UNSTABLE LOCALIZATION
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M. J. HOPKINS

Contents

1. Introduction

In the last lecture we discussed the (telescopic) chromatic localizations of stable homotopy theory. These are based on an orthogonal decomposition

$$\langle S^0\rangle = \langle v_0^{-1}W_0\rangle \vee \cdots \vee \langle v_n^{-1}W_n\rangle \vee \langle W_{n+1}\rangle \tag{1.1}$$

in which W_k is a finite p-local spectrum of type k and $v_k : \Sigma^{d_k}W_k \to W_k$ is a v_k self-map. (Recall that a finite p-local spectrum is of *type* k $K(k)_*W_k \neq 0$ and for $m < k$, $K(m)_*W_k = 0$.) We are interested in the localization L_n^f of homotopy theory at the maps which induce isomorphisms

$$\pi_* v_k^{-1}(W_k \wedge X) \to \pi_* v_k^{-1}(W_k \wedge Y) \tag{1.2}$$

for all $0 \leq k \leq n$. These maps are sometimes called L_n^f-equivalences. As proved in the previous talk, the decomposition (1.1) implies that one can also realise L_n^f-local homotopy by killing W_{n+1}.

There are several unstable analogues of the L_n^f-equivalences. The most naive is the collection of maps inducing L_n^f-equivalences of suspension spectra. This has the defect that it does not directly make use of unstable maps. There is another (non-equivalent) approach that does.

To describe it replace each W_k $(0 \leq k \leq n)$ by its Spanier-Whitehead dual DW_k and rewrite (1.2) as

$$v_k^{-1}\pi_*(X; DW_k) \to v_k^{-1}\pi_*(Y; DW_k) \tag{1.3}$$

in which

$$\pi_*(X; DW_k) = [DW_k, X]_* = [\Sigma^* DW_k, X].$$

Now the condition of whether or not (1.3) is an equivalence depends only on the Bousfield class of DW_k, so we might as well use W_k instead, as it is Bousfield equivalent by the thick subcategory theorem.

Definition 1.4. A map $X \to Y$ of spectra is a $T(k)$-equivalence if it induces an isomorphism

$$v_k^{-1}\pi_*(X; W_k) \to v_k^{-1}\pi_*(Y; W_k).$$

We are eventually interested in homotopy theory localized at the $T(k)$-equivalences. It turns out easier to begin with the L_n^f-equivalences, which, as just stated above are maps which are $T(k)$-equivalences for all $0 \le k \le n$.

Now we are led to two other notions. Let W_k be a *space* with a v_k self-map $v_k : \Sigma^{d_k} W_k \to W_k$. We define the periodic homotopy set $v_k^{-1}\pi_m(X; W_k)$ by

$$v_k^{-1}\pi_m(X; W_k) = \varinjlim_{t\to\infty} [\Sigma^{m+td_k} W_k, X]$$

in which the bonding maps are given by precomposition with (a suspension of) v_k. For $k = 0$ this is an abelian group if $m \ge 2$. When $k > 0$ this is always an abelian group because the terms in which W_k has been suspended an large number of times are cofinal. In fact, because of this, when $k > 0$ the abelian group $v_k^{-1}\pi_m(X; W_k)$ makes sense for a finite *spectrum* W_k and is functorial on the category of type k spectra. This will be spelled out more carefully when the Bousfield-Kuhn functor is defined.

Definition 1.5. A map $X \to Y$ of spaces is a v_k-periodic equivalence if for all choices of base point in X and all m,

$$v_k^{-1}\pi_m(X; W_k) \to v_k^{-1}\pi_m(Y; W_k)$$

is an isomorphism.

Using the Thick Subcategory Theorem one can easily check that when $k > 0$ this notion is independent of the choice of type k space (or spectrum) W_k. When $k = 0$ one can show that given W_0 and W_0' there is an N with the property that for $m > N$ a map is a $v_0^{-1}\pi_m(\,-\,; W_0)$-equivalence if an only if it is a $v_0^{-1}\pi_m(\,-\,; W_0')$-equivalence.

Our main object of study will eventually be based on the localization of the category of spaces at the maps f which are v_n-equivalences.

The second approach to unstable chromatic homotopy theory is based on "killing" a type $(n+1)$-space. Fix a space W_{n+1} which is a suspension and of type $(n+1)$. A space X is $P_{W_{n+1}}$-local if the inclusion

$$X \to \mathrm{Map}(W_{n+1}, X)$$

of the constant maps into the space of all maps is a weak equivalence. The category of $P_{W_{n+1}}$-local spaces is another category in which one might conceivably capture the v_k-periodic homotopy theory of spaces for $0 \le k \le n$. The hope would be to identify the localization of **Spaces** at the v_n-periodic equivalence with the category of spaces constructed as the fiber of a natural transformation from a $P_{W_{n+1}}$-local space to its P_{W_n}-localization. This works almost as well as it does in stable homotopy theory, and the only thing that needs to be added is the passage to a suitable connected cover. The picture will be described in a later lecture. The purpose of this talk is to establish the basic necessary supporting facts about unstable localization with respect to a finite complex.

2. Unstable localization

2.1. **Localization with respect to a map.** This was the subject of the second talk in this seminar, but let me summarize some of the main points here for easier reference. The theory of unstable localizations was developed by Bousfield and Dror/Farjoun in a series of papers [3, 2, 1, 4, 5, 6].

Let **Spaces** be the category of pointed topological spaces and for $X, Y \in \mathbf{Spaces}$, write

$$\operatorname{Map}_*(X, Y) = Y^X$$

for the space of basepoint preserving maps from X to Y and

$$\operatorname{Map}(X, Y) = Y^{X_+} = \operatorname{Map}_*(X_+, Y)$$

for the space of unpointed maps.

Suppose that $f : A \to D$ is a map in **Spaces**.

Definition 2.1. A space Y is *f-local* if the map

$$f^* : Y^{D_+} \to Y^{A_+}$$

is a weak equivalence.

