Strength of (infinite) Polynomials

Applied Algebra Seminar

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Arthur Bik

Different ranks of tensors

Notation

We visualize a tensor by putting its slices next to each other.

$$
\begin{array}{r}
\\
\text{Row 1} \\
\text{Row 2} \\
\\
\end{array}
\begin{array}{c}
\qquad\text{Layer 1} \qquad\qquad \text{Layer 2} \\
\left(\begin{array}{cc|cc}
T_{111} & T_{112} & T_{211} & T_{212} \\
T_{121} & T_{122} & T_{221} & T_{222}
\end{array}\right) \\
\text{Col 1} \quad \text{Col 2} \quad \text{Col 1} \quad \text{Col 2}
\end{array}
$$

Example

Tensor representing avarage speeding fines:

$$
\begin{array}{r}
\\
\text{US} \\
\text{UK} \\
\\
\end{array}
\begin{array}{c}
\qquad\text{Red car} \qquad \text{Blue car} \\
\left(\begin{array}{cc|cc}
100 & 80 & 50 & 40 \\
60 & 60 & 30 & 30
\end{array}\right) \\
\text{GGB} \quad \text{no} \quad \text{GGB} \quad \text{no}
\end{array}
$$

BBG = The Great British Bake Off

Different ranks of tensors

HUGE tensor $\rightsquigarrow$ search for structure

Definition

A *pure tensor* is any tensor of the form

$$(\ell_1 \ \ell_2) \otimes (r_1 \ r_2) \otimes (c_1 \ c_2) := \left(\begin{array}{cc|cc} \ell_1 r_1 c_1 & \ell_1 r_1 c_2 & \ell_2 r_1 c_1 & \ell_2 r_1 c_2 \\ \ell_1 r_2 c_1 & \ell_1 r_2 c_2 & \ell_2 r_2 c_1 & \ell_2 r_2 c_2 \end{array}\right)$$

The *tensor rank* of a tensor T is the minimum r such that

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

for some choices of u_i, v_i, w_i.

Different ranks of tensors

How do you recognize a pure tensor?

$$(\ell_1\ \ell_2)\otimes(r_1\ r_2)\otimes(c_1\ c_2) := \left(\begin{array}{cc|cc} \ell_1 r_1 c_1 & \ell_1 r_1 c_2 & \ell_2 r_1 c_1 & \ell_2 r_1 c_2 \\ \ell_1 r_2 c_1 & \ell_1 r_2 c_2 & \ell_2 r_2 c_1 & \ell_2 r_2 c_2 \end{array}\right)$$

Answer: Flattenings have rank 1

$$\begin{pmatrix} \ell_1 r_1 c_1 & \ell_1 r_1 c_2 & \ell_2 r_1 c_1 & \ell_2 r_1 c_2 \\ \ell_1 r_2 c_1 & \ell_1 r_2 c_2 & \ell_2 r_2 c_1 & \ell_2 r_2 c_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}(\ell_1 c_1\ \ell_1 c_2\ \ell_2 c_1\ \ell_2 c_2)$$

$$\begin{pmatrix} \ell_1 r_1 c_1 & \ell_1 r_1 c_2 \\ \ell_1 r_2 c_1 & \ell_1 r_2 c_2 \\ \ell_2 r_1 c_1 & \ell_2 r_1 c_2 \\ \ell_2 r_2 c_1 & \ell_2 r_2 c_2 \end{pmatrix} = \begin{pmatrix} \ell_1 r_1 \\ \ell_1 r_2 \\ \ell_1 r_1 \\ \ell_2 r_2 \end{pmatrix}(c_1\ c_2), \quad \begin{pmatrix} \ell_1 r_1 c_1 & \ell_2 r_1 c_1 \\ \ell_1 r_2 c_1 & \ell_2 r_2 c_1 \\ \ell_1 r_1 c_2 & \ell_2 r_1 c_2 \\ \ell_1 r_2 c_2 & \ell_2 r_2 c_2 \end{pmatrix} = \begin{pmatrix} r_1 c_1 \\ r_2 c_1 \\ r_1 c_2 \\ r_2 c_2 \end{pmatrix}(\ell_1\ \ell_2)$$

