Curves with a rational MLE and chipfiring games

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Discrete statistical models

Let $\Delta_n$ be $\{(p_0, p_1, \ldots, p_n) \in \mathbb{R}^{n+1}_{>0} \mid p_0 + p_1 + \ldots + p_n = 1\}$.

**Definition**

A *discrete statistical model* is a subset $\mathcal{M}$ of $\Delta_n$. The points of $\mathcal{M}$ represent probability distributions on the set $\{0, 1, \ldots, n\}$.

**Definition**

The *maximum likelihood estimator* (MLE) of $\mathcal{M}$ is the function

$$\Phi: \Delta_n \to \mathcal{M}$$

such that $(\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_n) = \Phi(u_0, u_n, \ldots, u_n)$ maximizes over $\mathcal{M}$ the chance that distribution $(u_0, u_1, \ldots, u_n)$ is observed from an experiment.
Example

Flip a biased coin. When $H$ flip again. Record the outcomes.

\[
\mathcal{M} = \{(p^2, p(1-p), 1-p) \mid p \in (0, 1)\}
\]
\[ \mathcal{M} = \{(p^2, p(1-p), 1-p) \mid p \in (0, 1)\} \]

Assume that \( a + b + c \) experiments results in outcomes:

\[ a \times \text{HH}, \quad b \times \text{HT}, \quad c \times \text{T} \]

What value of \( p \) maximizes the following?

\[
\left( \begin{array}{c} a + b + c \\ a, b, c \end{array} \right) \cdot (p^2)^a \cdot (p(1-p))^b \cdot (1-p)^c
\]

\[ \sim (2a + b)/\hat{p} - (b + 2c)/(1 - \hat{p}) = 0 \Rightarrow \\
\hat{p} = \frac{2a + b}{2a + 2b + c} \quad \text{and} \quad 1 - \hat{p} = \frac{b + c}{2a + 2b + c} \]
Discrete statistical models

Example

Flip a biased coin. When $H$ flip again. Record the outcomes.

$$
\mathcal{M} = \{(p^2, p(1-p), 1-p) \mid p \in (0, 1)\}
$$

$$
\Phi(a, b, c) = \left( \left( \frac{2a + b}{2a + 2b + c} \right)^2 , \frac{2a + b}{2a + 2b + c} \cdot \frac{b + c}{2a + 2b + c} , \frac{b + c}{2a + 2b + c} \right)
$$
Discrete statistical models

Example

Flip a biased coin twice. When same outcomes flip again.
Record $HHH$, $TTT$ or other.

$$M = \{(p^3, 3p(1-p), (1-p)^3) \mid p \in (0,1)\} \text{ and } \hat{p} = \frac{3a + b}{3a + 2b + 3c}$$
Theorem (Duarte, Marigliano, Sturmfels)

The following are equivalent:

1. The model $\mathcal{M}$ has a rational MLE.
2. There exists a Horn pair $(H, \lambda)$ such that $\mathcal{M}$ is the image of the Horn map.
3. There exists a discriminantal triple $(A, \Delta, m)$ such that $\mathcal{M}$ is the image of the associated map.

Question (Duarte, Marigliano, Sturmfels)

Can models with a rational MLE be classified?

Today (with Orlando Marigliano)

We focus on curves, i.e. models of dimension 1.
Curves with a rational MLE

**Theorem**

Let $\mathcal{M} \subseteq \Delta_n$ be a model of dimension 1 with a rational MLE. Then

$$\mathcal{M} = \{ (\lambda_0 t^{i_0} (1-t)^{j_0}, \lambda_1 t^{i_1} (1-t)^{j_1}, \ldots, \lambda_n t^{i_n} (1-t)^{j_n}) \mid t \in (0, 1) \}$$

for some $\lambda_\nu \in \mathbb{R}_{>0}$ and monomials $t^{i_\nu} (1-t)^{j_\nu}$ in $t, 1-t$ such that

$$\lambda_0 t^{i_0} (1-t)^{j_0} + \lambda_1 t^{i_1} (1-t)^{j_1} + \ldots + \lambda_n t^{i_n} (1-t)^{j_n} = 1$$

as polynomials.