By taking iterated pushouts along maps $\Sigma^t A_+ \to \Sigma^t D_+$, and passing to the colimit, Bousfield and Farjoun construct a localization functor

$$L_f : \mathbf{Spaces} \to \mathbf{Spaces}$$

and a natural transformation $X \to L_f X$ with the properties

(1) For all X, $L_f X$ is f-local.

(2) The map $X \to L_f X$ is universal for maps to an f-local space: if Y is f local then the map

$$\operatorname{Map}(L_f X, Y) \to \operatorname{Map}(X, Y)$$

is a weak equivalence.

Example 2.2. Suppose that f is the map $S^{n+1} \to \mathrm{pt}$. In this case a space Y is f-local if and only if Y is $(n+1)$ coconnected. The localization map $X \to P_{S^{n+1}} X$ is the map from X to its Postnikov section $P^n X$.

Example 2.3. If $f : \Sigma^d A \to A$ is a self-map and Y is f-local then

$$[A, Y] \to [\Sigma^d A, Y]$$

is a bijection and so the graded set $[A, Y]_*$ is a periodic graded abelian group in the sense that composition with f gives an isomorphism

$$[A, Y]_* \approx [A, Y]_{d+*}.$$

In this sense L_f can be thought of as some kind of "periodization."

There is also a close relationship between the localization L_f and the localization $P_{D \cup_f CA}$. I leave it to the reader to work this out using the results of this talk.

The case in which f is the unique map $f : A \to \mathrm{pt}$ has its own notation and terminology.

Definition 2.4. A space Y is *P_A-local* if it is f-local for the unique map $f : A \to \mathrm{pt}$.

The localization functor L_f is written P_A in this case. The use of the symbol P is in reference to the situation in Example 2.2. Sometimes P_A is referred to the "nullification" or "A-nullification" and a P_A-local space is called "A-null." The P in P_W is also in reference to the situation of "periodicity" in Example 2.3 (with W the mapping cone of the self-map) in which case the "W-periodic" and "W-periodization" are used.

In this lecture I'm going to stick with the terminology P_A-local and L_f-local. This is to avoid confusion with the class of spaces which localize to a point. There is a purpose to retaining both P_A and L_f. Some theorems are true for P_A but not for L_f. I've tried to stick to statements that indicate in the notation, the precise level of generality of the result.

2.2. **Local equivalences.**

Definition 2.5. A map $U \to V$ is an *L_f-equivalence* if for every L_f-local space Y the induced map of function spaces

$$\mathrm{Map}(V, Y) \to \mathrm{Map}(U, Y)$$

is a weak equivalence.

When f is the map $A \to \mathrm{pt}$ and L_f-equivalence will be called a P_A-equivalence.

It is more or less immediate from the definition that the class of L_f-equivalences is closed under finite products and homotopy colimits. Here are two instances of this we will use.

Proposition 2.6. *If*

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[dl] \\ & B & \end{tikzcd}$$

is a map of fibrations over B and for each $b \in B$ the induced map of fibers is an L_f-equivalence then $X_1 \to X_2$ is an L_f-equivalence. □

Remark 2.7. The "Fibration Lemma" of the previous lecture (Lemma 4.4 below) is the special case in which $X_2 \to B$ is the identity map.

Proposition 2.8. *Suppose that $G_1 \to G_2$ is a group homomorphism and*

$$\begin{tikzcd} G_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ G_2 \arrow[r] & X_2 \arrow[r] & B_2 \end{tikzcd}$$

is a map of principal bundles in the sense that the map $X_1 \to X_2$ is G_1-equivariant. If the left and middle vertical arrows are L_f-equivalences then the map $B_1 \to B_2$ is an L_f-equivalence.

Proof: This follows from realizing the map $B_1 \to B_2$ as the geometric realization of the map of simplical spaces (the bar construction) given in simplicial degree n by

$$X_1 \times (G_1)^n \to X_2 \times (G_2)^n.$$

□

Corollary 2.9. *If $X \to Y$ is map of $(n-1)$-connected spaces having the property that $\Omega^n X \to \Omega^n Y$ is an L_f-equivalence then $X \to Y$ is an L_f-equivalence.*

Proof: This follows by induction on n using Proposition 2.8 and the diagram

$$\begin{tikzcd}
\Omega^k X \arrow[r] \arrow[d] & P\Omega^{k-1} X \arrow[r] \arrow[d] & \Omega^{k-1} X \arrow[d] \\
\Omega^k Y \arrow[r] & P\Omega^{k-1} Y \arrow[r] & \Omega^{k-1} Y
\end{tikzcd}$$

□

2.3. **Localization with respect to a suspension: Bousfield's Theorem.** There is a somewhat surprising deeper analogy between P_A and Postnikov sections. We begin with a simple observation.

Lemma 2.10. *Suppose that $f : A \to D$ is a map of pointed spaces. If Y is f-local then Y is Σf-local.*

Proof: This looks like it might involve something until you actually start writing down the proof. Suppose that Y is f-local. The space $\mathrm{Map}(\Sigma A, Y)$ fits into a homotopy pullback square

$$\begin{tikzcd}
\mathrm{Map}(\Sigma A, Y) \arrow[r] \arrow[d] & \mathrm{Map}(S^1 \times A, Y) \arrow[d] \\
Y \arrow[r] & \mathrm{Map}(S^1 \vee A, Y)
\end{tikzcd}$$

and the space $\mathrm{Map}(S^1 \vee A, Y)$ fits into the pullback square

$$\begin{tikzcd}
\mathrm{Map}(S^1 \vee A, Y) \arrow[r] \arrow[d] & \mathrm{Map}(S^1, Y) \times \mathrm{Map}(A, Y) \arrow[d] \\
Y \arrow[r] & Y \times Y.
\end{tikzcd}$$

So, assuming Y to be f-local, in order to check that Y is Σf-local it suffices to check that

$$\mathrm{Map}(S^1 \times D, Y) \to \mathrm{Map}(S^1 \times A, Y)$$

is a weak equivalence. Rewriting this as

$$\mathrm{Map}(S^1, Y^{D_+}) \to \mathrm{Map}(S^1, Y^{A_+})$$

makes this obvious. □

Remark 2.11. The above argument actually proves that if Y if L_f-local then Y is $L_{S \wedge f}$ local for any pointed space S.

