Strength of polynomials via polynomial functors

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The strength of polynomials

Let f be a homogeneous polynomial of degree $d \geq 2$ over $\mathbb{C}$.

Definition

The *strength* of f is the minimal number $\mathrm{str}(f) := r \geq 0$ such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d-1$.

Examples

(0) $\mathrm{str}(0) = 0$

(1) $\mathrm{str}((x^2 + xy + y^2) \cdot (u^3 + uvw + v^3)) = 1$

(2) The polynomial

$$\begin{aligned} x^2 + y^2 + z^2 &= x \cdot x + y \cdot y + z \cdot z \\ &= (x + iy) \cdot (x - iy) + z \cdot z \end{aligned}$$

has strength 2. (It would be 3 over $\mathbb{R}$)

(3) $\mathrm{str}(x_1 \cdot g_1 + x_2 \cdot g_2 + \ldots + x_n \cdot g_n) \leq n$

Why care about strength?

A coordinate transformation of $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ is

$f(c_{11}y_1 + \ldots + c_{1m}y_m, \ldots, c_{n1}y_1 + \ldots + c_{nm}y_m) \in \mathbb{C}[y_1, \ldots, y_m]_d$

Let $\mathcal{P}$ be a property of degree-d polynomials such that

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{every coordinate transformation of } f \text{ has } \mathcal{P}$$

Example

$\mathcal{P} =$ "has strength $\leq k$" for fixed $k \geq 0$.

Example (Kazhdan-Ziegler)

$\mathcal{P} =$ "all partial derivatives have strength $\leq k$" for fixed $k \geq 0$.

Theorem (Kazhdan-Ziegler, B-Draisma-Eggermont)

One of the following holds:

(1) Every polynomial has $\mathcal{P}$.

(2) There exists an $\ell \geq 0$ such that f has $\mathcal{P} \Rightarrow \mathrm{str}(f) \leq \ell$.

Properties of strength

$\mathbf{Q}_{d,k,n}$: Is $\{f \in \mathbb{C}[x_1, \dots, x_n]_d \mid \mathrm{str}(f) \leq k\}$ closed?

For $k = 1$, yes. (Union of images of projective morphisms).

For $k = 2$, I don't know. (Conjecture: yes)

For $d = 2$, yes. (rank of symmetric matrices)

For $d = 3$, yes. (slice rank of polynomials)

Theorem (Ballico-B-Oneto-Ventura)

The $\{f \in \mathbb{C}[x_1, \dots, x_n]_4 \mid \mathrm{str}(f) \leq 3\}$ is not closed for $n \gg 0$.

Consider

$$^1/_t(x^2+tg)(y^2+tf) - {^1/_t}(u^2-tq)(v^2-tp) - {^1/_t}(xy-uv)(xy+uv)$$
$$=$$
$$x^2f + y^2g + u^2p + v^2q + t(fg - pq)$$

It has strength ≤ 3. For $t \to 0$, we get $x^2f + y^2g + u^2p + v^2q$.

Strength ≤ 3 is not closed

Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, there are polynomials $f, g, p, q \in \mathbb{C}[z_1, \dots, z_n]_2$ such that

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, z_1, \dots, z_n]_4$$

has strength 4.

Consider the polynomial

$$h := x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

where x, y, u, v have degree 1 and $\underbrace{f, g, p, q}_{\text{variables}}$ have degree 2.

Proposition

The polynomial h has strength 4.

Strength ≤ 3 is not closed

Definition

The strength of a polynomial $h \in \mathbb{C}[x, y, u, v, f, g, p, q]_d$ is the minimum number $r \geq 0$ (when this exists) such that

$$h = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

with $g_1, h_1, \ldots, g_r, h_r$ homogeneous polynomials of degree $\leq d - 1$.

Example

The polynomial

$$f \cdot g + x \cdot (uh + v^3)$$

is irreducible and hence has strength 2.

Example

When the g_i, h_i have degree 1, then

$$g_1 \cdot h_1 + \ldots + g_r \cdot h_r \in \mathbb{C}[x, y, u, v]_2$$

Hence the variable f has infinite strength.

Strength ≤ 3 is not closed

Proposition

The polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

1/4 of the proof

We need to show, for example, that

$$x^2 f + y^2 g + u^2 p + v^2 q \neq \ell_1 \cdot h_1 + \ell_2 \cdot h_2 + \ell_3 \cdot h_3$$

for all $\ell_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_1$ and $h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_3$.

Strength ≤ 3 is not closed

Proposition

The polynomial

$$x^2 f + y^2 g + u^2 p + v^2 q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

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We need to show, for example, that

$$x^2 f + y^2 g + u^2 p + v^2 q \neq \ell_1 \cdot h_1 + \ell_2 \cdot h_2 + \ell_3 \cdot h_3$$

for all $\ell_i \in \mathbb{C}[x, y, u, v]_1$ and $h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_3$.

Think of $R = \mathbb{C}[x, y, u, v]$ as the set of coefficients.
So $\ell_i \in R$ and $h_i \in R[f, g, p, q]$.