**Proof.**

($\Leftarrow$) Compute the MLE.
($\Rightarrow$) Models with rational MLE are unirational.
Model consists of data \((\lambda_\nu, i_\nu, j_\nu)\) for \(\nu = 0, \ldots, n\) such that
\[
\lambda_0 t^{i_0} (1 - t)^{j_0} + \lambda_1 t^{i_1} (1 - t)^{j_1} + \ldots + \lambda_n t^{i_n} (1 - t)^{j_n} = 1.
\]

**Reductions**

1. If \((i_\nu, j_\nu) = (0, 0)\), discard \((\lambda_\nu, i_\nu, j_\nu)\) and scale by \((1 - \lambda_n u)^{-1}\).
2. If \((i_\nu, j_\nu) = (i_\nu', j_\nu')\), combine them (by adding \(\lambda_\nu\) and \(\lambda_{\nu'}\)).

We assume the model is *reduced*, i.e. all \((i_\nu, j_\nu)\) distinct from \((0, 0)\) and from each other.

**Proposition**

The data \((\lambda_\nu, i_\nu, j_\nu)\) for \(\nu = 0, \ldots, n\) form a model \(\iff\)

\[
-1 + \lambda_0 x^{i_0} y^{j_0} + \lambda_1 x^{i_1} y^{j_1} + \ldots + \lambda_n x^{i_n} y^{j_n} = (x + y - 1) \sum_{i,j=0}^{\infty} f_{i,j} x^i y^j
\]

for some \(f_{i,j} \in \mathbb{R}\) almost all zero.
Chipsplitting games

Let $G = (V, E)$ be a (fixed) directed graph without loops. Let $v_0 \in V$ have at least 1 outgoing edge $(v_0, v) \in E$.

**Definition**

1. A chip configuration is a tuple $w = (w_v)_{v \in V} \in \mathbb{Z}^V$.
2. A chipsplitting move at $v_0$ sends $w$ to $\tilde{w}$ defined by
   \[
   \tilde{w}_v = \begin{cases} 
   w_v - 1 & \text{if } v = v_0, \\
   w_v + 1 & \text{if } (v_0, v) \in E, \\
   w_v & \text{otherwise}
   \end{cases}
   \]
   An unsplitting move at $v_0$ is its inverse.
3. The initial configuration $w$ is given by $w_v = 0$ for all $v \in V$.
4. A chipsplitting game $f$ is a finite sequence of moves.
5. The outcome of $f$ is the result of applying all moves starting from the initial configuration.
Chipsplitting games

Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$V_d := \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid \text{deg}(i, j) \leq d\}$$

$$E_d := \{(v, v + e) \mid v \in V_{d-1}, e \in \{(1, 0), (0, 1)\}\}$$

where $\text{deg}(i, j) := i + j$.

**Example**

We apply a splitting move at the red vertex.

\[
\begin{array}{ccccccc}
0 & & & & & & \\
0 & 0 & & & & & \\
0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Chipsplitting games

Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$V_d := \{ (i, j) \in \mathbb{Z}^2_\geq 0 \mid \text{deg}(i, j) \leq d \}$$

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\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

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where $\deg(i, j) := i + j$.

**Example**

We apply a splitting move at the red vertex.

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Let \( d \in \{1, 2, 3, \ldots, \infty\} \). Define

\[
\begin{align*}
V_d & := \{(i, j) \in \mathbb{Z}^2_{\geq 0} \mid \deg(i, j) \leq d\} \\
E_d & := \{(v, v + e) \mid v \in V_{d-1}, e \in \{(1, 0), (0, 1)\}\}
\end{align*}
\]

where \( \deg(i, j) := i + j \).

**Example**

We apply a splitting move at the red vertex.