Because of Lemma 2.10 there is a natural transformation

$$L_{\Sigma f} \to L_f,$$

or, in the case of $f : A \to \mathrm{pt}$ a natural transformation

$$P_{\Sigma A} \to P_A.$$

This leads to a tower of localization functors
$$\cdots \to P_{\Sigma^{k+1}A}X \to P_{\Sigma^k A}X \to \cdots .$$
When $A = S^0$ this is the Postnikov tower and the fibers are always Eilenberg-MacLane spaces. Surprisingly this is always true.

Theorem 2.12 (Bousfield [2, Theorem 7.2])**.** *If A has the properties*

i) *A is $(n-1)$-connected,*

ii) *the reduced homology groups $\tilde{H}_*(A;\mathbb{Z})$ are p-power torsion, so that $A \to pt$ is a $\mathbb{Z}[1/p]$-homology equivalence,*

iii) *$H_n(A;\mathbb{Z}/p) \neq 0$,*

then for $k \geq 1$, the fiber of
$$P_{\Sigma^{k+1}A}X \to P_{\Sigma^k A}X$$
is an Eilenberg-MacLane space $K(G, n+k)$ for some p-torsion abelian group G.

Most of this lecture will be devoted to the proof of this result.

2.4. **Unstable Bousfield classes.** The localization L_f depends only on the class of f-local spaces. In the case $f : A \to \text{pt}$ this is the class of Y for which the inclusion of the constant maps maps
$$Y \to Y^{A_+}$$
is a weak equivalence.

Definition 2.13. Two spaces A and B are *unstably Bousfield equivalent* if the class of P_A-local spaces coincides with the class of P_B-local spaces. The *unstable Bousfield class* of A, denoted $\langle A\rangle_{\text{un}}$, is the collection of all A' which are unstably Bousfield equivaelnt to A.

Remark 2.14. Bousfield uses the symbol $\langle A\rangle$ instead of $\langle A\rangle_{\text{un}}$, and uses the terms P-equivalent and P-equivalence class.

Just as in the stable case the unstable Bousfield equivalence classes have a partial ordering.

Definition 2.15. For spaces A and B the relation $\langle A\rangle_{\text{un}} \leq \langle B\rangle_{\text{un}}$ holds if the implication
$$Y \text{ is } P_B\text{-local} \implies Y \text{ is } P_A\text{-local}$$
is true.

When $\langle A\rangle_{\text{un}} \leq \langle B\rangle_{\text{un}}$ there is a unique diagram of natural transformation
$$X \to P_A X \to P_B X.$$

Example 2.16. From Lemma 2.10, there is an inequality
$$\langle \Sigma A\rangle_{\text{un}} \leq \langle A\rangle_{\text{un}}.$$

Also important is the class of spaces X for which $L_f X \sim *$. This is exactly the class of spaces X with the property that for every L_f-local Y the inclusion of the constant maps $Y \to Y^{X_+}$ is a weak equivalence. Bousfield calls such spaces f co-null, and they are related to Farjoun's "cellular spaces." I think it is easier to remember the property if we call these *L_f-acyclic spaces*, so that is what we will do. In case f is the map $f : A \to \text{pt}$ we will call this the class of P_A-acyclic spaces. The following is straightforward.

Lemma 2.17. *For spaces A and A' one has $\langle A\rangle_{un} \geq \langle A'\rangle_{un}$ if and only if every $P_{A'}$-acyclic space is P_A-acyclic. This in turn holds if and only if A' is P_A-acyclic.* $\square$

3. Application to unstable periodic homotopy theory

Bousfield applies Theorem 2.12 to show that if W is a finite p-local space then, up to connective covers, the localization P_W depends only on the (stable) type of W. I'll start with one form of the result and indicate more precise version later.

Proposition 3.1. *Suppose that W and W' are finite p-local complexes and that the suspension spectrum of W' is in the thick subcategory generated by the suspension spectrum of W. There is a $d > 0$ for which*

$$\langle W\rangle_{un} \geq \langle \Sigma^d W'\rangle_{un}.$$

Proof: Let's consider the full subcategory $\mathcal{C}$ of finite spectra consisting of those of the form $S^t \wedge W'$ with $t \in \mathbb{Z}$ and W' a space for which

$$\langle W\rangle_{\mathrm{un}} \geq \langle W'\rangle_{\mathrm{un}}.$$

The category $\mathcal{C}$ is non-empty since it contains the suspension spectrum of W, so it suffices to show that $\mathcal{C}$ is thick. Suppose that $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ is a cofibration sequence of finite spectra, with two of the terms in $\mathcal{C}$. We must show that the third is. Since $\mathcal{C}$ is closed under arbitrary suspensions we can assume that it is $\mathbf{A}$ and $\mathbf{B}$ that are in $\mathcal{C}$. Suspending a few times if necessary, we may assume that the cofibration sequence in question comes from cofibration sequence $A \to B \to C$ of finite p-local spaces, and that

$$\langle W\rangle_{\mathrm{un}} \geq \langle A\rangle_{\mathrm{un}}$$
$$\langle W\rangle_{\mathrm{un}} \geq \langle B\rangle_{\mathrm{un}}.$$

By Lemma 4.1 the space C is P_W-null and so $\langle W\rangle_{\mathrm{un}} \geq \langle C\rangle_{\mathrm{un}}$. The case of a retract is similar. $\square$

We now turn to the relationship between unstable localization and connective covers. For a space X let $X\langle d\rangle$ be the $(d-1)$-connected cover of X and define the order of precedence of this symbol by

$$P_W X\langle d\rangle = \big(P_W X\big)\langle d\rangle.$$

Proposition 3.2. *If W and W' are finite complexes with $\langle W\rangle = \langle W'\rangle$ (stable Bousfield classes), then there is a d and a natural weak equivalence*

$$P_{\Sigma W}(X)\langle d\rangle \approx P_{\Sigma W'}(X)\langle d\rangle.$$

Proof: By Lemma 3.1 there are integers k_1, k_2, k_3 with

$$\langle \Sigma W\rangle_{\mathrm{un}} \geq \langle \Sigma^{k_1} W'\rangle_{\mathrm{un}} \geq \langle \Sigma^{k_2} W\rangle_{\mathrm{un}} \geq \langle \Sigma^{k_3} W'\rangle_{\mathrm{un}}.$$

