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Semi-algebraic properties of Minkowski sums of a twisted cubic segement

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joint work with Adam Czapliński and Markus Wageringel

9th swiss-french workshop in AG, Charmey, 14 January 2020

Eigenvalues of orthogonal matrices

Question (Rubinstein, Sarnak)

What are the possible sets of eigenvalues of an orthogonal $(2n+1) \times (2n+1)$ matrix A given its first $k \leqslant n$ moments $\mathrm{tr}(A), \ldots, \mathrm{tr}(A^k)$?

Eigenvalues of orthogonal matrices

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Let A be an orthogonal $(2n+1) \times (2n+1)$ matrix A with eigenvalues

$$\det(A), e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_n}$$

for some $\theta_1, \ldots, \theta_n \in [0, \pi]$. Then

$$\begin{aligned} \operatorname{tr}(A^j) &= \det(A)^j + (e^{ij\theta_1} + e^{-ij\theta_1}) + \cdots + (e^{ij\theta_n} + e^{-ij\theta_n}) \\ &= \det(A)^j + 2\cos(j\theta_1) + \cdots + 2\cos(j\theta_n) \end{aligned}$$

Eigenvalues of orthogonal matrices

We know

$$\begin{aligned}\frac{1}{2}(\mathrm{tr}(A^j) - \det(A)^j) &= \cos(j\theta_1) + \cdots + \cos(j\theta_n) \\ &= T_j(\cos(\theta_1)) + \cdots + T_j(\cos(\theta_n))\end{aligned}$$

where T_j is the j-th Chebyshev polynomial of the first kind.

$(\forall j : \deg(T_j) = j) \Rightarrow$ We know $\cos(\theta_1)^j + \cdots + \cos(\theta_n)^j$ for $j \leqslant k$

Eigenvalues of orthogonal matrices

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We write $t_i = \cos(\theta_i) \in [-1, 1]$ for each $i \in \{1, \ldots, n\}$.

Question (Rubinstein, Sarnak)
What are all possible sets of $(t_1, \ldots, t_n) \in [-1, 1]^n$ with given power sums $t_1^j + \cdots + t_n^j$ for $j \leqslant k$?

Eigenvalues of orthogonal matrices

Consider the set

$$\mathcal{A}_{k,n} = \left\{ \begin{pmatrix} t_1 \\ t_1^2 \\ \vdots \\ t_1^k \end{pmatrix} + \cdots + \begin{pmatrix} t_{n-1} \\ t_{n-1}^2 \\ \vdots \\ t_{n-1}^k \end{pmatrix} + \begin{pmatrix} t_n \\ t_n^2 \\ \vdots \\ t_n^k \end{pmatrix} \,\middle|\, -1 \leqslant t_1, \ldots, t_n \leqslant 1 \right\} \subseteq \mathbb{R}^k$$

Idea using membership tests of the sets $\mathcal{A}_{k,n}$: Let $p \in \mathbb{R}^k$ be a point. If $p \notin \mathcal{A}_{k,n}$, then there are no solutions. Otherwise, find all $t_n \in [-1, 1]$ such that $p' = p - (t_n, t_n^2, \ldots, t_n^k) \in \mathcal{A}_{k,n-1}$. Now, for each t_n, find all solutions for p' recursively.

Eigenvalues of orthogonal matrices

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Problem (Rubinstein, Sarnak, Sturmfels)
Find semi-algebraic descriptions of the sets $\mathcal{A}_{k,n}$ for $k = 3$.

The semi-algebraic descriptions

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For integers $k, \ell > 0$, write

$$\begin{aligned} A_{k\ell} &= k\ell(k+\ell)^2 > 0 \\ B_{k\ell}(x,y) &= 2k\ell x(2x^2 - 3(k+\ell)y) \\ C_{k\ell}(x,y) &= x^6 - 3(k+\ell)x^4y + 3(k^2+k\ell+\ell^2)x^2y^2 - (k-\ell)^2(k+\ell)y^3 \\ f_{k\ell}(x,y,z) &= A_{k\ell}\cdot z^2 + B_{k\ell}(x,y)\cdot z + C_{k\ell}(x,y) \end{aligned}$$

and take

$$X = \bigcup_{k=1}^{n-1} \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} + (n-k-1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} y \leqslant k + (x+k)^2 \\ y \geqslant (k+1)^{-1}x^2 \\ y \leqslant 1 + k^{-1}(x-1)^2 \\ z \leqslant \frac{-B_{k1}(x,y)}{2A_{k1}} \text{ or } f_{k1}(x,y,z) \leqslant 0 \end{array} \right\}$$

$$Y = \bigcup_{\ell=1}^{n-1} \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} + (n-\ell-1)\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} y \leqslant \ell + (x-\ell)^2 \\ y \geqslant (\ell+1)^{-1}x^2 \\ y \leqslant 1 + \ell^{-1}(x+1)^2 \\ z \geqslant \frac{-B_{1\ell}(x,y)}{2A_{1\ell}} \text{ or } f_{1\ell}(x,y,z) \leqslant 0 \end{array} \right\}$$

Then we have $\mathcal{A}_{3,n} = X \cap Y$.

