Noetherianity of dualized adjoint representations up to locally diagonal groups

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Noetherianity up to a group action

A finite-dimensional vector space V over an infinite field K is Noetherian, i.e. every descending chain $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ of Zariski-closed subsets of V stabilizes.

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The vector space $V = K^{\mathbb{N}}$ is not Noetherian. (Take $X_i = Z(x_1, \dots, x_i)$.)

However, $S_{\mathbb{N}}$ acts linearly on $K^{\mathbb{N}}$ by permuting entries.

Theorem (Cohen, 1967)

The space $K^{\mathbb{N}}$ is $S_{\mathbb{N}}$-Noetherian, i.e. every descending chain

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$$

of $S_{\mathbb{N}}$-stable Zariski-closed subsets of $K^{\mathbb{N}}$ stabilizes.

Noetherianity up to a group action

Definition

A representation V of a group G is called G-Noetherian when every descending chain $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ of G-stable Zariski-closed subsets of V stabilizes.

We say that a G-stable Zariski-closed subset X of V is G-Noetherian when every descending chain starting with X stabilizes.

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Examples

- Finite-dimension affine varieties with the trivial group action.
- $K^{\mathbb{N}\times n} \oplus K^{m\times\mathbb{N}} \oplus K^k$ with $S_{\mathbb{N}}$ (or with GL).
- $K^{\mathbb{N}\times\mathbb{N}}$ with $\mathrm{GL}\times\mathrm{GL}$ acting by left and right multiplication.

Noetherianity up to a group action

Theorem (Draisma, 2017)

Let $P\colon \mathbf{Vec} \to \mathbf{Vec}$ *be a finite-degree polynomial functor. Then* P *is a Noetherian* $\mathbf{Vec}$*-topological space.*

Noetherianity up to a group action

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$$\begin{array}{ccccccccc}
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\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & & & \circlearrowleft \\
P(K) & \leftarrow & P(K^2) & \leftarrow & P(K^3) & \leftarrow & \dots & \leftarrow & \varprojlim P(K^n) \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
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Noetherianity up to a group action

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Theorem (Eggermont, Snowden, 2017)

Let V be an algebraic representation of $G \in \{\mathrm{GL}, \mathrm{Sp}, \mathrm{O}\}$. Then V^* is G-noetherian.

Locally diagonal groups

Consider

$$G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow G_4 \longrightarrow \dots$$

with $G_i \in \{\mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{O}_n\}$.

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Definition

Take $G, H \in \{\mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{O}_n\}$. A homomorphism $\iota\colon G \to H \subset \mathrm{GL}_n$ is called a diagonal embedding if there is a $T \in \mathrm{GL}_n$ such that

$$\iota(A) = T \,\mathrm{Diag}\left(A, \dots, A, A^{-T}, \dots, A^{-T}, 1, \dots, 1\right) T^{-1}$$

for all $A \in G$.

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for all $A \in G$. With $G \subseteq \mathrm{GL}_m$, this is equivalent to

$$K^n \cong (K^m)^{\oplus l} \oplus ((K^m)^*)^{\oplus r} \oplus K^{\oplus z}$$

as representations of G.

Locally diagonal groups

Let

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be a sequence of diagonal embeddings with limit G.

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of Lie algebras acted on by conjugation.

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of Lie algebras acted on by conjugation.

Theorem

Let V be the inverse limit of the dualized sequence

$$\mathfrak{g}_1^* \longleftarrow \mathfrak{g}_2^* \longleftarrow \mathfrak{g}_3^* \longleftarrow \mathfrak{g}_4^* \longleftarrow \dots$$

and suppose that $\mathrm{char}(K) \neq 2$ or $\#\{i \mid G_i \in \{\mathrm{SL}_n\}\} = \infty$. Then the space V is G-Noetherian.

Limits of special linear groups

Consider $\mathrm{SL}_{n_1} \longrightarrow \mathrm{SL}_{n_2} \longrightarrow \mathrm{SL}_{n_3} \longrightarrow \mathrm{SL}_{n_4} \longrightarrow \dots$ with maps

$$A \mapsto \mathrm{Diag}(A, \dots, A, A^{-T}, \dots, A^{-T}, 1, \dots, 1)$$

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is an isomorphism.

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Limits of special linear groups

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$$\mathfrak{gl}_{n_1}/\operatorname{span}(I_{n_1}) \leftarrow \mathfrak{gl}_{n_2}/\operatorname{span}(I_{n_2}) \leftarrow \mathfrak{gl}_{n_3}/\operatorname{span}(I_{n_3}) \leftarrow \dots$$

with maps

$$\begin{array}{ccc} \mathfrak{gl}_{(l+r)n+z}/\operatorname{span}(I_{(l+r)n+z}) & \to & \mathfrak{gl}_n/\operatorname{span}(I_n) \\ \begin{pmatrix} P_{11} & \dots & P_{1l} & \bullet & \dots & \bullet & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ P_{l1} & \dots & P_{ll} & \bullet & \dots & \bullet & \bullet \\ \bullet & \dots & \bullet & Q_{11} & \dots & Q_{1r} & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \bullet & \dots & \bullet & Q_{r1} & \dots & Q_{rr} & \bullet \\ \bullet & \dots & \bullet & \bullet & \dots & \bullet & \bullet \end{pmatrix} & \mapsto & \sum_{i=1}^{l} P_{ii} - \sum_{j=1}^{r} Q_{jj}^T \end{array}$$

Preserved algebraic invariants

$$\mathfrak{gl}_{(l+r)n+z} / \operatorname{span}(I_{(l+r)n+z}) \quad \to \quad \mathfrak{gl}_n / \operatorname{span}(I_n)$$

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- If $(l, r) = (1, 0)$, then $\{\min_\lambda(\operatorname{rk}(\bullet - \lambda I)) \leqslant k\}$ is preserved.