This gives natural transformations

$$P_{\Sigma^{k_3} W'} \to P_{\Sigma^{k_2} W} \to P_{\Sigma^{k_1} W'} \to P_{\Sigma W}.$$

By Bousfield's Theorem (Theorem 2.12) there exists a d with the property that for all X, the two-fold composites in

$$P_{\Sigma^{k_3} W'} X\langle d\rangle \to P_{\Sigma^{k_2} W} X\langle d\rangle \to P_{\Sigma^{k_1} W'} X\langle d\rangle \to P_{\Sigma W} X\langle d\rangle$$

are weak equivalences, as are the maps

$$P_{\Sigma^{k_1}W}X\langle d\rangle \to P_{\Sigma W}X\langle d\rangle$$
$$P_{\Sigma^{k'_1}W'}X\langle d\rangle \to P_{\Sigma W'}X\langle d\rangle.$$

The result follows directly from this. □

With a more care one can make a more precise statement.

Proposition 3.3. *The integer d in Proposition 3.2 can be taken to be*

$$1 + \max\{\mathrm{conn}(\Sigma W), \mathrm{conn}(\Sigma W')\}.$$

To go further we remind the reader of a result from the previous lecture (the proposition at the end of page 23). This result also appears to be a consequence of Bousfield's theorem, but it is actually used in the proof. It is true with P_W replaced with L_f. A generalization, using basically the same proof will be described later (Proposition 4.5).

Proposition 3.4. *If $\tilde{H}_d(W;\mathbb{Z}/p) \neq 0$ then for $d' \geq d$, the space $K(\mathbb{Z}/p,d)$ is P_W-acyclic.* □

Proposition 3.3 can be deduced directly from Proposition 3.4, though it is somewhat cleaner to do so in terms of Bousfield's determination of unstable Bousfield classes of finite p-local spaces. The main tools used in the proof are the Thick Subcategory Theorem and Bousfield's Theorem 2.12.

Proposition 3.5. *If W is an $(n-1)$-connected space with $H_*(W;\mathbb{Z}[1/p]) = 0$ and $H_n(W;\mathbb{Z}/p) \neq 0$ then for every $k > 0$*

$$\langle \Sigma W\rangle_{un} = \langle \Sigma^{k+1}W\rangle_{un} \vee \langle K(\mathbb{Z}/p, n+1)\rangle_{un}.$$

Proof: It suffices to establish the case $k = 1$. Since $H_{n+1}(\Sigma W;\mathbb{Z}/p) \neq 0$, the space $K(\mathbb{Z}/p, n+1)$ is $P_{\Sigma W}$-acyclic (Proposition 3.4 above). Since $\Sigma^2 W$ is also $P_{\Sigma W}$-acyclic this gives the inequality

$$\langle \Sigma^2 W\rangle_{\mathrm{un}} \vee \langle K(\mathbb{Z}/p, n+1)\rangle_{\mathrm{un}} \leq \langle \Sigma W\rangle_{\mathrm{un}}.$$

For the reverse inequality it suffices, by Lemma 2.17, to show that

$$P_{\Sigma^2 W \vee K(\mathbb{Z}/p,n+1)}(\Sigma W) \sim *.$$

For this note that by Theorem 2.12 the fiber of

$$P_{\Sigma^2 W}\Sigma W \to P_{\Sigma W}\Sigma W \tag{3.6}$$

is an Eilenberg-MacLane space $K(G, n+1)$ for some p-torsion abelian group G, and so is $P_{K(\mathbb{Z}/p,n+1)}$-acyclic. Since $P_{\Sigma W}\Sigma W \sim *$ this means that ΣW is $P_{\Sigma^2 W \vee K(\mathbb{Z}/p,n+1)}$-acyclic. □

Combining Proposition 3.5 and Proposition 3.2 gives

Corollary 3.7 (Bousfield [2, Theorem 9.15])**.** *Suppose that W and W' are finite CW complexes of type k and k' with $k, k' > 0$. The unstable Bousfield classes are in the relation*

$$\langle W\rangle_{un} \geq \langle W'\rangle_{un}$$

if and only the stable Bousfield classes are related by

$$\langle W\rangle \geq \langle W'\rangle$$

and the connectivities satisfy

$$\operatorname{conn}(W) \le \operatorname{conn}(W'). \quad \square$$

4. Proof of Bousfield's Theorem

We now turn to the proof of Theorem 2.12. We begin with a list of elementary facts about unstable localization.

4.1. **The list of elementary properties.** Here are a few facts about unstable localization that are easily verified from the definition. The proofs are no harder than the proof of Lemma 2.10. This list will be referred to as the *List of Elementary Properties.*

(1) For a map $f : A \to D$, a space X is L_f local if and only if for every choice of base point in X the map

$$\operatorname{Map}_*(D, X) \to \operatorname{Map}_*(A, X)$$

of pointed mapping spaces is a weak equivalence.

(2) If $f : A \to D$ is a map of connected spaces then X is L_f-local if and only if every path component of X is L_f-local.

(3) The map $L_f(X \times Y) \to L_f(X) \times L_f(Y)$ is a weak equivalence.

(4) If f is the identity map then every space is L_f-local

(5) A space X is both L_f-local and L_g-local if and only if X is $L_{f \vee g}$-local

(6) A space X is $L_{\Sigma f}$-local if and only if for every choice of base point, ΩX is f-local.

(7) If $L_f X \sim *$ then $L_{\Sigma f} \Sigma X \sim *$ (this follows directly from the construction).

(8) If $f : A \to D$ is a map of connected spaces and $L_f X$ is connected then X is connected (this follows from the construction).

(9) If $f : A \to D$ is a $(k-1)$-connected (in the sense that (D, A) is $(k-1)$-connected) then

$$X \to L_f X$$

is $(k-1)$-connected. (This follows from the construction of L_f and the relative Hurewicz theorem.)