Different ranks of tensors

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Example

Tensor representing avarage speeding fines:

$$\begin{array}{cc} & \text{Red car} \quad \text{Blue car} \\ \begin{array}{c}\text{US}\\ \text{UK}\end{array} & \left(\begin{array}{cc|cc} 100 & 80 & 50 & 40 \\ 60 & 60 & 30 & 30 \end{array}\right) \\ & \text{GGB} \quad \text{no} \quad \text{GGB} \quad \text{no} \end{array}$$

BBG = The Great British Bake Off

Only 1 out of 3 flattenings has rank $1 \Rightarrow$ not a pure tensor

Definition

The *strength* of a tensor T is the minimum r such that

$$T = T_1 + \ldots + T_r$$

where each T_i has a rank-1 flattening.

Different ranks of tensors

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Given linear maps $L_i\colon \mathbb{C}^{n_i} \to \mathbb{C}^{m_i}$, we get the linear map

$$\begin{aligned} L_1 \otimes \cdots \otimes L_d\colon \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d} &\to \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_d} \\ v_1 \otimes \cdots \otimes v_d &\mapsto L_1(v_1) \otimes \cdots \otimes L_d(v_d) \end{aligned}$$

We call $(L_1 \otimes \cdots \otimes L_d)(T)$ a *coordinate transform* of T.

Theorem (B-Draisma-Eggermont)

Let $\mathcal{P}$ be a property of tensors such that

$$T \text{ has } \mathcal{P} \Rightarrow \text{ all coordinate transforms of } T \text{ have } \mathcal{P}$$

holds. Then either

$$\{T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d} \mid T \text{ has } \mathcal{P}\}$$

is Zariski-dense for all $n_1, \ldots, n_d \geq 1$ or there exists a C such that

$$T \text{ has } \mathcal{P} \Rightarrow \mathrm{str}(T) \leq C$$

Different ranks of tensors

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How difficult is strength?

The set of pure tensors is a variety with 1 component.

The set of d-way tensors with a rank-1 flattening has $2^{d-1} - 1$ components.

How about symmetric tensors/homogenous polynomials?

The *strength* of a homogeneous polynomial f is the minimum r such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

where $\deg(g_i), \deg(h_i) < \deg(f)$.

The set of reducible polynomials has $\lfloor d/2 \rfloor$ components.

Definition

The *strength* of a homogeneous polynomial f is the minimum r such that

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

where $\deg(g_i), \deg(h_i) < \deg(f)$.

Example

What is the strength of $f = x^2 + y^2 + z^2$?

- We have $\mathrm{str}(f) \leq 3$ since $f = x \cdot x + y \cdot y + z \cdot z$.
- We have $\mathrm{str}(f) \neq 0$ since $f \neq 0$.
- We have $\mathrm{str}(f) \neq 1$ since f is not reducible.
- Note that $f = (x + iy) \cdot (x - iy) + z \cdot z$.

So $\mathrm{str}(f) = 2$ over $\mathbb{C}$ (but over $\mathbb{R}$ it would be 3).

Universality

Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ and $\ell_1, \ldots, \ell_n$ be linear forms in $y_1, \ldots, y_m$. The polynomial

$$f(\ell_1, \ldots, \ell_n) \in \mathbb{C}[y_1, \ldots, y_m]_d$$

is a coordinate transform of f.

Let $\mathcal{P}$ be a property of degree-d polynomials such that

$$f \text{ has } \mathcal{P} \Rightarrow \text{every coordinate transform of } f \text{ has } \mathcal{P}$$

Theorem (Kazhdan-Ziegler, B-Danelon-Draisma-Eggermont)

Either all f have $\mathcal{P}$ or there exists a $k \geq 0$ such that

$$f \text{ has } \mathcal{P} \Rightarrow \mathrm{str}(f) \leq k$$

Remark

Choosing $\mathcal{P} =$"is a limit of strength k polynomials over $\overline{K}$" yields that $\mathrm{str}_K(f) \leq P(\mathrm{str}_{\overline{K}}(f))$ for some polynomial P.

