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The monic rank
and instances of Shapiro's Conjecture

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A Conjecture by Shapiro

Conjecture (Boris Shapiro)

Every homogeneous polynomial $f \in \mathbb{C}[x, y]$ of degree $d \cdot e$ is the sum of at most d d-th powers of polynomials of degree e.

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Why believe this?

- True when $e = 1$, when $d = 1$ and when $d = 2$.
- The projective variety

$$\{[h^d] \mid h \in \mathbb{C}[x, y]_{(e)}\} \subseteq \mathbb{P}(\mathbb{C}[x, y]_{(d \cdot e)})$$

  has dimension e in a projective space of dimension $d \cdot e$.
  $\Rightarrow$ Its d-th secant variety is expected to be everything.
- True for $(d, e) = (3, 2)$ by Lundqvist, Oneto, Reznick and Shapiro.

Example: $\{\text{deg } d\} = \{\text{sum of } d\text{-th powers of deg } 1\}$

Consider

$$(x + a_1 y)^d + (x + a_2 y)^d + \cdots + (x + a_d y)^d$$

$$=$$

$$dx^d + \binom{d}{1} b_1 x^{d-1} y + \binom{d}{2} b_2 x^{d-2} y^2 + \cdots + \binom{d}{d} b_d y^d$$

with $b_k = a_1^k + \cdots + a_d^k$.

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Fact (Hilbert): The map $(a_1, \ldots, a_d) \mapsto (b_1, \ldots, b_d)$ is a finite morphism.

Using coordinate transformations, this implies:

$$\mathbb{C}[x,y]_{(d)} = \left\{ \ell_1^d + \cdots + \ell_d^d \;\middle|\; \ell_1, \ldots, \ell_d \in \mathbb{C}[x,y]_{(1)} \right\}$$

The monic rank

- V a finite-dimensional vector space
- $X \subseteq V$ a non-degenerate irreducible Zariski-closed cone
- $h\colon V \to \mathbb{C}$ a non-zero linear function and $H = h^{-1}(1) \subseteq V$

Definition

The monic rank of a vector $v \in V \setminus h^{-1}(0)$ is the minimal r such that

$$\frac{r}{h(v)} \cdot v = w_1 + \cdots + w_r$$

with $w_1, \ldots, w_r \in X \cap H$.

Theorem

monic rank $\leqslant 2 \cdot ($*the generic monic rank*$) < \infty$

Shapiro's Conjecture (Monic Version)

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Every $f \in \mathbb{C}[x,y]_{(d \cdot e)}$ with leading coefficient d has monic rank $\leqslant d$.

$$X = \{d\text{-th powers of homogeneous polynomials of degree } e\}$$

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Goal: We want to show that

$$\begin{aligned} \prod_{i=1}^{d} \{f \in \mathbb{C}[x,y]_{(e)} \text{ monic}\} &\rightarrow \mathbb{C}[x,y]_{(d\cdot e)} \\ (f_1,\ldots,f_d) &\mapsto f_1^d + \cdots + f_d^d \end{aligned}$$

is a finite morphism.

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is a finite morphism.

Proposition: This is true if $(c_{ij})_{ij} = 0$ is the only solution of the equation

$$dx^{de} = \sum_{i=1}^{d} \left(x^e + c_{i1}x^{e-1}y + \cdots + c_{ie}y^e\right)^d$$

Reduction to a Gröbner basis computation

Assume that $(c_{ij})_{ij}$ satisfies

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We have $c_{ie} = 0$ for all i. Divide by x^d.

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After permuting summands and scaling y, we get $c_{1e} = 1$.

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The computation finished for $(d,e) = (3,2), (3,3), (3,4), (4,2)$.

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Other examples of (monic) rank

Some other objects that have a rank:

- Matrices
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Question: How do the maximal rank and monic rank compare?

2 x 2 x 2 Tensors

The space of $2 \times 2 \times 2$ tensors:

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \left\{ \left(\begin{array}{cc|cc} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{array} \right) \;\middle|\; a_{ij}, b_{ij} \in \mathbb{C} \right\}$$

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The tensors of rank $\leqslant 1$:

$$X = \left\{ (A \mid B) \;\middle|\; \begin{array}{c} \operatorname{rk}(A), \operatorname{rk}(B) \leqslant 1 \\ A, B \text{ are linearly dependent} \end{array} \right\}$$

Fact: The maximal rank of a $2 \times 2 \times 2$ tensor is 3.

