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Strength and polynomial functors

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The rank of infinite-by-infinite matrices

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Definition: The rank of an $\mathbb{N} \times \mathbb{N}$ matrix A is

$$\mathrm{rk}(A) := \sup\{\mathrm{rk}(B) \mid \text{finite submatrices } B \text{ of } A\} \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$$

Lemma

$A \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ has rank $\leqslant k \Leftrightarrow A = \sum_{i=1}^{k} v_i w_i^T$ with $v_i, w_i \in \mathbb{C}^{\mathbb{N}}$

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Proof. An equation on $\mathbb{C}^{\mathbb{N} \times \mathbb{N}}$ uses only finitely many rows and columns. So non-zero equations on $\mathrm{GL}_\infty \cdot A \cdot \mathrm{GL}_\infty$ give rank constraints on A. $\square$

Fact: An $n \times m$ matrix A has rank $\min(n, m) \Leftrightarrow \overline{\mathrm{GL}_n \cdot A \cdot \mathrm{GL}_m} = \mathbb{C}^{n \times m}$

Other Examples/Theorems

Definition: The rank of a tuple of $\mathbb{N} \times \mathbb{N}$ matrices $A_1, \dots, A_k$ is

$$\operatorname{rk}(A_1, \dots, A_k) := \inf\{\operatorname{rk}(\lambda_1 A_1 + \cdots + \lambda_k A_k) \mid (\lambda_1 : \cdots : \lambda_k) \in \mathbb{P}^{k-1}\}$$

Example/Theorem (Draisma-Eggermont)

$$\operatorname{rk}(A_1, \dots, A_k) = \infty \Leftrightarrow \overline{\mathrm{GL}_\infty \cdot (A_1, \dots, A_k) \cdot \mathrm{GL}_\infty} = (\mathbb{C}^{\mathbb{N} \times \mathbb{N}})^k$$

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Definition: The q-rank of a series

$$f = a_{111}x_1^3 + a_{112}x_1^2x_2 + \cdots + a_{ijk}x_ix_jx_k + \ldots$$

is the minimal $k \leqslant \infty$ such that $f = \ell_1 q_1 + \cdots + \ell_k q_k$ with $\deg(\ell_i) = 1$.

Example/Theorem (Derksen-Eggermont-Snowden)
$\mathrm{qrk}(f) = \infty \Leftrightarrow \overline{\mathrm{GL}_\infty \cdot f} = \varprojlim_n \mathbb{C}[x_1, \ldots, x_n]_{(3)}$

Other Examples/Theorems

Take $d \geqslant 2$.

Definition (Ananyan-Hochster)
The strength of a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]_{(d)}$ is the minimal k such that

$$f = g_1 h_1 + \cdots + g_k h_k$$

with $g_1, \ldots, g_k, h_1, \ldots, h_k \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous of degree $< d$.

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Example/Theorem (B-Draisma-Eggermont, Kazhdan-Ziegler)
For every n, let $X_n \subseteq \mathbb{C}[x_1, \ldots, x_n]_{(d)}$ be a closed subset such that:

$(*)$ We have $f \circ \ell \in X_m$ for all $f \in X_n$ and all linear maps $\ell \colon \mathbb{C}^m \to \mathbb{C}^n$.

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Then either $X_n = \mathbb{C}[x_1, \dots, x_n]_{(d)}$ for all $n \geqslant 0$ or there is a $k < \infty$ such that $\operatorname{str}(f) \leqslant k$ for all $f \in X_n$ and $n \geqslant 0$.

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Remark: This version implies the infinite version using Lang's theorem.

Polynomial functors

Vec = category of finite-dimensional vector spaces over $\mathbb{C}$.

Definition

A polynomial functor P assigns to $V \in \mathrm{Vec}$ a $P(V) \in \mathrm{Vec}$ and to $(V, W) \in \mathrm{Vec}^2$ a polynomial map $\mathrm{Hom}_{\mathbb{C}}(V, W) \to \mathrm{Hom}_{\mathbb{C}}(P(V), P(W))$ such that $P(\mathrm{id}_V) = \mathrm{id}_{P(V)}$ for all $V \in \mathrm{Vec}$ and $P(\varphi \circ \psi) = P(\varphi) \circ P(\psi)$ for all linear maps $\psi \colon V \to W$ and $\varphi \colon W \to U$.

Examples

- Constants: $V \mapsto U$ for $U \in \mathrm{Vec}$ fixed.
- Linear functors: $V \mapsto U \otimes V$ for $U \in \mathrm{Vec}$ fixed.
- Matrices: $V \mapsto V \otimes V$
- Polynomials: $V \mapsto S^d V$

Remark: The class of polynomial functors is closed under direct sums, tensor products, quotients and subfunctors. Polynomial functors have a degree. (This can be infinite, but we don't consider such poly functors.)

Definition

Let P, Q be polynomial functors. A polynomial transformation $\alpha\colon Q \to P$ is a family $(\alpha_V\colon Q(V) \to P(V))_{V \in \mathrm{Vec}}$ of polynomial maps such that

$$\begin{array}{ccc} Q(V) & \xrightarrow{\alpha_V} & P(V) \\ \downarrow{\scriptstyle Q(\ell)} & & \downarrow{\scriptstyle P(\ell)} \\ Q(W) & \xrightarrow{\alpha_W} & P(W) \end{array}$$

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Polynomial transformations and Closed subsets of polynomial functors

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Definition

A closed subset $X \subseteq P$ of a polynomial functor assigns to each $V \in \mathrm{Vec}$ a closed subset $X(V) \subseteq P(V)$ such that $p(\varphi)(X(V)) \subseteq X(W)$ for all linear maps $\ell\colon V \to W$.

The dichotomy

Let P, Q be polynomial functors. Write $Q \prec P$ when $Q_{(d)}$ is a quotient of $P_{(d)}$ where d is maximal with $Q_{(d)} \ncong P_{(d)}$.

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Theorem (B-Draisma-Eggermont-Snowden)
Let $X \subseteq P$ be a closed subset. Then $X = P$ or there are $Q_1, \ldots, Q_k \prec P$ and $\alpha_i \colon Q_i \to P$ such that $X \subseteq \bigcup_i \mathrm{im}(\alpha_i)$.

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Examples

- {matrices of rank $\leqslant k$} $= \{v_1 w_1^T + \cdots + v_k w_k^T \mid v_i, w_i$ vectors}
- {degree d polynomials that are zero on a codim k subspace} $=$ $\{\ell_1 g_1 + \cdots + \ell_k g_k \mid \deg(\ell_i) = 1, \deg(g_i) = d - 1\}$

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  Proof. Using induction on P:
  Take $Q_1, \dots, Q_k \prec P$ and $\alpha_i \colon Q_i \to P$ such that $X_1 \subseteq \bigcup_i \mathrm{im}(\alpha_i)$ and pull back the chain of closed subsets along each α_i. The resulting chains all have to stabilize.

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- **Theorem** (B-Draisma-Eggermont-Snowden)
  The map $\alpha \mapsto \overline{\mathrm{im}(\alpha)}$ is a surjection from
  $\{$polynomial transformations into $P\}$ to
  $\left\{\text{closures of } \mathrm{GL}_\infty\text{-orbits in } \varprojlim_n P(\mathbb{C}^n)\right\}$.

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