 299–316.