Some Tricks

1. We have $\mathrm{str}(f+g) \leq \mathrm{str}(f) + \mathrm{str}(g)$.
2. For $f \in \mathbb{C}[x_1,\dots,x_n]_d$, we define the singular locus:

$$\mathrm{Sing}(f) := \left\{ \frac{\partial f}{\partial x_1} = \ldots = \frac{\partial f}{\partial x_n} = 0 \right\}$$

   When $f = g_1 \cdot h_1 + \ldots + g_k \cdot h_k$, then

$$\{g_1 = h_1 = \ldots = g_k = h_k = 0\} \subseteq \mathrm{Sing}(f)$$

   and so $\dim \mathrm{Sing}(f) \geq n - 2\,\mathrm{str}(f)$.
3. Every polynomial in $\mathbb{C}[x,y]_d$ is reducible. Hence

$$f \in \mathbb{C}[x,y]_d \Rightarrow \mathrm{str}(f) \leq 1$$

Example

Consider $f = x_1^d + \ldots + x_n^d$.

We have

$$f = \begin{cases} (x_1^d + x_2^d) + \ldots + (x_{2k-1}^d + x_{2k}^d) & \text{if } n = 2k \\ (x_1^d + x_2^d) + \ldots + (x_{2k-1}^d + x_{2k}^d) + x_{2k+1}^d & \text{if } n = 2k+1 \end{cases}$$

and so $\operatorname{str}(f) \leq \lceil n/2 \rceil$.

The singular locus

$$\operatorname{Sing}(f) = \{dx_1^{d-1} = \ldots = dx_n^{d-1} = 0\} = \{(0, \ldots, 0)\} \subseteq \mathbb{C}^n$$

has dimenion $0 \geq n - 2\operatorname{str}(f)$. So $\operatorname{str}(f) \geq \lceil n/2 \rceil$.

So $\operatorname{str}(f) = \lceil n/2 \rceil$.

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Theorem

For every $k \geq 0$, the set $\{A \in \mathbb{C}^{n\times m} \mid \mathrm{rk}(A) \leq k\}$ is closed.

What about $\{f \in \mathbb{C}[x_1, \ldots, x_n]_d \mid \mathrm{str}(f) \leq k\}$?

For $k = 1$, yes. (union of images of projective morphisms).

For $k = 2$, I don't know.

For $d = 2$, yes. (rank of symmetric matrices)

For $d = 3$, yes. (slice rank of polynomials)

Example ($k = 3, d = 4$)

$$^1/_t(x^2+tg)(y^2+tf) - {}^1/_t(u^2-tq)(v^2-tp) - {}^1/_t(xy-uv)(xy+uv)$$

1. It has strength ≤ 3.
2. For $t \to 0$, we get $x^2 f + y^2 g + u^2 p + v^2 q$.

Strength vs matrix rank

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Theorem (Ballico-B-Oneto-Ventura)

For $n \gg 0$, we have

$$\mathrm{str}(x^2 f + y^2 g + u^2 p + v^2 q) = 4$$

for some $x, y, u, v \in \mathbb{C}[x_1, \ldots, x_n]_1$ and $f, g, p, q \in \mathbb{C}[x_1, \ldots, x_n]_2$.

Corollary

The set $\{f \in \mathbb{C}[x_1, \ldots, x_n]_4 \mid \mathrm{str}(f) \leq 3\}$ is not closed for $n \gg 0$.

Question

Which n are high enough?

Question

What is the strength of $x^2a^2 + y^2b^2 + u^2c^2 + v^2d^2$?

An easier case

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Proposition

We have

$$\mathrm{str}(x^2 f + y^2 g + u^2 p + v^2 q) = 4$$

when x, y, u, v and f, g, p, q are variables of degrees 1 and 2.