(10) If $p : X \to B$ is a map between L_f-local spaces then for every $b \in B$ the homotopy fiber of p over b is L_f-local.

(11) If $p : X \to B$ is a map, B is P_A-local, and for every $b \in B$ the homotopy fiber of p over b is P_A-local, then X is P_A-local.

(12) If A is n-connected and $\pi_i X = 0$ for $i > n$ then X is P_A-local.

Sampling of proof: For part (1) consider the map of fibration sequences

$$\begin{tikzcd}
\operatorname{Map}_*(D, X) \arrow[r] \arrow[d] & \operatorname{Map}(D, X) \arrow[r] \arrow[d] & X \arrow[d, equal] \\
\operatorname{Map}_*(A, X) \arrow[r] & \operatorname{Map}(A, X) \arrow[r] & X \,.
\end{tikzcd}$$

The assumption is that for every choice of base point in X the map of fibers on the right is a weak equivalence. It follows that the map of total spaces is a weak equivalence.

Now let's also do part (6). By (2) we may assume that X is connected. Suppose that X is $L_{\Sigma f}$-local. Note that for each choice of base point in X one has

$$\operatorname{Map}(A, \Omega X) \approx \operatorname{Map}_*(A_+, \Omega X) \approx \operatorname{Map}_*(\Sigma A_+, X)$$

so that by (1) ΩX is L_f-local for every choice of base point in X if and only if X is $L_{S^1 \wedge f_+}$-local. But $S^1 \wedge f_+ \sim S^1 \vee \Sigma f$ so the claim follws from (5) and (4). $\square$

The difference in the statements of (10) and (11) is responsible for the fact that Bousfield's Theorem is formulated for P_W and not L_f.

4.2. **Further basic properties of localization.** Now we turn to some less trivial properties of localization (except for the first one which is still trivial).

Lemma 4.1. *If*

$$T \to U \to V$$

is a cofibration sequence and $L_f T \sim *$ then the map $L_f U \to L_f V$ is a weak equivalence. In particular, if T and one of U and V is L_f-acyclic then so is the other. $\square$

Lemma 4.2. *Suppose that $f : A \to D$ is a map of pointed spaces and W is L_f-acyclic. If X is P_W-acyclic then X is L_f-acyclic.*

Proof: The assertion is equivalent to the implication

$$Z \text{ is } L_f\text{-local} \implies Z \text{ is } P_W\text{-local.}$$

Suppose that Z is L_f-local. Then since $L_f W \sim *$, the inclusion of the constant maps

$$Z \to \operatorname{Map}(W, Z)$$

is a weak equivalence. This means that Z is P_W-local. $\square$

The following corollary implies that many results hold for trivial reasons in the non-connected case.

Corollary 4.3. *If there exists an L_f-acyclic space W having more than one path component, then for all X, $L_f X \sim *$.*

Proof: If W has more than one path component then S^0 is a retract of W and so S^0 is L_f-acyclic. Lemma 4.2 then implies any P_{S^0}-acyclic space is L_f-acyclic. But every space is P_{S^0}-acyclic. $\square$

In the previous lecture we established

Lemma 4.4 (Fibration Lemma)**.** *If $p : X \to B$ is a map and for every $b \in B$ the homotopy fiber of p over b is L_f-acyclic then $L_f X \to L_f B$ is a weak equivalence.*

Here is another useful result

Proposition 4.5. *If $L_f X \sim *$ then for every $k \geq 0$,*

i) *For any $(k-1)$-connected space Z, $L_{\Sigma^k f} X \wedge Z \sim *$*
ii) *For any $(k-1)$-connected spectrum Z, $L_{\Sigma^k f} \Omega^\infty (X \wedge Z) \sim *$.*

Proof: We will use the implication

$$L_f X \sim * \implies L_{\Sigma^k f}\Sigma^k X \sim *,$$

which follows from Property (7) of the *List of Elementary Properties*. Let $Z^{(n)}$ be the n-skeleton in a CW approximation to Z. We prove by induction on n that

$$L_{\Sigma f}(X \wedge Z^{(n)}) \sim *.$$

Since the class of X for which $L_g X \sim *$ is closed under filtered colimits this proves the result. Since Z is $(k-1)$-connected, $Z^{(k)} \sim \bigvee S^k$ and the induction starts. For the induction step use the cofibration sequence

$$X \wedge Z^{(n-1)} \to X \wedge Z^{(n)} \to \bigvee X \wedge S^n$$

and Lemma 4.1. By Corollary 4.3 it suffices to establish the second assertion in case X is connected. In that case one can proceed with a similar induction. To do so one must first show that for connected X one has

$$L_f X \sim * L_f QX \sim *.$$

This follows from the configuration space model, for instance, as in the proof of the proposition on page 23 of the last lecture. Combined with Property (7) this gives the implications that for all connected X and all $n \geq 0$

$$L_f X \sim * \implies L_{\Sigma^n f}\Sigma^n X \sim * \implies L_{\Sigma^n f} Q\Sigma^n X \sim *.$$

One now proceeds as in the proof of i) using the fibrations

$$QX \wedge Z^{(n-1)} \to QX \wedge Z^{(n)} \to Q\left(\bigvee X \wedge S^n\right)$$

and the fibration lemma. □

Remark 4.6. Proposition 3.4 is deduced from Proposition 4.5 by taking $k = 0$ and $Z = H\mathbb{Z}/p$ in part ii), and observing that the hypotheses imply that $K(\mathbb{Z}/p, d)$ is a retract of $\Omega^\infty X \wedge H\mathbb{Z}/p$.

4.3. **Fiberwise localization.** Given a fibration $X \to B$ one can localize each fiber to obtain a new map $L_f(X/B) \to B$ whose fiber over b is the L_f-localization of the fiber of X over B (this a consequence of the functoriality of L_f). This is called the *fiberwise localization* but could equally well be called the *relative localization*. If B is already L_f-local then $L_f(X/B) \to B$ is a fibration over an L_f-local base with L_f-local fibers. In the context of a general map $f : A \to D$ one can't go much further with this. But in the case of $f : A \to \mathrm{pt}$ one can avail oneself of part (11) of the *List of Elementary Properties* and conclude

Lemma 4.7. *If $X \to B$ is a fibration over a P_A-local space B then the total space of the fiberwise localization*

$$P_A(X/B) \to B$$

is P_A-local. □

As an application one has the following non-trivial theorem.