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]
Chipsplitting games

Let \( d \in \{1, 2, 3, \ldots, \infty\} \). Define

\[
V_d := \{(i, j) \in \mathbb{Z}_{\geq 0}^2 | \deg(i, j) \leq d\}
\]

\[
E_d := \{(v, v + e) | v \in V_{d-1}, e \in \{(1, 0), (0, 1)\}\}
\]

where \( \deg(i, j) := i + j \).

Example

We apply a splitting move at the red vertex.

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]
Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$V_d := \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid \deg(i, j) \leq d\}$$

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where $\deg(i, j) := i + j$.

**Example**

We apply a splitting move at the red vertex.

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]
Let $d \in \{1, 2, 3, \ldots, \infty\}$. Define

$$V_d := \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid \text{deg}(i, j) \leq d\}$$

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where $\text{deg}(i, j) := i + j$.

**Example**

We apply a splitting move at the red vertex.

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
$$
Proposition

The data \((\lambda_\nu, i_\nu, j_\nu)\) for \(\nu = 0, \ldots, n\) form a model ⇔

\[-1 + \lambda_0 x^0 y^0 + \lambda_1 x^1 y^1 + \ldots + \lambda_n x^n y^n = (x+y-1) \sum_{i,j=0}^{\infty} f_{i,j} x^i y^j\]

for some \(f_{i,j} \in \mathbb{R}\) almost all finite.

Assume the model is reduced and set

\[w_{i,j} = \begin{cases} 
\lambda_\nu & \text{if } (i,j) = (i_\nu, j_\nu), \\
-1 & \text{if } (i,j) = (0,0), \\
0 & \text{otherwise}
\end{cases}\]

Then \((w_{i,j})_{(i,j) \in V_d}\) is the outcome of the chipsplitting game where \((i,j)\) is split \(f_{i,j}\) times (where unsplitting moves count negatively).
Chipsplitting games

Definition

1. A chip configuration $w$ is valid when $w_{i,j} \geq 0$ for all $(i, j) \neq (0, 0)$.
2. The *positive support* of $w$ is $\text{supp}^+(w) := \{(i, j) \mid w_{i,j} > 0\}$.
3. The *degree* of $w$ is $\text{deg}(w) := \max\{\text{deg}(i, j) \mid w_{i,j} \neq 0\}$.

Conjecture

Let $w$ be a valid outcome. Then $\text{deg}(w) \leq 2 \cdot \# \text{supp}^+(w) - 3$.

Why is the nice?

- The conjecture gives a bound of the degree of the parametrisation of a dimension-1 curve with a rational MLE.
- The conjecture shows that there are finitely many ”fundamental” models in $\Delta_n$, which can be used to get any other model.
Composite models

Definition
A model \( \{ (\lambda_\nu, i_\nu, j_\nu) \mid \nu = 0, \ldots, n \} \) is fundamental when the \( \lambda_\nu \) are unique given the \( i_\nu, j_\nu \).

Composition
Let \( \mu \in (0, 1) \). The \( \mu \)-composite of models

\[
\{ (\lambda_{i,j}, i, j) \mid (i, j) \in S \}, \quad \{ (\lambda'_{i,j}, i, j) \mid (i, j) \in S' \}
\]

is the model

\[
\{ (\lambda_{i,j} + \lambda'_{i,j}, i, j) \mid (i, j) \in S \cup S' \}
\]

where \( \lambda_{i,j} := 0 \) for all \( (i, j) \not\in S \) and \( \lambda'_{i,j} := 0 \) for all \( (i, j) \not\in S' \).

Theorem
Every reduced model in \( \Delta_n \) is a composite of \( \leq n \) fundamental models (from \( \Delta_m \) with \( m < n \)).
Chipsplitting games

Conjecture

Let \( w \) be a valid outcome. Then \( \deg(w) \leq 2 \cdot \# \text{supp}^+(w) - 3 \).

Why believe the conjecture?