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Fact: The maximal rank of a $2 \times 2 \times 2$ tensor is 3.

Let a_{11} be the leading coefficient. The maximal monic rank is $\geqslant 3$.

Question: Is the maximal monic rank equal to 3?

Orbits of tensors

We have 3 commuting actions of $\mathbb{C}$:

- $(v_1\ v_2 \mid w_1\ w_2) \rightsquigarrow (v_1\ v_2 + \lambda v_1 \mid w_1\ w_2 + \lambda w_1)$
- $\begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} r_1 & s_1 \\ r_2 + \lambda r_1 & s_2 + \lambda s_1 \end{pmatrix}$
- $(A \mid B) \rightsquigarrow (A \mid B + \lambda A)$

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Remark: These operations do not change ranks or leading coefficients.

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Remark: These operations do not change ranks or leading coefficients.

Lemma: Every $2 \times 2 \times 2$ tensor with a non-zero leading coefficient lies in the orbit of a tensor of the form

$$\left(\begin{array}{cc|cc} c & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & \lambda \end{array}\right)$$

with $c, \lambda, \mu_1, \mu_2, \mu_3 \in \mathbb{C}$.

Tensors with monic rank $\leqslant$ 2

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Claim: The set of sums of two monic tensors with rank 1 is

$$\mathbb{C}^3 \cdot \left\{ \left(\begin{matrix} 2 & 0 \\ 0 & \mu_3 \end{matrix} \middle| \begin{matrix} 0 & \mu_1 \\ \mu_2 & 0 \end{matrix}\right) \;\middle|\; \begin{matrix} \mu_1, \mu_2, \mu_3 \in \mathbb{C}, \\ \#\{i \mid \mu_i = 0\} \neq 1 \end{matrix} \right\}$$

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Proof:

$$\left(\begin{array}{cc|cc} 2 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & \lambda \end{array}\right) = \left(\begin{array}{cc|cc} 1 & b & c & bc \\ a & ab & ac & abc \end{array}\right) + \left(\begin{array}{cc|cc} 1 & -b & -c & bc \\ -a & ab & ac & -abc \end{array}\right)$$

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Idea: Write every tensor with leading coefficient 3 as

$$\left(\begin{array}{cc|cc} 2 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & 0 \end{array}\right) + \left(\begin{array}{cc|cc} 1 & b & c & bc \\ a & ab & ac & abc \end{array}\right)$$

with $\mu_1, \mu_2, \mu_3 \in \mathbb{C} \setminus \{0\}$ and $a, b, c \in \mathbb{C}$.

Tensors with monic rank $\leqslant 3$

Start with a tensor with leading coefficient 3 in standard form.

$$\left(\begin{array}{cc|cc} 3 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & \lambda \end{array}\right)$$

Tensors with monic rank $\leqslant$ 3

Start with a tensor with leading coefficient 3 in standard form.

$$(\tfrac{1}{3}a, \tfrac{1}{3}b, \tfrac{1}{3}c) \cdot \left(\begin{array}{cc|cc} 3 & 0 & 0 & \mu_1 \\ 0 & \mu_3 & \mu_2 & \lambda \end{array}\right)$$

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$$\left(\begin{array}{cc|cc} 2 & 0 & 0 & \mu_1 - \frac{2}{3}bc \\ 0 & \mu_3 - \frac{2}{3}ab & \mu_2 - \frac{2}{3}ac & \lambda + \frac{1}{3}(a\mu_1 + b\mu_2 + c\mu_3) - \frac{8}{9}abc \end{array}\right)$$

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Want:

- $\lambda + \frac{1}{3}(a\mu_1 + b\mu_2 + c\mu_3) - \frac{8}{9}abc = 0$
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This is doable unless $\lambda = \mu_1 = \mu_2 = \mu_3 = 0$ (and that case is easy).

Maximal rank vs maximal monic rank

Theorem

- *For an* $n \times m$ *matrix, we have*

$$\textit{maximal rank} = \textit{maximal monic rank} = \min(n, m)$$

- *For a symmetric* $n \times n$ *matrix, we have*

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Assume that h is a highest weight vector.
Question: Are the maximal rank and maximal monic rank always equal?

References

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