The coefficients of f, g, p, q on the right are all in (ℓ_1, ℓ_2, ℓ_3).
The coefficients x^2, y^2, u^2, v^2 on the left are not all (ℓ_1, ℓ_2, ℓ_3). $\square$

Strength ≤ 3 is not closed

Proposition

The polynomial

$$x^2f + y^2g + u^2p + v^2q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

$\vdots$

$\vdots$

Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, there are polynomials $f, g, p, q \in \mathbb{C}[z_1, \ldots, z_n]_2$ such that

$$x^2f + y^2g + u^2p + v^2q \in \mathbb{C}[x, y, u, v, z_1, \ldots, z_n]_4$$

has strength 4.

How to bridge the gap?

Polynomial functors

Definition

The polynomial functor $S^d\colon \mathrm{Vec} \to \mathrm{Vec}$ is the functor

$$\begin{array}{rcl} V & \mapsto & S^d(V) \\ (L\colon V \to W) & \mapsto & \left(S^d(L)\colon S^d(V) \to S^d(W)\right) \\ \mathbb{C}\,x_1 \oplus \cdots \oplus \mathbb{C}\,x_n & \mapsto & \mathbb{C}[x_1,\dots,x_n]_d \\ (x_i \mapsto \sum_j c_{ij} y_j) & \mapsto & (x_i \mapsto \sum_j c_{ij} y_j) \end{array}$$

Definition

A polynomial transformation

$$\alpha\colon S^{d_1} \oplus \cdots \oplus S^{d_k} \to S^{e_1} \oplus \cdots \oplus S^{e_\ell}$$

is of the form

$$(f_1,\dots,f_k) \mapsto (F_1(f_1,\dots,f_k),\dots,F_\ell(f_1,\dots,f_k))$$

Here $F_j \in \mathbb{C}[X_1,\dots,X_k]_{e_j}$ are fixed forms with $\deg(X_i) = d_i$.

Polynomial functors

Example

$$(g_1, h_1, g_2, h_2, g_3, h_3) \mapsto g_1 \cdot h_1 + g_2 \cdot h_2 + g_3 \cdot h_3$$

defines a polynomial transformation

$$\alpha\colon (S^{d_1} \oplus S^{4-d_1}) \oplus (S^{d_2} \oplus S^{4-d_2}) \oplus (S^{d_3} \oplus S^{4-d_3}) \to S^4$$

for all fixed $1 \leq d_1 \leq d_2 \leq d_3 \leq 2$.

Definition

We define the inverse limit

$$S^d_\infty := \{\text{degree-}d \text{ series in } x_1, x_2, \dots\} \ni x_1^d + x_2^d + x_3^d + \dots$$

Proposition (B-Draisma-Eggermont-Snowden)

Let $p \in S^d_\infty$ be a series with projections $p_n \in \mathbb{C}[x_1, \dots, x_n]_d$ and $\alpha\colon P \to S^d$ a polynomial transformation. Then

$$p \in \mathrm{im}(\alpha_\infty) \Leftrightarrow p_n \in \mathrm{im}(\alpha_n) \text{ for all } n$$

Take $p = x^2 f + y^2 g + u^2 p + v^2 q$ for series some $f, g, p, q \in S^2_\infty$.

Polynomial functors

Definition

Write $D^d \subseteq S^d_\infty$ for the subspace of finite strength series.
A system of variables consists of a basis of S^d_∞ / D^d for every $d \geq 1$.

Proposition (B-Draisma-Eggermont-Snowden)

Let $\beta\colon S^{e_1} \oplus \cdots \oplus S^{e_k} \to S^d$ and $\alpha\colon P \to S^d$ be polynomial transformations. Let $f_1 \in S^{e_1}_\infty, \ldots, f_k \in S^{e_k}_\infty, p \in P_\infty$ be a series. Assume that $\beta_\infty(f_1, \ldots, f_k) = \alpha_\infty(p)$ and that $(f_1, \ldots, f_k)$ is part of a system of variables. Then there exists a polynomial transformation $\gamma\colon S^{e_1} \oplus \cdots \oplus S^{e_k} \to P$ such that $\beta = \alpha \circ \gamma$.

Example (which closes the gap)

Take

$$\beta(x, y, u, v, f, g, p, q) = x^2 f + y^2 g + u^2 p + v^2 q$$
$$\alpha(g_1, h_1, g_2, h_2, g_3, h_3) = g_1 \cdot h_1 + g_2 \cdot h_2 + g_3 \cdot h_3$$

Strength ≤ 3 is not closed

Proposition

The polynomial

$$x^2f + y^2g + u^2p + v^2q \in \mathbb{C}[x, y, u, v, f, g, p, q]_4$$

has strength 4.

$\Big\Downarrow$ Polynomial functors

Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, there are polynomials $f, g, p, q \in \mathbb{C}[z_1, \dots, z_n]_2$ such that

$$x^2f + y^2g + u^2p + v^2q \in \mathbb{C}[x, y, u, v, z_1, \dots, z_n]_4$$

has strength 4.

Thanks for your attention!

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