- Computer search for low degree. \( \left( \frac{1}{2}(\deg(w) + 3) \leq \# \text{supp}^+(w) \right) \)
- Take \( d = 2k + 1 \). Let \( w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d} \) be defined by

\[
\begin{align*}
    w_{0,0} &= -1, \\
    w_{0,2k+1} &= 1, \\
    w_{2i+1,k-i} &= \frac{2k + 1}{2i + 1} \binom{k + i}{2i}, \quad i \in \{0, 1, \ldots, k\}
\end{align*}
\]

and \( w_{i,j} = 0 \) otherwise. Then \( w \) is a valid outcome.

\[
\deg(w) = 2k + 1 = 2 \cdot (k + 2) - 3 = 2 \cdot \# \text{supp}^+(w) - 3
\]
Main results

Conjecture
Let $w$ be a valid outcome. Then $\deg(w) \leq 2 \cdot \# \text{supp}^+(w) - 3$.

Main result
The conjecture holds when $\# \text{supp}^+(w) \leq 5$.

Corollary
Let
\[ \mathcal{M} = \{(\lambda_0 t^{i_0}(1-t)^{i_0}, \lambda_1 t^{i_1}(1-t)^{i_1}, \ldots, \lambda_n t^{i_n}(1-t)^{i_n}) \mid t \in (0, 1)\} \]
be a model with a rational MLE.

1. If $n = 1$, then $\max_{\nu}(i_{\nu} + j_{\nu}) \leq 1$.
2. If $n = 2$, then $\max_{\nu}(i_{\nu} + j_{\nu}) \leq 3$.
3. If $n = 3$, then $\max_{\nu}(i_{\nu} + j_{\nu}) \leq 5$.
4. If $n = 4$, then $\max_{\nu}(i_{\nu} + j_{\nu}) \leq 7$. 
The proof

Conjecture

Let $w$ be a valid outcome. Then $\deg(w) \leq 2 \cdot \# \supp^+(w) - 3$.

We aim to prove that certain chip configurations cannot be the outcome of a chipsplitting game.

Here are the tools:

1. Invertibility Criterion
2. Hyperfield Criterion
3. Hexagon Criterion
4. A computer
Pascal equations

For \((k, \ell) \in V_{d-1}\), take \(E^{(k, \ell)} \in \mathbb{Z}^V_d\) so that

\[
E^{(k, \ell)}_{i,j} = \begin{cases} 
1 & \text{when } (i, j) \in \{(k + 1, \ell), (k, \ell + 1)\}, \\
-1 & \text{when } (i, j) = (k, \ell), \\
0 & \text{otherwise}
\end{cases}
\]

Then \(\text{span}_{\mathbb{Z}} \{E^{(k, \ell)} \mid (k, \ell) \in V_{d-1}\}\) is the space of outcomes.

Definition

A Pascal equation on \(\mathbb{Z}^V_d\) is a linear form

\[
\sum_{(i,j) \in V_d} c_{i,j} x_{i,j}
\]

such that \(c_{i,j} = c_{i+1,j} + c_{i,j+1}\) for all \((i, j) \in V_{d-1}\).

We have \(\{\text{outcomes}\} = V(\text{Pascal equations})\).
The Invertibility Criterium

For \(a, b \geq 0\) with \(a + b = d\), define

\[
\varphi_{a,b} := \sum_{i=0}^{a} \sum_{j=0}^{b} \left( d - (i + j) \right) x_{i,j} = \sum_{i=0}^{a} \sum_{j=0}^{b} \left( d - (i + j) \right) x_{i,j}
\]

For \(w \in \mathbb{Z}^V_d\), define \(\text{supp}(w) := \{(i, j) \in V_d \mid w_{i,j} \neq 0\} \subseteq V_d\).

**Invertibility Criterium**

Let \(S \subseteq V_d\) and \(E \subseteq \{(a, b) \in V_d \mid a + b = d\}\) be subsets of the same size. Let \(w \in \mathbb{Z}^V_d\) be an outcome. Suppose that the matrix

\[
A_{E,S} := \left( \begin{pmatrix} d - (i + j) \\ a - i \end{pmatrix} \right)_{a \in E, (i,j) \in S}
\]

is invertible. Then \(\text{supp}(w) \neq S\).
The Invertibility Criterium

How to apply it?