Theorem 4.8 (Farjoun [6, Theorem H.2], Bousfield [2, Corollary 4.8 (iii)])**.** *If F is the fiber of the map*

$$X \to P_W X$$

over any point, then $P_W F \sim *$.

Proof: Start with the fibration $F \to X \to P_W X$ and apply the functor P_W fiberwise to obtain a diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & P_W X \arrow[dl, dashed] \arrow[d, equal] \\ P_W F \arrow[r] & \bar{X} \arrow[r] & P_W X \end{tikzcd}$$

By item (11) of the list of basic properties, the space $\bar{X}$ is P_W-local. It follows from the universal property that the dotted arrow exists and both triangles commute. This implies that the map $P_W F \to \bar{X}$ is null. But the bottom sequence is a fibration with a section, so this implies that $P_W(F)$ is contractible. □

Remark 4.9. Some folks did not find the last assertion obvious. The claim (with different notation) is that if $F \to X \to B$ is a fibration with a section and $F \to X$ is null then F is contractible. Look at the fibration

$$\operatorname{Map}(F, F) \to \operatorname{Map}(F, X) \to \operatorname{Map}(F, B). \tag{4.10}$$

The section $B \to X$ gives this fibration a section. The assumption is that the identity map of F is connected to the base point in $\operatorname{Map}(F, X)$. This means that they are in the same orbit of the action of $\pi_1 \operatorname{Map}(F, B)$. But the section tells us that this action is trival and so the identity map is in the same path component of $\pi_0 \operatorname{Map}(F, F)$ as the null map.

4.4. **Localization and algebraic structures.** The fact that L_f is (weakly) product preserving has many useful consequences. For instance

Proposition 4.11. *If X is a loop space then $L_f X$ is also a loop space and the localization map $X \to L_f X$ is a loop map.*

Proof: By [7, Proposition 1.5] to give a loop space structure on X is equivalent to producing a simplicial space $X_\bullet$ in which X_0 is weakly contractible, X_1 is weakly equivalent to X and the map $X_n \to (X_1)^n$, corresponding to the order preserving inclusion $[1] \to [n]$ taking 0 to 0, is a weak equivalence. If $X_\bullet$ satisfies these properties, then $L_f X_\bullet$ does as well. □

Similarly one has

Proposition 4.12. *If X is an infinite loop space then $L_f X$ is an infinite loop space in a natural way, as is the map $X \to L_F X$.*

Proof: The fact that L_f is product preserving implies that if X is a (special) Γ-space then so is $L_f(X)$. □

There is a more refined version of this statement.

Proposition 4.13. *If that R is a (-1)-connected ring spectrum and that M is a (-1)-connected R-module then $L_f \Omega^\infty M$ is the naturally the 0^{th} space of a (-1)-connected R-module.*

Proof: Note that Proposition 4.12 gives us a functor $\tilde{L}_f$ from spectra to (-1)-connected spectra, characterized by a functorial weak equivalence

$$\Omega^\infty \tilde{L}_f X = L_f \Omega^\infty X.$$

I claim that $\tilde{L}_f$ has the property that for (-1)-connected spectra X and Z the map

$$Z \wedge X \to Z \wedge \tilde{L}_f(X)$$

becomes an equivalence after applying $\tilde{L}_f$. By the characterizing property it suffice to check that the map

$$\Omega^\infty(Z \wedge X) \to \Omega^\infty(Z \wedge \tilde{L}_f X)$$

is an L_f-equivalence. Using the Fibration Lemma (Lemma 4.4) and induction on the skeleton filtration of Z reduces one to the case of $Z = S^n$, which is a consequence of Corollary 2.9. With this in hand the proof is easy. If M is an R-module then using the inverse of the equivalence

$$\tilde{L}_f(R \wedge M) \to \tilde{L}_f(R \wedge \tilde{L}_f(M)),$$

$\tilde{L}_f M$ becomes an R-module via

$$R \wedge \tilde{L}_f M \to L_f(R \wedge \tilde{L}_f M) \xrightarrow{\approx} \tilde{L}_f(R \wedge M) \to \tilde{L}_f(M).$$

□

Remark 4.14. There is another approach to Proposition 4.13 that does not use Proposition 4.12 and in fact specializes to it in case $R = S^0$. It requires assuming R to be A_∞ and M to be a left A_∞ module, but it shows that $\tilde{L}_f M$ is a left A_∞ module. The idea is to make use of the ∞ version of Lawvere theories. I don't claim to have worked out the details and for all I know they might be worked out somewhere. In fact it might be contained in §6 of [2] under the name of "convergent functors." Let $\Theta_R^{\mathrm{op}} \subset \mathbf{Mod}_R$ be the full ∞-sub-category generated by the objects $R \wedge S_+$ with S a finite set. In other words we take Θ_R to have objects finite sets and in which the space of space of maps $S \to T$ is the space of left R-module maps $R \wedge T_+ \to R \wedge S_+$. A left R-module gives a product preserving functor

$$\operatorname{res} M : \Theta_R \to \mathbf{Spaces}_*$$

sending M to the functor

$$S \mapsto \mathbf{Mod}_R\big(R \wedge S_+, M\big).$$

On the other hand, given a product preserving functor

$$\mathcal{M} : \Theta_R \to \mathbf{Spaces}_*$$

one can construct an R-module M by the coend

$$\int_{\Theta_R} R \wedge S_+ \wedge \mathcal{M}(S).$$

These form an adjoint equivalence of categories. Put more succinctly, the left Kan extension of the inclusion along the Yoneda embedding

$$\begin{tikzcd} \Theta^{\mathrm{op}} \arrow[r, "\text{inclusion}"] \arrow[d, "\text{Yoneda}"'] & \mathbf{Mod}_R \\ (\mathbf{Spaces}_*)^{\Theta_R}_{pr} & \end{tikzcd}$$

is the left adjoint in an adjoint equivalence of the ∞-category, $\mathbf{Mod}_R$, of left R-modules, and the ∞-category $(\mathbf{Spaces}_*)^{\Theta}_{pr}$ of product preserving functors from Θ_R to pointed spaces. Under this equivalence, the value of a product preserving functor on the 1-point set is the zeroth space of the corresponding R-module. Armed with this, the proposition is obvious: post-composition with L_f sends a product preserving functor $\Theta_R \to \mathbf{Spaces}$, to another product preserving functor. The value on the set $S = \{\mathrm{pt}\}$ is $\Omega^\infty \tilde{L}_f X$, showing that $\tilde{L}_f$ is naturally a left R-module.