Some observations

Things we "see" with our eyes:

- The boundary consists of two shells $\mathcal{B}_n^+$ and B_n^- that project down to the (x, y)-plane injectively and with the same image.
- A point $p \in \mathbb{R}^3$ lies in $\mathcal{A}_{3,n} \Leftrightarrow p$ lies below $\mathcal{B}_n^+$ and above $\mathcal{B}_n^-$.
- Both shells have $n - 1$ ridges.

Things we don't "see" with our eyes:

- For fixed (x, y), the z^+ such that $(x, y, z^+) \in \mathcal{B}_n^+$ is the highest root of a parabola (with positive leading coefficient).
- For fixed (x, y), the z^- such that $(x, y, z^-) \in \mathcal{B}_n^-$ is the lowest root of a parabola (with positive leading coefficient).

Theorem
All of these observations are true.

Overview of the proof

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(1) Find a collection of ridges whose union is a superset the boundary.

(2) Look at $\mathcal{A}_{3,4}$ in more detail and exclude all unnecessary ridges from the union.

(3) Use the previous step to do the same for $\mathcal{A}_{3,n}$.

(4) Find semi-algebraic descriptions of the ridges.

(5) Finish up.

Finding ridges

Let $p \in \mathbb{R}^3$ be a point on the boundary of $\mathcal{A}_{3,n}$ and write

$$p = \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{pmatrix} + \cdots + \begin{pmatrix} t_n \\ t_n^2 \\ t_n^3 \end{pmatrix}$$

for some tuple $(t_1, \ldots, t_n) \in [-1, 1]^n$.

Theorem

The set $\{t_1, \ldots, t_n\} \backslash \{-1, 1\}$ *has at most two elements.*

Proof.

If $-1 < t_i < t_j < t_k < 1$, fix the other entries and consider the map

$$\begin{aligned} \varphi\colon [-1,1]^3 &\to \mathbb{R}^3 \\ (s,r,t) &\mapsto \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \begin{pmatrix} r \\ r^2 \\ r^3 \end{pmatrix} + \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + \sum_{\ell \notin \{i,j,k\}} \begin{pmatrix} t_\ell \\ t_\ell^2 \\ t_\ell^3 \end{pmatrix} \end{aligned}$$

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At the point (t_i, t_j, t_k), its Jacobian

$$\begin{pmatrix} 1 & 1 & 1 \\ 2s & 2r & 2t \\ 3s^2 & 3r^2 & 3t^2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ s & r & t \\ s^2 & r^2 & t^2 \end{pmatrix}$$

is invertible. So φ is locally invertible at $p = \varphi(t_i, t_j, t_k)$ by the Inverse Function Theorem. So p lies in the interior of $\mathcal{A}_{3,n}$. $\square$

Finding ridges

Consequence: The boundary of $\mathcal{A}_{3,n}$ is contained in the union of

$$\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \middle| -1 \leqslant s \leqslant t \leqslant 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

where $k, \ell \geqslant 1$ and $a, b \geqslant 0$ have sum n.

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where $k, \ell \geqslant 1$ and $a, b \geqslant 0$ have sum n.

Next Goal: Prove that this is still true when we leave out the following:

- the cases where $k, \ell \geqslant 2$
- the cases where $a, b > 0$
- the cases where $\ell \geqslant 2$ and $a > 0$
- the cases where $k \geqslant 2$ and $b > 0$

Eliminating ridges in $\mathcal{A}_{3,4}$

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Proposition

Take $-1 < s < t < 1$. Then there exist distinct $-1 < t_1, t_2, t_3, t_4 < 1$ such that

$$2\begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + 2\begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_1^2 \\ t_1^3 \end{pmatrix} + \begin{pmatrix} t_2 \\ t_2^2 \\ t_2^3 \end{pmatrix} + \begin{pmatrix} t_3 \\ t_3^2 \\ t_3^3 \end{pmatrix} + \begin{pmatrix} t_4 \\ t_4^2 \\ t_4^3 \end{pmatrix}$$

(and hence the point cannot lie on the boundary of $\mathcal{A}_{3,4}$.)

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Proposition

Take $-1 < s < t < 1$. Then there exist distinct $-1 < t_1, t_2, t_3, t_4 < 1$ such that

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(and hence the point cannot lie on the boundary of $\mathcal{A}_{3,4}$.)

Proof.