1. Split into pieces.
2. Use symmetry:
   We have an action of $S_3$ on $\mathbb{Z}^{V_d}$ given by

$$
(12) \cdot (w_{i,j})_{(i,j) \in V_d} := (w_{j,i})_{(i,j) \in V_d} \\
(13) \cdot (w_{i,j})_{(i,j) \in V_d} = ((-1)^{d-j} w_{d-(i+j),j})_{(i,j) \in V_d}
$$

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The Hyperfield Criterium

Definition

A hyperfield is a tuple \((H, +, \cdot, 0, 1)\) where ...

Example (Sign hyperfield)

Take \(H = \{1, 0, -1\}\) with usual multiplication and

\[
s + r := \{\text{sign}(x + y) \mid x, y \in \mathbb{R}, \text{sign}(x) = s, \text{sign}(y) = r\}
\]

for all \(s, r \in H\).

We have \(0 + s = s, s + s = s\) and \(1 + (-1) = H\).
The Hyperfield Criterium

Definition

A hyperfield is a tuple \((H, +, \cdot, 0, 1)\) where

\[
- + - : H \times H \to 2^H \setminus \{\emptyset\}, \quad - \cdot - : H \times H \to H
\]

are symmetric maps satisfying the following relations:

1. The tuple \((H \setminus \{0\}, \cdot, 1)\) is a group.
2. We have \(0 \cdot x = 0\) and \(0 + x = \{x\}\) for all \(x \in H\).
3. We have \(a \cdot (x + y) = (a \cdot x) + (a \cdot y)\) for all \(a, x, y \in H\).
4. For every \(x \in H\) there is an unique element \(-x \in H\) such that \(x + (-x) \ni 0\).

A subset of \(H^n\) is Zariski-closed when it is of the form

\[
\{(s_1, \ldots, s_n) \in H^n \mid f_1(s_1, \ldots, s_n), \ldots, f_k(s_1, \ldots, s_n) \ni 0\}
\]

for some polynomials \(f_1, \ldots, f_k\) over \(H\) in variables \(x_1, \ldots, x_n\).
Example (Sign hyperfield)

Take \( H = \{1, 0, -1\} \) with usual multiplication and

\[
0 + s = s, \quad s + s = s, \quad 1 + (-1) = H
\]

Take \( f = x_1 + x_2 - x_3 - x_4 \) and \( s_1, s_2, s_3, s_4 \in H \). Then

\[
f(s_1, s_2, s_3, s_4) \ni 0 \iff \begin{cases} s_1 = s_2 = s_3 = s_4 = 0 \\
\text{or} \\
1, -1 \in \{s_1, s_2, -s_3, -s_4\} \\
f(s_1, s_2, s_3, s_4) = 0 \\
\text{or} \\
f(s_1, s_2, s_3, s_4) = H
\end{cases}
\]
For $f = \sum_i c_i x_i \in \mathbb{R}[x_1, \ldots, x_n]$, take $\text{sign}(f) := \sum_i \text{sign}(c_i)x_i$.

**Hyperfield Criterium**

Let $w \in \mathbb{Z}^V_d$ be an outcome and $s \in H^V_d$. Suppose that $\text{sign}(\phi)$ does not vanish at $s$ for some Pascal equation $\phi$ on $\mathbb{Z}^V_d$. Then $\text{sign}(w) \neq s$.

**How to apply it?**
The Hexagon Criterium

Let $\ell_1, \ell_2 \geq d' \geq 1$ be integers such that $d' + \ell_1 + \ell_2 \leq d$.

Let $w = (w_{i,j})_{(i,j) \in V_d} \in \mathbb{Z}^{V_d}$ and write $w' = (w_{i,j})_{(i,j) \in V_{d'}} \in \mathbb{Z}^{V_{d'}}$.

Hexagon Criterium

Suppose that $w'$ is not an outcome and

$$\text{supp}(w) \subseteq V_{d'} \cup \{(i,j) \in V_d \mid j > d - \ell_1\} \cup \{(i,j) \in V_d \mid i > d - \ell_2\}$$

holds. Then $w$ is not an outcome.