The case $R = H\mathbb{Z}$ of Proposition 4.13 gives

Corollary 4.15. *If X is a product of Eilenberg-MacLane spaces $K(G,n)$ with G abelian then $L_f(X)$ is a product of abelian Eilenberg-MacLane spaces.*

Proof: This is more or less immediate but one must observe that a space is a product of abelian Eilenberg-MacLane spaces if and only if it can be written as the zeroth space of a left $H\mathbb{Z}$-module (in the whatever sense of module you've used in proving Proposition 4.13). □

Remark 4.16. If you're really checking proofs it might help to make this observation: if the map $X \to H\mathbb{Z} \wedge X$ admits a left homotopy inverse $r : H\mathbb{Z} \wedge X \to X$ then X has the homotopy type of a wedge of Eilenberg-MacLane spectra. For the proof just choose for each n a Moore spectrum M_n with $H_n M_n = \pi_n X$ and map it to X by a map inducing an isomorphism on π_n. Then the map

$$H\mathbb{Z} \wedge \left(\bigvee M_n\right) \to H\mathbb{Z} \wedge X \to X$$

induces an isomorphism of homotopy groups.

5. Proof of Bousfield's Theorem

5.1. **A result of Farjoun and Smith.** In this section we will establish special case of Bousfield's theorem in which X is $L_{\Sigma f}$-acyclic. The result I'm quoting for this is Theorem A in the paper [5] of Farjoun and Smith.

We begin with a result about the localization of loop spaces. Recall from Proposition 4.11 that if X is a loop space then $L_f(X)$ is also a loop space. Proposition 5.2 below (which appears in Farjoun-Smith [6, Theorem A.1] and Bousfield [2, Theorem 3.1]) identifies which loop space.

Let $f : A \to D$ be a map. Starting with the localization map

$$X \to L_{\Sigma f} X$$

pass to loop spaces to get

$$\Omega X \to \Omega L_{\Sigma f} X.$$

By property (6) of the *List of Elementary Properties*, the space $\Omega L_{\Sigma f} X$ is L_f-local. This gives a unique natural factorization

$$\Omega X \to L_f \Omega X \to \Omega L_{\Sigma f} X. \tag{5.1}$$

Proposition 5.2. *If $f : A \to D$ is any map, and X is connected, then*

$$L_f \Omega X \to \Omega L_{\Sigma f} X$$

is a weak equivalence.

Proof: We first produce a map going the other direction. Note that our assumption that X is connnected implies that $L_{\Sigma f}X$ is connected. By Proposition 4.11 the map

$$\Omega X \to L_f \Omega X$$

is a map of loop spaces. Passing to classifying spaces gives

$$X \to BL_f\Omega X.$$

Property (6) implies that $BL_f\Omega X$ is $L_{\Sigma f}$-local and so this map factors uniquely as

$$X \to L_{\Sigma f}X \to BL_f\Omega X.$$

Passing to loop spaces gives a diagram

$$\Omega X \to \Omega L_{\Sigma f}X \to L_f\Omega X.$$

The rightmost map is our candidate for an inverse. Since the composite above is the localization map, this shows that

$$L_f\Omega X \to \Omega L_{\Sigma f}X \to L_f\Omega X$$

is the identity. For the other composite deloop the diagram

$$\Omega X \to L_f\Omega X \to \Omega L_{\Sigma f}X$$

to observe that

$$X \to BL_f\Omega X \to L_{\Sigma f}X$$

factors the localization map. This means that the composite

$$L_{\Sigma f}X \to BL_f\Omega X \to L_{\Sigma f}X$$

is the identity and hence so is the induced map of loop spaces. □

Proposition 5.3. *Suppose that $A \to D$ is a map of connected spaces. If X is $L_{\Sigma f}$-acyclic, then the map $L_{\Sigma^2 f}X \to L_{\Sigma^2 f}QX$ is the inclusion of a retract.*

We need a lemma.

Lemma 5.4. *If X is a connected $L_{\Sigma f}$-acyclic space (i.e., $L_{\Sigma f}X \sim *$) and Y is simply connected, then the map $X \vee Y \to X \times Y$ is an $L_{\Sigma^2 f}$-equivalence.*

Proof: The homotopy fiber is homotopy equivalent to

$$\Sigma\Omega X \wedge \Omega Y$$

which we write as $\big(\Omega X\big) \wedge Z$ in which Z is simply connected. Now

$$L_{\Sigma f}X \sim * \implies L_f\Omega X \sim *$$

and so Proposition 4.5 implies that $L_{\Sigma^2 f}\big(\Omega X\big) \wedge Z$ and the result follows from the Fibration Lemma (Lemma 4.4). □

Proof of Proposition 5.3: It suffices to show that $L_f X$ is an infinite loop space. Note that since $L_{\Sigma f}X \sim *$ then X is connected (Property (8) or (9) of the *List of Elementary Properties*), and so by Proposition 5.2, $L_f\Omega X \sim \Omega L_{\Sigma f}X \sim *$. Property (8) then shows that ΩX is connected and so X is simply connected. This allows us to use Lemma 5.4 to conclude that for every n the map

$$L_{\Sigma^2 f}\big(X \vee \cdots \vee X\big) \to L_{\Sigma^2 f}X^n$$

is a weak equivalence. Now either observe that this makes $L_{\Sigma^2 f}X$ into a Γ space, or, equivalently use the fold map $X \vee \cdots \vee X \to X$ to make X into an algebra over the infinite loop space operad. $\square$