1/4 of the proof

We need to show, for example, that

$$x^2 f + y^2 g + u^2 p + v^2 q \neq \ell_1 \cdot h_1 + \ell_2 \cdot h_2 + \ell_3 \cdot h_3$$

for all $\ell_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_1$ and $h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_3$.

An easier case

Proposition

We have

$$\mathrm{str}(x^2 f + y^2 g + u^2 p + v^2 q) = 4$$

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for all $\ell_i \in \mathbb{C}[x, y, u, v]_1$ and $h_i \in \mathbb{C}[x, y, u, v, f, g, p, q]_3$.

Think of $R = \mathbb{C}[x, y, u, v]$ as the set of coefficients.
So $\ell_i \in R$ and $h_i \in R[f, g, p, q]$.

The coefficients of f, g, p, q on the right are all in (ℓ_1, ℓ_2, ℓ_3).
The coefficients x^2, y^2, u^2, v^2 on the left are not all (ℓ_1, ℓ_2, ℓ_3). $\square$

Theorem

We have $\{\mathrm{rk}(A) \mid A \in \mathbb{C}^{n\times m}\} = \{0, 1, \ldots, \min(n, m)\}$.

What about strength in $\mathbb{C}[x_1, \ldots, x_n]_d$?

1. We can write any polynomial f as $x_1 \cdot g_1 + \ldots + x_n \cdot g_n$.
   $\Rightarrow \mathrm{str}(f) \in \{0, 1, \ldots, n\}$
2. Suppose that f has maximal strength and write

$$f = \sum_{i=1}^{\mathrm{str}(f)} g_i \cdot h_i$$

   Then $g_1 \cdot h_1 + \ldots + g_r \cdot h_r$ has strength r for $r = 0, \ldots, \mathrm{str}(f)$.
   $\Rightarrow \{\mathrm{str}(f) \mid f \in \mathbb{C}[x_1, \ldots, x_n]_d\}$ is an interval $\{0, 1, \ldots, r\}$.

Strength vs matrix rank

Take $d \geq 3$ and r minimal such that

$$r(n-r) \geq \binom{n-r+d-1}{d}.$$

This means that $r \approx n - \sqrt[d-1]{d!n}$.

Theorem (Harris)

A polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$ can be written as

$$\ell_1 h_1 + \ldots + \ell_r h_r$$

with $\ell_1, \ldots, \ell_r$ linear.

Theorem (Ballico-B-Oneto-Ventura)

A generic polynomial in $\mathbb{C}[x_1, \ldots, x_n]_d$ has strength r.

Polynomial series

For $d \geq 1$, we define

$$S_\infty^d = \left\{ \sum_{1 \leq i_1 \leq \cdots \leq i_d} a_{i_1 \cdots i_d} x_{i_1} \cdots x_{i_d} \,\middle|\, a_{i_1 \cdots i_d} \in \mathbb{C} \right\}$$

to be the set of degree-d *polynomial series*.

Now $S_\infty = \mathbb{C} \oplus \bigoplus_{d \geq 1} S_\infty^d$ is a ring.

Definition

A *system of variables* is collection $(f_i)_{i \in I}$ such that

$$\begin{aligned} \mathbb{C}[y_i \mid i \in I] &\to S_\infty \\ y_i &\mapsto f_i \end{aligned}$$

is an isomorphism. A *part of a system of variables* is a subcollection of a system of variables.

Theorem (Erman-Sam-Snowden)

A system of variables exists (in more general settings).

Proof

Let F^d be the subspace of finite-strength elements of S^d_∞ and take a collection $(f_i)_{i\in I_d}$ that maps to a basis of S^d_∞/F^d.

Take $I = \bigcup_{d\geq 1} I_d$. Then $(f_i)_{i\in I}$ is a system of variables. □

Proposition

Let $(x, y, u, v, f, g, p, q) \in (S^1_\infty)^4 \times (S^2_\infty)^4$ be part of a system of variables. Then $\mathrm{str}(x^2 f + y^2 g + u^2 p + v^2 q) = 4$.