How to apply it?
Main results

Conjecture
Let $w$ be a valid outcome. Then $\deg(w) \leq 2 \cdot \# \text{supp}^+(w) - 3$.

Main result
The conjecture holds when $\# \text{supp}^+(w) \leq 5$.

Corollary
Let
\[ \mathcal{M} = \{(\lambda_0 t^{i_0} (1-t)^{j_0}, \lambda_1 t^{i_1} (1-t)^{j_1}, \ldots, \lambda_n t^{i_n} (1-t)^{j_n}) \mid t \in (0,1)\} \]
be a model with a rational MLE.

1. If $n = 1$, then $\max_\nu (i_\nu + j_\nu) \leq 1$.
2. If $n = 2$, then $\max_\nu (i_\nu + j_\nu) \leq 3$.
3. If $n = 3$, then $\max_\nu (i_\nu + j_\nu) \leq 5$.
4. If $n = 4$, then $\max_\nu (i_\nu + j_\nu) \leq 7$. 
Main results

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\]
be a model with a rational MLE.

1. If $n = 1$, then $\max_\nu(i_\nu + j_\nu) \leq 1$. $\iff$ Invertibility Criterium
2. If $n = 2$, then $\max_\nu(i_\nu + j_\nu) \leq 3$. $\iff$ Invertibility Criterium
3. If $n = 3$, then $\max_\nu(i_\nu + j_\nu) \leq 5$.
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\]
be a model with a rational MLE.

1. If \( n = 1 \), then \( \max_{\nu} (i_{\nu} + j_{\nu}) \leq 1 \). \(\Leftarrow\) Invertibility Criterium
2. If \( n = 2 \), then \( \max_{\nu} (i_{\nu} + j_{\nu}) \leq 3 \). \(\Leftarrow\) Invertibility Criterium
3. If \( n = 3 \), then \( \max_{\nu} (i_{\nu} + j_{\nu}) \leq 5 \). \(\Leftarrow\) Hyperfield Criterium
4. If \( n = 4 \), then \( \max_{\nu} (i_{\nu} + j_{\nu}) \leq 7 \).
Main results

Conjecture
Let $w$ be a valid outcome. Then $\deg(w) \leq 2 \cdot \# \text{supp}^+(w) - 3$.

Main result
The conjecture holds when $\# \text{supp}^+(w) \leq 5$.

Corollary
Let

$$M = \{(\lambda_0 t^{i_0} (1-t)^{j_0}, \lambda_1 t^{i_1} (1-t)^{j_1}, \ldots, \lambda_n t^{i_n} (1-t)^{j_n}) | t \in (0, 1)\}$$

be a model with a rational MLE.

1. If $n = 1$, then $\max_{\nu}(i_{\nu} + j_{\nu}) \leq 1$. $\Leftarrow$ Invertibility Criterium
2. If $n = 2$, then $\max_{\nu}(i_{\nu} + j_{\nu}) \leq 3$. $\Leftarrow$ Invertibility Criterium
3. If $n = 3$, then $\max_{\nu}(i_{\nu} + j_{\nu}) \leq 5$. $\Leftarrow$ Hyperfield Criterium
4. If $n = 4$, then $\max_{\nu}(i_{\nu} + j_{\nu}) \leq 7$. $\Leftarrow$ HypC + InvC + HexC
Main results

Conjecture
Let $w$ be a valid outcome. Then $\deg(w) \leq 2 \cdot \# \text{supp}^+(w) - 3$.

Some computations
The conjecture holds when $\deg(w) \leq 9$.

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</table>

$\#\{"fundamental"\ \text{outcomes with} \ \# \text{supp}^+(w) = n, \deg(w)) = d\}$
Curves with a rational MLE

Thank you for your attention!
Eliana Duarte, Orlando Marigliano, Bernd Sturmfels

*Discrete statistical models with rational maximum likelihood estimator*

Bernoulli 27 (2021), pp. 135–154

Arthur Bik, Orlando Marigliano

*Discrete statistical curves with rational maximum likelihood estimator*

in preparation