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- If $r = z = 0$ and $\mathrm{char}(K) \mid n$, then $\{\mathrm{Tr}(\bullet) = \mu\}$ is preserved.

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- If $(l, r) = (1, 0)$, then $\{\min_\lambda(\mathrm{rk}(\bullet - \lambda I)) \leqslant k\}$ is preserved.
- If $r = z = 0$ and $\mathrm{char}(K) \mid n$, then $\{\mathrm{Tr}(\bullet) = \mu\}$ is preserved.
- If $z = 0$ and $\mathrm{char}(K) = 2 \mid n$, then $\{\mathrm{Tr}(\bullet) = \mu\}$ is preserved.

Classification of closed G-stable subsets

Theorem

The closed G-stable subsets of V are

- $\varnothing$, $\{0\}$, V,
- $\{P \in V \mid \min_\lambda(\mathrm{rk}(P - \lambda I)) \leqslant k\}$ *for* $k \in \mathbb{N}$, *and*
- $\{P \in V \mid \mathrm{Tr}(P) = \mu\}$ *for* $\mu \in K$

whenever these sets make sense.

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whenever these sets make sense.

Remark

Here "make sense" means that the maps

$$\mathfrak{gl}_{(l+r)n+z} / \mathrm{span}(I_{(l+r)n+z}) \to \mathfrak{gl}_n / \mathrm{span}(I_n)$$

preserve the property. We ask that the property holds in every space.

Classification of closed G-stable subsets

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The space of $\infty \times \infty$ matrices $K^{\mathbb{N} \times \mathbb{N}}$ is the inverse limit of

$$\mathfrak{gl}_1 \leftarrow \mathfrak{gl}_2 \leftarrow \mathfrak{gl}_3 \leftarrow \mathfrak{gl}_4 \leftarrow \dots$$

and is acted on by GL by conjugation.

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Theorem

The irreducible closed GL-stable subsets of $K^{\mathbb{N} \times \mathbb{N}}$ are

- V,
- $\{P \in K^{\mathbb{N} \times \mathbb{N}} \mid \mathrm{rk}(P - \lambda I) \leqslant k\}$ *for* $\lambda \in K$ *and* $k \in \mathbb{N}$, *and*
- $\{P \in K^{\mathbb{N} \times \mathbb{N}} \mid \min_\lambda(\mathrm{rk}(P - \lambda I)) \leqslant k\}$ *for* $k \in \mathbb{N}$.

In particular, $K^{\mathbb{N} \times \mathbb{N}}$ *is* GL*-Noetherian.*

Proof

Let $X \subsetneq K^{\mathbb{N} \times \mathbb{N}}$ be a closed GL-stable subset. Let X_n be the closure of its projection onto $K^{n \times n}$.

Claim: $X \subseteq \{P \in K^{\mathbb{N} \times \mathbb{N}} \mid \min_\lambda(\mathrm{rk}(P - \lambda I)) \leqslant k\}$ for some $k \in \mathbb{Z}_{\geqslant 0}$.

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Pick an $n \in \mathbb{N}$ such that $X_n \neq K^{n \times n}$ and let $f(P) \in I(X_n)$ be non-zero and of degree d. Consider f as a function on bigger matrices.

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Let $X \subsetneq K^{\mathbb{N}\times\mathbb{N}}$ be a closed GL-stable subset. Let X_n be the closure of its projection onto $K^{n\times n}$.

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$$\begin{pmatrix} I_n & \lambda I_n & \\ & I_n & \\ & & I_\bullet \end{pmatrix} \begin{pmatrix} P & Q & \bullet \\ R & S & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \begin{pmatrix} I_n & \lambda I_n & \\ & I_n & \\ & & I_\bullet \end{pmatrix}^{-1} = \begin{pmatrix} P+\lambda R & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

for $\lambda \in K$.

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$\Rightarrow \mathrm{span}\{f(P+\lambda R) \mid \lambda \in K\} \subseteq \mathrm{span}\{$orbit of $f(P)\} \subseteq I(X_{2n+m})$

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The degree of f is d. Let f_d be the homogenous part of f of degree d.

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The field K is infinite.

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- $f_d(R)$ is a non-zero polynomial in the ideal of X_{2n+m} of degree at most d only dependent on R.

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for $A, B \in \mathrm{GL}_n$.

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This proves the claim.

Proof

Claim: $X \subseteq \{P \in K^{\mathbb{N} \times \mathbb{N}} \mid \min_\lambda(\mathrm{rk}(P - \lambda I)) \leqslant k\}$ for some $k \in \mathbb{Z}_{\geqslant 0}$.

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Then

$$X = \{P \in K^{\mathbb{N} \times \mathbb{N}} \mid \min_\lambda(\mathrm{rk}(P - \lambda I)) \leqslant \ell\} \cup \bigcup_{i=1}^{N} \{\mathrm{rk}(\bullet - \lambda I) \leqslant r_i\}$$

for some $\ell < r_1, \ldots, r_N \leqslant k$. □

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Thank you for your attention!

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