Setting $x_{n+1}, x_{n+2}, \ldots = 0$ for $n \gg 0$ yields the counter example.

Systems of variables

What does a coordinate transform means in this setting?

Non-example

Take $f = x_1 + x_2 + \dots$ and set $x_i \mapsto x_1$ for all $i \in \mathbb{N}$.

Definition

Let $f \in S^d_\infty$. Then a *coordinate transform* of f is

$$f(\ell_1, \ell_2, \dots) \in S^d_\infty$$

where $\ell_1, \ell_2, \dots$ are linear forms in $x_1, x_2, \dots$ so that every variable x_i only appears in finitely many linear forms ℓ_j.

Example

$(x_1 + x_2 + \dots)^2$ is a coordinate transform of x_1^2.

Systems of variables

Definition

Let $f \in S_\infty^d$. Then a *coordinate transform* of f is

$$f(\ell_1, \ell_2, \ldots) \in S_\infty^d$$

where $\ell_1, \ell_2, \ldots$ are linear forms in $x_1, x_2, \ldots$ so that every variable x_i only appears in finitely many linear forms ℓ_j.

Definition

We say that f *specializes* to g when g is a coordinate transform of f. We say that f, g are *isogenous* when they specialize to each other.

Question

What is the structure of the poset of isogenous classes?

Systems of variables

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Example ($d = 1$)

The nonzero elements of S^1_∞ form one isogeny class.

Example ($d = 2$)

f, g are isogeneous $\Leftrightarrow$ associated matrices have same rank.

Proposition

If f specializes to g, then $\mathrm{str}(g) \leq \mathrm{str}(f)$.

Theorem (B-Danelon-Snowden)

The poset of infinite-strength isogeny classes in S^3_∞ is $\mathbb{N} \cup \{\infty\}$.

Residual rank

Proposition

$x_1^3 + x_2^3 + \ldots$ does not specialize to $x_1 \cdot (x_2^2 + x_3^2 + \ldots)$.

Proof.

Let $\ell_1^3 + \ell_2^3 + \ldots$ be a specialization of $x_1^3 + x_2^3 + \ldots$. Then the set

$$J := \{j \in \mathbb{N} \mid x_1 \text{ occurs in } \ell_j\}$$

is finite. The series

$$\frac{\partial}{\partial x_1}(\ell_1^3 + \ell_2^3 + \ldots) = \sum_{j \in J} 3 \frac{\partial \ell_j}{\partial x_1} \ell_j^2$$

has strength $\leq \# J < \infty$. □

Definition

The *residual rank* of $f \in S_\infty^d$ is

$$\mathrm{rrk}(f) = \dim \operatorname{span} \left\{ \frac{\partial}{\partial x_i} f \text{ mod } F^d \,\middle|\, i \in \mathbb{N} \right\}$$

where $F^d \subseteq S_\infty^d$ is the subspace of finite-strength elements.

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Definition

The *residual rank* of $f \in S_\infty^d$ is

$$\operatorname{rrk}(f) = \dim \operatorname{span} \left\{ \frac{\partial}{\partial x_i} f \text{ mod } F^d \,\middle|\, i \in \mathbb{N} \right\}$$

where $F^d \subseteq S_\infty^d$ is the subspace of finite-strength elements.

Theorem (B-Danelon-Snowden)

The map rrk is an isomorphism between the poset of isogeny classes of S_∞^3 and $\mathbb{N} \cup \{\infty\}$.

Sketch of proof for finite rrk

Set $r = \operatorname{rrk}(f)$ and put the series f in standard form

$$f \simeq x_1 g_1 + \ldots + x_r g_r + h$$

where $(g_1, \ldots, g_r, h)$ part of a system of variables and $\operatorname{rrk}(h) = 0$. Then show that all such tuples are isogenous.

Thank you for your attention!