Theorem 5.5. *Suppose that $f : A \to D$ is a map of connected spaces. If $L_{\Sigma f}X \sim *$ then $L_{\Sigma^2 f}X$ is weakly equivalent to a product of abelian Eilenberg-MacLane spaces.*

Proof: Since a retract of a product of abelian Eilenberg-MacLane spaces is an abelian Eilenberg-MacLane space (Remark 4.16) it suffices by Lemma 5.3 to show that $L_{\Sigma^2 f}Q(X)$ is a product of Eilenberg-Maclane spaces. Define a spectrum Z by the fibration sequence $Z \to S^0 \to H\mathbb{Z}$, so that there is a fibration sequence of spaces

$$\Omega^\infty Z \wedge X \to Q(X) \to \Omega^\infty(H\mathbb{Z} \wedge X).$$

From the fact that $\pi_0 S^0 = \mathbb{Z}$ we learn that Z is connected. This implies that $L_{\Sigma^2 f}\Omega^\infty Z \wedge X \sim *$, and by the Fibration Lemma that

$$L_{\Sigma^2 f}Q(X) \to L_{\Sigma^2 f}\Omega^\infty(H\mathbb{Z} \wedge X)$$

is an equivalence. $\square$

Proof of Theorem 2.12: By the trick of replacing A by $\Sigma^{k-1}A$ it suffices to show that the fiber F of

$$P_{\Sigma^2 A}X \to P_{\Sigma A}X$$

is an Eilenberg-MacLane space of the form $K(G, n+1)$ with G abelian. Now Theorem 4.8 implies $P_{\Sigma A}F \sim *$, placing us in the situation of Theorem 5.5. From there we can conclude that F is a product of Eilenberg-MacLane spaces $K(G, k)$ for abelian G (this is the step that does not apply to L_f). We now need to show that this product consists of just one Eilenberg-MacLane space $K(G, n+1)$ and that G is a p-torsion abelian group.

Since A is $(n-1)$-connected, the map $\Sigma A \to \mathrm{pt}$ is $(n+1)$-connected, and by property (9) (of the *Esteemed List of Elementary Properties*) so is the map $P_{\Sigma^2 A}X \to P_{\Sigma A}X$. This implies that F is n-connected. Write

$$F = \prod_{m \geq n+1} K(G_m, m).$$

We must show that for $m > (n+1)$ the space $K(G_m, m)$ is contractible and that G_{n+1} is p-torsion. Being retracts of F, the spaces $K(G_m, m)$ inherit two properties: they are $P_{\Sigma^2 A}$-local and $P_{\Sigma A}$-acyclic. We will deduce from the first property that for $m > n+1$, the group G_m is p-torsion free, and from the second property that for $m \geq n+1$, $G_m \otimes \mathbb{Z}[1/p] = 0$. The Theorem follows from this.

Since $H_n(A; \mathbb{Z}/p) \neq 0$, Proposition 3.4 implies

$$\begin{aligned} P_{\Sigma A}K(\mathbb{Z}/p, m) &\sim * \qquad m \geq n+1 \\ P_{\Sigma^2 A}K(\mathbb{Z}/p, m) &\sim * \qquad m > n+1. \end{aligned}$$

The fact that localization preserves finite products and filtered colimits then implies that

$$\begin{aligned} P_{\Sigma A}K(V, m) &\sim * \qquad m \geq n+1 \\ P_{\Sigma^2 A}K(V, m) &\sim * \qquad m > n+1 \end{aligned}$$

when V is any vector space over $\mathbb{Z}/p$. Making use of the Fibration Lemma (Lemma 4.4) one concludes that for $m > n+1$ the map

$$K(G_m, m) \to K(G_m/p\text{-torsion}, m)$$

is a $P_{\Sigma^2 A}$-equivalences, and hence an equivalence. This implies for $m > n+1$, G_m is torsion free.

To see that for $m > n+1$, $G_m \otimes \mathbb{Z}[1/p] = 0$ look at the fibration sequence

$$K(G_m \otimes \mathbb{Z}/p, m-1) \to K(G_m, m) \xrightarrow{p} K(G_m, m).$$

Again using the fibration lemma, this time for $P_{\Sigma A}$, one learns that

$$K(G_m, m) \xrightarrow{p} K(G_m, m)$$

is a $P_{\Sigma A}$-equivalence. Passing to the limit gives an $P_{\Sigma A}$-equivalence

$$K(G_m, m) \to K(G_m \otimes \mathbb{Z}[1/p], m).$$

On the other hand $K(G_m \otimes \mathbb{Z}[1/p], m)$ is ΣA-local. Indeed, since $A \to \mathrm{pt}$ is a $\mathbb{Z}[1/p]$-homology equivalence the inclusion of the constant maps

$$K(G_m \otimes \mathbb{Z}[1/p], m) \to \mathbf{Spaces}(\Sigma A, K(G_m \otimes \mathbb{Z}[1/p], m))$$

is a weak equivalence. But $K(G_m, m)$ has been assumed to be $P_{\Sigma A}$-acyclic. This implies that for $m > n+1$, $K(G_m \otimes \mathbb{Z}[1/p], m)$ is both $P_{\Sigma A}$-local and $P_{\Sigma A}$-acyclic. This means that $G_m \otimes \mathbb{Z}[1/p] = 0$.

It remains to show that G_{n+1} is a p-torsion group. Note as above that

$$K(G_{n+1}, n+1) \to K(G_{n+1}/p\text{-tors}, n+1)$$

is a $P_{\Sigma A}$-equivalence and so $K(G_{n+1}/p\text{-tors}, n+1)$ is $P_{\Sigma A}$-acyclic. Arguing as above one sees that for M p-torsion free the space $K(M, n+1)$ is $P_{\Sigma A}$-local. This implies that $G_{n+1}/p\text{-tors} = 0$. $\square$

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Department of Mathematics, Harvard University, Cambridge, MA 02138
E-mail address: mjh@math.harvard.edu