Try to find a solution satisfying the additional condition that

$$t_1 + t_2 = t_3 + t_4 = s + t$$

holds. This is doable. □

Eliminating ridges in $\mathcal{A}_{3,4}$

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Lemma

Take $\delta > 0$. Then there exists an $\varepsilon > 0$ such that

- *for all* $-1 < s < t < 1$ *and* $0 \leqslant \varepsilon' \leqslant \varepsilon$ *with* $1-t, t-s, s-(-1) \geqslant \delta$
$$(s, s^2, s^3) + (t, t^2, t^3) + (1,1,1) - (0,0,\varepsilon') \in \mathcal{A}_{3,3}$$
- *for all* $-1 < s < t < 1$ *and* $0 \leqslant \varepsilon' \leqslant \varepsilon$ *with* $1-t, t-s, s-(-1) \geqslant \delta$
$$(s, s^2, s^3) + (t, t^2, t^3) + (-1,1,-1) + (0,0,\varepsilon') \in \mathcal{A}_{3,3}$$
- *for all* $-1 \leqslant s < t < 1$ *and* $0 \leqslant \varepsilon' \leqslant \varepsilon$ *with* $1-t, t-s \geqslant \delta$
$$(s, s^2, s^3) + 2(t, t^2, t^3) + (0,0,\varepsilon') \in \mathcal{A}_{3,3}$$
- *for all* $-1 < s < t \leqslant 1$ *and* $0 \leqslant \varepsilon' \leqslant \varepsilon$ *with* $t-s, s-(-1) \geqslant \delta$
$$2(s, s^2, s^3) + (t, t^2, t^3) - (0,0,\varepsilon') \in \mathcal{A}_{3,3}$$

Sketch of proof.

Take $-1 \leqslant s < t < 1$. We want to show that

$$(s, s^2, s^3) + 2(t, t^2, t^3) + (0, 0, \varepsilon) \in \mathcal{A}_{3,3}$$

for small $\varepsilon > 0$. For $0 \leqslant \lambda \ll 1$, consider

$$(t_1, t_2, t_3) = (s + 2\lambda, t - \lambda + \mu, t - \lambda - \mu)$$

where $\mu = \sqrt{2(t-s)\lambda - 3\lambda^2}$ and finds that

$$(s, s^2, s^3) + 2(t, t^2, t^3) + (0, 0, \varepsilon) = \sum_{i=1}^{3} (t_i, t_i^2, t_i^3) \in \mathcal{A}_{3,3}\,.$$

for $\varepsilon = 6(t-s)^2\lambda - 24(t-s)\lambda^2 + 24\lambda^3$. $\square$

Eliminating ridges in $\mathcal{A}_{3,n}$

Take $p \in \mathcal{A}_{3,n}$ and $q \in \mathcal{A}_{3,4}$. Assume that q lies in the interior of $\mathcal{A}_{3,4}$.
$\Rightarrow p + q$ lies in the interior of $p + \mathcal{A}_{3,4} \subseteq \mathcal{A}_{3,n+4}$.
$\Rightarrow p + q$ does not lie on the boundary of $\mathcal{A}_{3,n+4}$.
This eliminates the sets

$$\left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \middle| -1 \leqslant s \leqslant t \leqslant 1 \right\} + a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

where

- the cases where $k, \ell \geqslant 2$
- the cases where $a, b > 0$
- the cases where $\ell \geqslant 2$ and $a > 0$
- the cases where $k \geqslant 2$ and $b > 0$

Semi-algebraic descriptions of the ridges

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Theorem

Suppose that $k > \ell$. Then we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \left\{ k \begin{pmatrix} s \\ s^2 \\ s^3 \end{pmatrix} + \ell \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} \;\middle|\; -1 \leqslant s \leqslant t \leqslant 1 \right\}$$

if and only if

$$\begin{array}{rcl} 0 & \leqslant & |x|, y, |z| \leqslant k + \ell \\ 0 & \leqslant & k\ell D_{k\ell}(x,y) \leqslant k^2(k+\ell+x)^2, \ell^2(k+\ell-x)^2 \\ 0 & = & A_{k\ell} \cdot z^2 + B_{k\ell}(x,y) \cdot z + C_{k\ell}(x,y) \\ z & \geqslant & \dfrac{-B_{k\ell}(x,y)}{2A_{k\ell}} \end{array}$$

Finishing up

What is left:

- Prove that the leftover ridges form two shells $\mathcal{B}_n^+$ and B_n^- that project down to the (x, y)-plane injectively and with the same image.
- Prove that $\mathcal{B}_n^+$ and B_n^- intersect in their boundary and that $\mathcal{B}_n^+$ lies above B_n^- otherwise.
- Use the semi-algebraic descriptions of the ridges to find the semi-algebraic description of $\mathcal{A}_{3,n}$.

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- Use the semi-algebraic descriptions of the ridges to find the semi-algebraic description of $\mathcal{A}_{3,n}$.

This finished the proof and the talk.

Thank you for listening!

References

- Bik, Czapliński, Wageringel, *Semi-algebraic properties of Minkowski sums of a twisted cubic segment*, in preparation.
- Rubinstein, Sarnak, *The underdetermined matrix moment Problem I*, in preparation.